# INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 1

Due Tuesday September 21 in class. No lates will be accepted, so Ana-Maria will be able to return it in time for Thursday's class.

1. Prove that the points of the form $(t, t^2, t^3)$ in $\mathbb{A}^3$ form an algebraic set. In other words, find a set of functions that vanish on these points, and no others.
2. If $I$ is an ideal, show that $\sqrt{I}$ is an ideal.
3. (a) Let $V$ be an algebraic set in $\mathbb{A}^n$, $P$ a point not in $V$. Show that there is a polynomial $F$ in $\overline{k}[x_1, ..., x_n]$ such that $F(Q) = 0$ for all $Q$ in $V$, but $F(P) = 1$. Hint: $I(V) \neq I(V \cup P)$.
   (b) Let $\{P_1, ..., P_r\}$ be a finite set of points in $\mathbb{A}^n(\overline{k})$. Show that there are polynomials $F_1, ..., F_r \in \overline{k}[x_1, ..., x_n]$ such that $F_i(P_j) = 0$ if $i \neq j$, and $F_i(P_i) = 1$.
4. Show that for any ideal $I$ in $\overline{k}[x_1, ..., x_n]$, $V(I) = V(\sqrt{I})$, and $\sqrt{I}$ is contained in $I(V(I))$. (You aren't allowed to use the Nullstellensatz.)
5. (a) If $I_1$ and $I_2$ are ideals of some ring, show that $\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}$.
   (b) If $I_1$ and $I_2$ are radical ideals, show that their intersection is also radical.
6. Find the ideal in $\overline{k}[x_1, x_2, x_3]$ corresponding to the $x_1$-axis union the point (1,1,1).
7. Without descending into horrific algebra, prove that there are three points $a, b, c \in \mathbb{A}^2$ such that
   $$\sqrt{(x^2 - 2xy^4 + y^6, y^3 - y)} = \mathfrak{m}_a \cap \mathfrak{m}_b \cap \mathfrak{m}_c.$$
   Hint: interpret both sides geometrically.
   (The following part is not for credit, but the answer will give you some insight into schemes.) Can you think of some reason why you would know that $(x^2 - 2xy^4 + y^6, y^3 - y)$ is not a radical ideal?
8. (a) If $S$ is a multiplicative subset of $R$ containing 0, show that $S^{-1}R$ is the zero-ring (i.e. the ring with one element, which you can call 0 or 1).
   (b) Let $A$ be a (commutative) ring. Prove that an element $a \in A$ is nilpotent (that is, $a^n = 0$ for some $n > 0$) if and only if $a$ belongs to every prime ideal of $A$.
9. Consider the following two conditions on a ring $R$.
   (i) Every ascending chain of ideals of $R$ eventually stabilizes (i.e. $R$ is Noetherian, by definition).
   (ii) Every ideal of $R$ is finitely generated.
   Prove that $R$ has condition (i) if and only if $R$ has condition (ii).

*Date*: September 14, 1999.