INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 5

Due Thursday October 21 in class (no lates). Hand in seven of the following questions. You're strongly encouraged to collaborate (although write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument). If you skip any of the non-scheme ones, ask someone else how to do them.

A miscellaneous question.

1. Suppose X is an affine variety, so $X \times X$ is affine as well. Let $\Delta : X \to X \times X$ be the diagonal morphism. Explicitly describe the ring map $A(X \times X) \to A(X)$.

Topology.

2. Prove that an affine variety (considred as a closed subset of $\mathbb{A}^n$) is irreducible if and only if its projective closure is irreducible.
3. A topological space X is *Hausdorff* if for every two points x and y, there are non-intersecting open sets U and V containing x and y respectively. Prove X is Hausdorff if and only if the diagonal subset of $X \times X$, i.e. $\{(x,x) \in X \times X\}$, is a closed subset. (Recall that $X \times X$ has a base of open sets of the form $U \times V$ where U and V are open; this is the definition of the product topology.) This criterion for "Hausdorffness" will parallel a criterion for a prevariety to be separated.
4. Show that the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$ is not the same as the (Zariski) topology on $\mathbb{A}^2$.

Projective prevarieties. Assume that the characteristic is not 2 in the following two problems.

5. Show that the projective subprevarieties X and Y of $\mathbb{P}^3$ given by $wx - yz = 0$ and $w^2 + x^2 + y^2 = z^2$ respectively are isomorphic. Describe both one-parameter families of lines in Y.
6. (a) Show all projective prevarieties in $\mathbb{P}^2$ defined by a polynomial of degree 2 are isomorphic. (More precisely, suppose $p(x_0, x_1, x_2)$ is a degree 2 polynomial whose zero-locus V in $\mathbb{P}^2$ is a projective (pre)variety. Show that V is isomorphic to $\mathbb{P}^1$.) You can use the classification of quadratic forms (over an algebraically closed field), see e.g. Lang's Algebra Ch. XIV: given any quadratic form $p(x_0, x_1, x_2)$, there is a change of basis such that $p = \sum_{1 \le i \le j} y_i^2$ for some $j \in \{0, 1, 2\}$.

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(b) Give two degree 2 polynomials in 4 variables that define projective prevarieties in $\mathbb{P}^3$ that are *not* isomorphic. (Neither may be the square of a linear form.) You don't have to prove that they are not isomorphic.

7. Consider the morphism $\mathbb{P}^1 \to \mathbb{P}^n$ given by

$$[u; v] \mapsto [u^n; u^{n-1}v; \dots; v^n] = [x_0; \dots; x_n]$$

(where n is a postive integer). Let V be the image.

(a) Let I be the homogeneous ideal of the graded ring $R = \overline{k}[x_0, \dots, x_n]$ generated by the 2×2 minors of the matrix

$$\begin{bmatrix} x_0 & x_1 & \dots & x_{n-1} \\ x_1 & x_2 & \dots & x_n \end{bmatrix}$$

Show that V is a projective prevariety, with homogeneous coordinate ring R/I.

(b) Show that the map from $\mathbb{P}^1$ onto its image is an isomorphism. (The image is called a *rational normal curve*. The case $n = 2$ is a *smooth conic*. The case $n = 3$, which we've seen before, is a *twisted cubic*.)

Playing around with schemes.

8. (a) If $\mathfrak{p}$ is the point of $\operatorname{Spec} \overline{k}[x, y]$ corresponding to the prime ideal $(y - 3x^2)$, show that $\{\mathfrak{p}\}$ is not a closed subset in the Zariski topology, i.e. $\mathfrak{p}$ is not a *closed point*.

(b) Suppose $\mathfrak{p}$ is a prime ideal of some ring R. Then the ring morphism $R \to R_{\mathfrak{p}}$ corresponds to a map of schemes $\pi : \operatorname{Spec} R_{\mathfrak{p}} \to \operatorname{Spec} R$. Show that π is injective. Thus the points of $R_{\mathfrak{p}}$ form a subset of the points of R; which prime ideals of R do they correspond to? (Feel free to quote results from commutative algebra if you want.)

(c) Describe the points of $\operatorname{Spec} \mathbb{R}[x]$ (and compare to the points of $\mathbb{C}$ and $\mathbb{R}$).

9. The ring morphism $\mathbb{Z} \to \mathbb{Z}[i]$ corresponds to a map of schemes $f : \operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$. Suppose (p) is a prime ideal of $\mathbb{Z}$ (warning: p could be 0). Find the points of $f^{-1}(p)$ in $\operatorname{Spec} \mathbb{Z}[i]$. Compare the degree of the residue field extensions with the number of points of $f^{-1}(p)$; one prime (not 0) will be a special case.