INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 9

RAVI VAKIL

Contents

From an e-mail of Oct. 6.

I thought I'd say in detail how you glue two prevarieties together along isomorphic open subprevarieties.

Suppose we're just gluing two prevarieties X and Y along open sets U in X and V in Y. In order to glue, we need a choice of homeomorphism between U and V, along with a choice of isomorphism of sheaves $\mathcal{O}_X$ restricted to U and $\mathcal{O}_Y$ restricted to V. (In other words, we've chosen an isomorphism of U and V as prevarieties.)

i) Sets. You can probably see how to glue X and Y together along $U = V$ as sets. Let $X \cup Y$ be this set; we can consider X and Y as subsets of $X \cup Y$ in a natural way.

ii) Topological spaces. A subset of $X \cup Y$ is declared to be open if its intersection with X is an open subset of X, and its intersection with Y is an open subset of Y. You can check that i) this is a topology, and ii) the restriction of the topology to X (resp. Y) is the original topology on X (resp. Y), and iii) that X (resp. Y) is an open set in this topology. (This is the definition of quotient topology.)

iii) Structure sheaves. Suppose W is an open subset of $X \cup Y$ (defined in ii)). I'll now tell you what the structure sheaf on W is. Let W_X be the intersection of W and X, and W_Y similarly, and W_U the intersection of W and X and Y. Sections over W correspond to the data of a section f of $\mathcal{O}_X$ over W_X, and a section g of $\mathcal{O}_Y$ over W_Y, such that the restriction of f to U is the same as the restriction of g to V (using the isomorphism of U and V as prevarieties). You can check that i) this is a sheaf, and ii) the restriction of the sheaf to the open subset X is our original sheaf $\mathcal{O}_X$ (and the same thing with X replaced by Y).

Class begins.

Date: Thursday, October 7, 1999.

Unrelated exercise, also about C; I probably won't mention this in class. If p is a point of $\mathbb{P}^1$, show that $\pi^{-1}p$ is two points unless p is one of two points of $\mathbb{P}^1$, when $\pi^{-1}p$ is one point.

Then if $\overline{k} = \mathbb{C}$, we have the following picture. (Give it.)

We will later see the *Riemann-Hurwitz formula*, which will tell us the following facts (even if $\overline{k} \neq \mathbb{C}$). Suppose we knew the genus of the base was 0 (we have to define genus in general), and that the cover is generally 2-1 (we have to define "generally"), except that it was 1-1 at two points. Then we could compute that the genus of the top curve was also 0.

Copies of Eisenbud-Harris are available in my office.

Schemes next day. Problem set: don't do scheme problems.

1. Projective Prevarieties

The original motivation for introducing $\mathbb{P}^n$ was to add some points to affine space $\mathbb{A}^n$, that correspond to extra "points at infinity". If $\overline{k} = \mathbb{C}$, then this is a compactification of $\mathbb{A}^n$. Here's the classical picture.

Suppose you were interested in a subset of $\mathbb{A}^2$, for example the curve $x^3+y^3 = xy$, a curve that comes up in Problem Set 4. Then if I draw it, it might look something vaguely like this. It's a real picture; really there are 3 directions in which the curve goes off to infinity; more about that soon. We'll embed it in projective space as follows. We'll basically add points at infinity corresponding to directions, i.e. we'll add a $\mathbb{P}^1$.

(Draw a picture; put $\mathbb{A}^2$ at the plane $x_0 = 1$. $(x, y) \mapsto (1, x, y)$.)

Take the closure. (This is called taking the *projective closure*, which we'll describe algebraically soon.) We get a curve in projective space, $x_1^3 + x_2^3 = x_1x_2x_0$. The open set U_0 is just the affine space we started with.

Now you can see that if you compactify $\mathbb{A}^2$ in this way, you add points of infinity that are really a $\mathbb{P}^1$: the points of projective space $(x_0; x_1; x_2)$ where $x_0 = 0$. Note that you can see the 3 points at infinity we've added to this curve, i.e. where $x_0 = 0$ (ask them).

The cone I've described is the *affine cone over a projective variety*; I'll rigorously define that soon.

As a good thought experiment, you might think about what happens when you do the same process with a parabola, e.g. $y = x^2$; how does it meet the line at ∞? In how many points?

Back to prevarieties.

These are various constructions that it would be good to be familiar with.

Definition. A projective prevariety is a prevariety that is isomorphic to a closed subprevariety of $\mathbb{P}^n$. A quasi-projective prevariety is a prevariety that is isomorphic to an open subprevariety of a projective prevariety.

There are many analogies between affine varieties and projective varieties, and we'll explore them now.

Consider a closed subprevariety X of $\mathbb{P}^n$, and say the projective coordinates on $\mathbb{P}^n$ are $x_0, \dots, x_n$.

Consider the ring $R = \overline{k}[x_0, \dots, x_n]$. Assign a grading to monomials in this ring, so the degree of $x_0^{a_0} \dots x_n^{a_n}$ is $\sum a_i$. An element of R is *homogeneous of degree d* if it is a linear combination of monomials of degree d.

From the problem set, you've seen that a non-zero homogeneous element of R of degree d cuts out a closed subset of $\mathbb{P}^n$, and if this subset is irreducible, we get a closed subprevariety of $\mathbb{P}^n$; such a subset (cut out by one equation) is called a *hypersurface of degree d.*

There is a projective analog of the correspondence between affine varieties and coordinate rings.

Definition. A *homogeneous ideal* of R is an ideal generated by homogeneous elements of R.

Lemma. Any homogeneous ideal A has a finite number of homogeneous generators.

Proof. (You won't be surprised that at some point we'll use the fact that R is Noetherian.) A is generated by some set of homogeneous elements, (f_a). We'll find a finite set of generators in this set. Pick f_1 to be any f_a of lowest degree. If $(f_1) = A$, we're done. Otherwise, pick f_2 of lowest degree in $A \setminus (f_1)$. If $(f_1, f_2) = A$, we're done. Otherwise, pick f_3 of lowest degree in $A \setminus (f_1, f_2)$. Etc. etc.

If we eventually stop, we're happy. But if we never stop, then we have a strictly ascending chain of ideals

$$(f_1) \subset (f_1, f_2) \subset (f_1, f_2, f_3) \subset \dots .$$

But any ascending chain must stabilize because R is Noetherian, giving a contradiction. ∎

If A is a homogeneous ideal, define $V(A)$ to be the points of $\mathbb{P}^n$ where $(x_0; \dots; x_n)$ where $f(x_0, \dots, x_n) = 0$ for each element f of A; it suffices to check the generators, and by the lemma, there are a finite number of generators.

If $S \subset \mathbb{P}^n$ is a closed algebraic subset, then define $I(S)$ to be the ideal generated by all the homogeneous polynomials that vanish identically on S.

Theorem. V and I set up a bijection between the set of closed algebraic subsets of $\mathbb{P}^n$ and the set of all homogeneous ideals $A \subset \overline{k}[x_0, \ldots, x_n]$ such that $A = \sqrt{A}$, except for the one ideal $A = (x_0, \ldots, x_n)$, called the *irrelevant ideal*.

(Note why the irrelevant ideal defines the empty set in projective space.)

Proof. Omitted; it uses the Nullstellensatz; see Mumford p. 13. □

Just as in the affine case, we have:

Corollary. In the bijection of the Theorem above, the irreducible algebraic sets correspond exactly to the homogeneous prime ideals (excepting the irrelevant ideal). Moreover, every closed algebraic set S in $\mathbb{P}^n$ can be written in a unique way as the union of distinct irreducibles.

Definition. The *projective coordinate ring* is the ring $\overline{k}[x_0, \ldots, x_n]/A$.

Caution. In the affine case, the data of an affine variety is the same as the data of the coordinate ring. In the projective case, the data of an abstract projective prevariety is *not* the same as the data of the projective coordinate ring, although the data of a projective prevariety with the choice of embedding in some $\mathbb{P}^n$ *is* the same as the data of the projective coordinate ring.

As an example, consider the two projective prevarieties $\mathbb{P}^1$ (with projective coordinates a and b, say) and the conic C given by $x^2 + y^2 = z^2$ in $\mathbb{P}^2$. These two are isomorphic, as you'll see in Problem Set 4.

In the first case, the ring is given by $\overline{k}[a, b]$.

In the second case, the ring you have is $\overline{k}[x, y, z]/(x^2 + y^2 - z^2)$.

Then if you look at the part of the first ring with grading 1, you have a vector space of rank 2; in the second case, you have a vector space of rank 3. So they aren't the same.

Classical constructions.

We can now define the classical constructions I mentioned earlier.

Affine cone. Suppose you have a projective prevariety corresponding to an ideal $A \subset \overline{k}[x_0, \ldots, x_n]$. Then the affine cone corresponds precisely to the same thing.

Projectivizing affine varieties.

Suppose you have an ideal $I \subset \overline{k}[x_1, \ldots, x_n]$, defining an affine variety $V \subset U_0 \cong \mathbb{A}^n$, and you want to find the projective closure $\overline{V}$ of V in $\mathbb{P}^n$.

First note: as V is an irreducible topological space, $\overline{V}$ is also irreducible. (This is a general topological fact, and not hard to prove; I might put it on the next problem set.)

So what's the homogeneous ideal $A \subset \overline{k}[x_0, \ldots, x_n]$? Define a "homogenization operator" that takes an element $f(x_1, \ldots, x_n) \in \overline{k}[x_1, \ldots, x_n]$, and makes it homogeneous in $\overline{k}[x_0, \ldots, x_n]$:

$$f(x_1, \ldots, x_n) \mapsto x_0^d f(x_1/x_0, \ldots, x_n/x_0)$$

where d is the largest degree that appears in f. (Example: $x_2^3 + x_1^3 = x_2 x_1$.)

Then to cook up A, just take *all* the elements of I (not just generators), homogenize them all, and look that the ideal A they generate in $\overline{k}[x_0, \ldots, x_n]$. (Question: why can't you just take generators? What goes wrong?)

An affine base of open sets.

Theorem. Suppose g is a homogeneous element of the projective coordinate ring of a projective prevariety $V \subset \mathbb{P}^n$, of *positive* degree. Define $D(g)$ to be the locus where g vanishes. Then $D(g)$ is affine. Moreover, these $D(g)$ form a base of the topology.

For simplicity, I'll give the proof in the case where $V = \mathbb{P}^n$; but the general proof is actually the same; if you're feeling really secure, then you should try to understand what I'm saying in general.

Theorem. $\mathbb{P}^n$ minus a hypersurface of positive degree is affine.

For example, in $\mathbb{P}^2$ with projective coordinates x, y, z, if you discard the locus $x = 0$, you get the standard open U_x, which is isomorphic to the affine plane. Ditto for $y = 0$ and $z = 0$. But it turns out that you can discard *any* line (e.g. $x + y = 0$), and still get something isomorphic to the affine plane. The moral reason is that, given some random line L there is an automorphism taking $\mathbb{P}^2$ to itself, and sending L to the x-axis. (In this case, $(x; y; z) \mapsto (x + y; y; z)$ works.)

Rough sketch of one proof. You've already done much of the work in the problem set. Suppose the hypersurface is defined by $f = 0$, and the prevariety is $W = \mathbb{P}^n \setminus \{f = 0\}$. The global section of the structure sheaf form a ring, and elements are of the form p/f^m where p is a polynomial homogeneous of degree md. This is a ring, finitely generated over $\overline{k}$, with no nilpotents, etc. etc., so it corresponds to an affine variety V. Then you just have to check that (i) this defines a morphism to V, and (ii) that this is an isomorphism, using affine opens. □

This is annoying. Far better is to re-define projective space.

Let R be the projective coordinate ring $\overline{k}[x_0, \ldots, x_n]$, which has a grading. (If you're following this proof for a general variety, then R is a quotient of this.) For each element $f \in R$ of positive homogeneous degree, define $(R_f)_0$ to be the zero-th graded piece of the ring R_f. Explicitly, it is the ring defined earlier. R is nilpotent-free finitely-generated domain over $\overline{k}$; you can check quickly that hence so is R_f, hence so is $(R_f)_0$. Thus this corresponds to an affine variety which I'll suggestively call $D(f)$. Glue together all these affines (as f varies over all homogeneous elements of R of positive degree) as follows. $D(f)$ and $D(g)$ are glued together along open subset $D(fg)$.

Let me make this somewhat explicit. The ring for $D(f)$ are elements of the form p/f^m, where $\deg p = m \deg f$. The ring for $D(g)$ are elements of the form q/g^n, where $\deg q = n \deg g$. The ring for $D(fg)$ are elements of the form $r/(fg)^l$, where $\deg r = l \deg(fg)$. Clearly the first two rings lie in the third; this defines the restriction map.

So this explains how to glue together 2 of these affine opens. Then you must check that the gluing data for 3 "agree", which isn't too hard for $\mathbb{P}^n$ (although you think more carefully if you're doing this for general V, even though the argument is the same). (Do this if enough time.)

Finally, we should check that we get the same prevariety as when we defined $\mathbb{P}^n$ before, which involved gluing together $n+1$ opens $U_0, \ldots, U_{n+1}$, not an infinite number of affine opens (including these). If you think about it, this new definition involves gluing together these $n+1$, plus a whole lot more, so we just need to check that we arne't adding any more points than the ones we already have. Equivalently, this says that given any $D(g)$, the open sets $U_i \cap D(g)$ cover it.

$D(g) = \{p/g^q\} \Leftrightarrow V$

Exercise (not on PS). Idea why this isn't surprising: Think of this as lying in $\mathbb{P}^n$. $\square$

Corollary. The $D(g)$'s (where g is a homogeneous element of positive degree) form a base (of affine opens) of the topology.

Proof. We show it first for $\mathbb{P}^n$. Then the Zariski topology on any closed subprevariety is induced by the topology on $\mathbb{P}^n$, so a base restricts to a base.

We show that it includes the base on the opens $U_0, \ldots, U_n$; this forms a base of the topology on their union. (Do this in a special case for convenience of exposition.) $\square$

Describing morphisms between projective varieties. Morphisms from projective prevarieties can be painful to describe, especially if you deal with them purely in terms of affine opens. There is often an easier way to describe them.

Suppose $V \subset \mathbb{P}^n$ is a projective prevariety, where the projective coordinates on $\mathbb{P}^n$ are $(x_0; \ldots; x_n)$. Suppose $p_0, \ldots, p_m$ are homogeneous polynomials in the x_i of degree d, that don't have any common vanishing set on V. Then $(p_0; \ldots; p_m)$ describes a morphism from V to $\mathbb{P}^m$.

The best way to convince you of this is to describe it in a particular case.

For example, consider the projective variety $\mathbb{P}^1$ with projective coordinates x, y. Then $(x^2; xy; y^2)$ defines a morphism to $\mathbb{P}^2$, with projective coordinates x_0, x_1, x_2. (You might notice that the image is given by $x_1^2 = x_0 x_2$.) First, (i) I'll describe this map as sets. It's well-defined, because $(x; y) \to (x^2; xy; y^2)$ and $(ax; ay) \to$ the same thing.

(ii) Next, I'll show it is a continuous map. A base for the topology on $\mathbb{P}^2$ is given by $D(f(x_0, x_1, x_2))$ where f is homogeneous, so we need to show that the pullback of the zero-locus of f is closed in $\mathbb{P}^1$ for all f. But the pullback of $f(x_0, x_1, x_2) = 0$ is $f(x^2, xy, y^2) = 0$, which is a closed set (as any set that is the vanishing set of a polynomial is closed).

(iii) Finally, we need to deal with structure sheaves. It suffices to consider the cover $\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$, and to show that regular functions on U_i pullback to regular functions on the preimage of U_i. Regular functions on U_0: $p(x_0, x_1, x_2)/x_0^{\deg p}$, where $x_0 \neq 0$. Pull back to get $p(x^2, xy, y^2)/(x^2)^{\deg p}$ where $x^2 \neq 0$.

The general argument is precisely the same. We have a map on sets, we show it is continuous by the same argument, and we deal with structure sheaves using the standard affine cover.

Warning: Not every morphism of projective prevarieties is of this form. But it makes a lot of them easy to describe.

If time: Classify degree 1 hypersurfaces. Classify degree 2 curves that are varieties.

End of lecture.

Collect problem sets. Hand out new sets.

Coming in the next few lectures (in some order):

1. Tues: schemes
2. Rational maps (why equivalent to maps of function fields?). Degree of rational maps. Birationality.
3. products
4. separatedness; defining varieties
5. properness?