INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 3

Due Thursday October 7 in class. No lates will be accepted, so Ana-Maria will be able to return it in time for the next Tuesday's class. Do any 7 of the following 9 problems, including Problem 9 (which is, I think, the hardest). You're strongly encouraged to collaborate (although obviously write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument). Ask someone about the solutions to the two problems you don't hand in too.

More straightforward problems.

1. Prove that every prevariety X is an irreducible topological space. (In particular, by a result on problem set 2, every open set is dense.) Prove that every prevariety is a Noetherian space.
2. Let X be the prevariety that is the line with the "double origin". Find the global sections of $\mathcal{O}_X$.
3. Find the global sections of $\mathcal{O}_{\mathbb{P}^n}$.
4. Prove that $\mathbb{A}^2 \setminus \{(0,0)\}$ is not an affine variety.

Sheaf problems.

5. Suppose $f : X \to Y$ is a (continuous) morphism of topological spaces, and F is a sheaf on X. To each open set U on Y, define $G(U) = F(f^{-1}(U))$. Show that G is a sheaf on Y. (The sheaf G is denoted $f_* F$, and is called the *pushforward* of F. There is also a *pullback*, but it is more complicated to define.)
6. In class, I mentioned that you can recover a sheaf from knowing its sections over the elements of a base, and the restriction maps between them. This exercise makes this precise.
   (a) Suppose F is a sheaf on a topological space X, with open subset U that is the union of open sets U_i (where i varies over some set A). Show that
   $$F(U) \cong \{(f_i \in F(U_i))_{i \in A} | \operatorname{res}_{U_i, U_i \cap U_j} f_i = \operatorname{res}_{U_j, U_i \cap U_j} f_j \text{ for all } i, j \in A \}.$$
   (b) Suppose furthermore that $U_i \cap U_j$ is the union of open sets U_{ijk} where k varies over some set A_{ij}. Show that
   $$F(U) \cong \{(f_i \in F(U_i))_{i \in A} | \operatorname{res}_{U_i, U_{ijk}} f_i = \operatorname{res}_{U_j, U_{ijk}} f_j \text{ for all } i, j \in A, k \in A_{ij} \}.$$
   (c) Explain how you can recover a sheaf from knowing its sections over the elements of a base of the topology, and the restriction maps between them. (Translation: convince Ana-Maria you understand how (b) above gives this fact.)

Date: September 28, 1999.

Theory.

7. In the course of proving that if X is an affine variety and f is a regular function, then $D(f)$ is an affine variety, we used a fact we didn't prove. We'll correct this oversight now. Suppose X lies in $\mathbb{A}^n$ with co-ordinates $x_1, \ldots,$ x_n, and the corresponding (radical) ideal is $I = I(X)$. As $f \in \overline{k}[x_1, \ldots, x_n]/I$, let f_1 be an element of $\overline{k}[x_1, \ldots, x_n]$ restricting to f (i.e. choose a polynomial "representing" f). Let J be the ideal in $\overline{k}[x_1, \ldots, x_{n+1}]$ generated by I and $1 - f_1 x_{n+1}$. (We showed that J is prime.) Define a morphism (of affine algebraic sets) from $V(J)$ to X by
$$\pi : (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n).$$
Show that π gives a homeomorphism from $V(J)$ to $D(f)$.
8. *Projection from a point.* Explain how the map $\mathbb{P}^n \setminus \{(1; 0; \ldots; 0)\} \to \mathbb{P}^{n-1}$ given by $(x_0; x_1; \ldots; x_n) \mapsto (x_1; \ldots; x_n)$ corresponds to a morphism of prevarieties.
9. Let $f(x_0, x_1, \ldots, x_n)$ be a polynomial of *homogeneous degree* d (i.e. x_i has degree 1, and each monomial appearing in f has total degree d).
(a) Note that the set $S = \{f(x_0, \ldots, x_n) = 0 : (x_0; \ldots; x_n) \in \mathbb{P}^n\}$ is well-defined: if $\lambda \in \overline{k}^*$, $f(x_0, \ldots, x_n) = 0$ if and only if $f(\lambda x_0, \ldots, \lambda x_n) = 0$. Show that S is a closed subset of $\mathbb{P}^n$.
(b) Let U be its complement; calculate $\mathcal{O}_{\mathbb{P}^n}(U)$. (Hint: calculate the sections over the $n+1$ standard affine open sets, and try to patch them together.)
(c) (Not for credit; tricky but doable given what you know.) Think about how you would prove that $(U, \mathcal{O}_{\mathbb{P}^n}|_U)$ is an affine variety if the degree of f is positive. (We'll do this in class; if you understand this, you'll have a good handle on projective varieties.)