INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 10

Due Tuesday November 30 in class; this will possibly be extended. You're strongly encouraged to collaborate (although write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument).

Nakayama's Lemma. Hand in three of these problems. (In solving them, feel free to invoke the one you skip.) Hint: they all have solutions that are one or two lines long.

1. If $\mathfrak{a} \subset A$ is an ideal such that every element of $1+\mathfrak{a}$ is invertible (which holds for example if $A/\mathfrak{a}$ is a local ring), M a finitely generated A-module and $M' \subset M$ any submodule, then $M' + \mathfrak{a}M = M$ implies that $M' = M$.
2. If $\mathfrak{a} \subset A$ is an ideal such that every element of $1+\mathfrak{a}$ is invertible, M a finitely generated A-module, show that the elements $u_1, \dots, u_n \in M$ generate M if and only if their images generate $M/\mathfrak{a}M$.
3. Let A be a Noetherian ring, and $\mathfrak{a} \subset A$ an ideal such that every element of $1+\mathfrak{a}$ is invertible in A. Then $\cap_{n>0}\mathfrak{a}^n = (0)$. *Tougher. Shafarevich's proof is wrong: How does he know that $\mathfrak{a} \cap_n \mathfrak{a}^n = \cap_n \mathfrak{a}^n$? Use Artin-Rees.*
4. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Suppose $a_1, \dots, a_n \in \mathfrak{m}$ generate $\mathfrak{m}/\mathfrak{m}^2$ (as a vector space). Show that $a_1, \dots, a_n$ generate $\mathfrak{m}$ (as an ideal).

Normalization.

5. Define the normalization $\tilde{X} \to X$ of an arbitrary variety X (and show that it is well-defined). (Hint: prove the lemma I mentioned in class, which shows that normalization "behaves well with respect to restriction to distinguished opens".)
6. *Explicit examples.* Hand in two of these three. In each case, explicitly identify the curve. (I *think* that in each case you get $\mathbb{A}^1$ minus some points.)
   (a) *A cusp.* Find the normalization of $y^2 = x^3$. (In other words, find an additional element — or additional elements — of the function field that satisfy an integral equation over the ring of regular functions. Use it to construct the candidate of the normalization. Show that the candidate is nonsingular, hence normal.)
   (b) *A triple point.* Find the normalization of $y^3 = x^3 + x^4$.
   (c) *A tacnode.* Find the normalization of $y^2 = x^4 + x^5$.
7. *A singular normal point.* (We know that the lowest dimension such an example can have is 2.) Show that the cone $z^2 = x^2 + y^2$ is normal but singular. (One method: follow the suggestions of Hartshorne Ex. II.6.4, p. 147.)

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Line bundles. This explicit problem will hopefully make line bundles and invertible sheaves (a central idea for the rest of the course) more clear.

8. Prove that a non-zero meromorphic section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(m)$ (over $\mathbb{P}^1$) has total sum of orders of vanishing m. Hence these no two of these line bundles are isomorphic. (Hint: Describe the meromorphic section over the standard open U_0, and figure out what it must be over the standard open U_1.)