INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 11

Due Thursday December 9 in class. Hand in 6 of these problems. Ask someone about the ones you skip. Problem 8 has a tricky step, but the punchlines are worth it, so I encourage you to try it. (You might want to try it earlier than the rest, to give yourself time to talk to me about it.) You're strongly encouraged to collaborate (although write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument).

I'll use "line bundles" and "invertible sheaves" interchangeably.

1. *Secretly trivial line bundles.* Suppose U_i is a finite open cover of X, and for each i, h_i is an invertible function on U_i. Prove that the data of the open cover U_i and the transition functions $f_{ij} = h_i/h_j$ defines a line bundle, and prove that this is isomorphic to the trivial line bundle. In fancy language, this shows that a zero-element of $H^1(X, \mathcal{O}^*)$ indeed induces a trivial line bundle. (One possibility: Use the proposition, proved in Class 23, that any invertible sheaf on X with a nowhere vanishing section is isomorphic to the trivial sheaf $\mathcal{O}_X$.) Conversely, suppose that the line bundle given by U_i and f_{ij} is trivial. Prove that there are invertible functions h_i on the U_i such that $f_{ij} = h_i/h_j$ on $U_i \cap U_j$.
2. Give (with proof) the identification between the vector space of global sections of $\mathcal{O}_{\mathbb{P}^n}(m)$ (where $\mathbb{P}^n$ has projective coordinates $x_0, \dots, x_n$) and the vector space of polynomials in $x_0, \dots, x_n$ of homogeneous degree m.
3. *Torsion in the Picard group.* Let X be the variety $\mathbb{P}^2$ minus an irreducible conic. Let $\mathcal{O}_X(m)$ be the restriction of $\mathcal{O}_{\mathbb{P}^2}(m)$ to X. Show that $\mathcal{O}_X(2)$ is trivial, but that $\mathcal{O}_X(1)$ isn't. Hence $\mathcal{O}_X(1)$ is a 2-torsion element of $\operatorname{Pic} X$. (Hint: Show that $\mathcal{O}_X(2)$ has a global section vanishing nowhere, and that every global section of $\mathcal{O}_X(1)$ vanishes somewhere. Use the proposition, proved in Class 23, that any invertible sheaf on X with a nowhere vanishing section is isomorphic to the trivial sheaf $\mathcal{O}_X$.
4. Prove that $\mathcal{O}_{\mathbb{P}^1}(p)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$. *A small addition, thanks to a good question in Thursday's class.* If you choose coordinates so $p = [1; 0]$, then the global sections of $\mathcal{O}_{\mathbb{P}^1}(p)$ can be identified with linear combinations of the rational functions 1 and x_0/x_1. You might think that the global section corresponding to 1 has no zeros or poles, but this isn't true! (For example, you know from Problem Set 10 that the sums of orders of zeros and poles of $\mathcal{O}_{\mathbb{P}^1}(1)$ is 1.) So: find the zeros and poles corresponding to 1.
5. Consider the morphism $\pi : \mathbb{P}^1 \to \mathbb{P}^3$ given by $[u; v] \mapsto [u^4; u^3v; uv^3; v^4]$ which expresses $\mathbb{P}^1$ as a closed subvariety C of $\mathbb{P}^3$. (You don't need to verify that.) By Problem 2, the space of global sections of $\mathcal{O}_{\mathbb{P}^3}(1)$ has dimension 4. Show

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that this induces a 4-dimensional vector space of global sections $\pi^*\mathcal{O}_{\mathbb{P}^3}(1)$. (Hint: show that no nonzero section of $\mathcal{O}_{\mathbb{P}^3}(1)$ vanishes on all of C. This may involve exhibiting 4 points of C that are not coplanar.) Show that $\pi^*\mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_{\mathbb{P}^1}(4)$, so (by Problem 2), $\pi^*\mathcal{O}_{\mathbb{P}^3}(1)$ has a 5-dimensional vector space of global sections.

Hence if $\mathcal{L}$ is an invertible sheaf on a variety Y, and X is a subvariety of Y, then the restriction of Y to X may have more sections than those sections restricted from Y.

6. *Pullbacks of divisors and pullbacks of line bundles.* Suppose $\pi : X \to Y$ is a dominant morphism of nonsingular curves, and D is a divisor $\sum_{q \in Y} n_q q$ on Y. Define the divisor $\pi^* D$ on X by $\sum_{p \in X} m_p n_{\pi(p)} p$ where m_p is defined as follows. Let u be a uniformizer in the stalk $\mathcal{O}_{Y,\pi(p)}$. Then u pulls back to give an element $\pi^* u$ of $\mathcal{O}_{X,p}$. Then $m_p = v_p(\pi^* u)$. Prove that m_p is independent of the choice of u. Then prove that $\pi^*\mathcal{O}_Y(D) \cong \mathcal{O}_X(\pi^* D)$.
7. Suppose X is a nonsingular projective curve, and $\mathcal{L}$ is an invertible sheaf on X of degree d. Prove that the sections of $\mathcal{L}$ form a *finite-dimensional* $\overline{k}$-vector space. (Hint: Prove it if $d < 0$. Show by induction that if $d \geq 0$, then $\mathcal{L}$ has at most $d+1$ linearly independent sections.) This fact is true more generally, whenever X is a projective scheme. It is not necessarily true if X isn't projective (consider $\mathbb{A}^1$ for example).
8. *Fun with elliptic curves.* Let C be the curve $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2$. (Everything will work for $y^2 z = x^3 - \alpha x z^2$ for any nonzero α.) Let $\pi : C \to \mathbb{P}^2$ be the inclusion. Let $\mathcal{L} = \pi^*\mathcal{O}(1)$ (an invertible sheaf on C).
(a) Show that $\deg \mathcal{L} = 3$.
(b) Show that $\mathcal{L}$ has a 3-dimensional family of global sections, and that they are all restrictions of global sections of $\mathcal{O}_{\mathbb{P}^2}(1)$. Hence any nonzero global section of $\mathcal{L}$ vanishes at collinear points (where C is considered as lying in the plane). This step might get tricky. If you get stuck after setting up what you want to prove, come by on Wednesday afternoon or earlier.
(c) Use (a), (b) and Problem 2 to show that C is not isomorphic to $\mathbb{P}^1$.
(d) *A nontrivial degree 0 line bundle.* Let p and q be two distinct points of C. Let $\mathcal{M} = \mathcal{O}_C(p-q)$ (a degree 0 line bundle). Show that $\mathcal{M}$ is not isomorphic to the trivial line bundle as follows. Let q, r, s be distinct collinear points on C so that $p \neq r$, $p \neq s$. Show that $\mathcal{O}_C(q+r+s) \cong \mathcal{L}$. Hence $\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_C(p+r+s)$, so there is a global section of $\mathcal{L} \otimes \mathcal{M}$ with divisor $p+r+s$. Show that $\mathcal{L} \otimes \mathcal{M}$ can't be isomorphic to $\mathcal{L}$ using (b), and hence that $\mathcal{M}$ isn't trivial. (Once again, this proves that C is not isomorphic to $\mathbb{P}^1$, as the only degree 0 invertible sheaves on $\mathbb{P}^1$ is the trivial sheaf.)
(e) *The group law on the elliptic curve.* We can put a group structure on the curve as follows. Call the point $[0;1;0]$ o; this will be the zero-point. Prove that $\mathcal{O}(3o) \cong \mathcal{L}$. Define a map $i : C \to \mathrm{Cl}(C)$ by $p \mapsto p - o$. Show that this map is injective (as sets), using (d). Given two points p and q on C, define a point r as follows. Let L be the line joining p and q in the plane; say it meets C at r' as well. Let M be the line joining o and r'; let r be the third point of intersection. (Questions to ask yourself: what if L is tangent to C at p? What if $p = q$? Think about these and similar issues. In writing up the problem, feel free to avoid them, but say clearly what cases you aren't dealing with, and what assumptions you make.) Prove that $i(p) + i(q) = i(r)$.

Given a point $p \in C$, describe a point p' such that $i(p) + i(p') = i(o)$ (with proof). Show that this gives the points of C the structure of a group. (We will see, on the last day of class or in PS12, that the image of i are precisely the degree 0 line bundles.)