# MODULI SPACES AND DEFORMATION THEORY, CLASS 11

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## 1. First order deformations of nonsingular varieties $X/k$

**Definition.** *First-order deformations* of $f : X \to k$ are precisely fibered diagrams of the form

$$\begin{array}{ccc} X & \to & \tilde{X} \\ \downarrow & & f \downarrow \\ \operatorname{Spec} k & \to & \operatorname{Spec} k[\epsilon]/\epsilon^2 \end{array}$$

where $f$ is flat.

(Note that the left side *is* $X/k$; automorphisms of $X$ don't come into it!)

Denote these $\operatorname{Def}(X/k)$. I'll define deformations (with no adjective) later today.

*Exercise.* Suppose we have some nice moduli stack, e.g. $\mathcal{M}_g$. Sow that there is a bijection between first-order deformations and the tangent space to the Deligne-Mumford stack.

Note that the tangent space to a Deligne-Mumford stack has a natural $k$-vector space structure, but it isn't clear that these diagrams do!

**Theorem.** $\operatorname{Def}(X/k)$ is naturally in bijection with $H^1(X, T_X)$.

(Note that the right side also has a vector space structure!)

Proof in a few minutes.

*Exercise.* If $X$ is a nonsingular curve of genus at least 2, then $h^1(X, T_X) = 3g-3$.

We'll later see that this means that $\mathcal{M}_g$ is smooth.

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*Remark.* Note that the automorphism group of the curve acts on $H^1(X, T_X)$. (Make geometric comment about $M_g$.)

*Flatness lemma.* (Eisenbud Cor. 6.2, p. 163 — this is one of the five basic things one should know about flatness.) If $M$ is a $k[\epsilon]/\epsilon^2$-module, then $M$ is flat iff

$$M/\epsilon M \stackrel{\epsilon\times}{\to} \epsilon M$$

is an isomorphism. (Clearly it is surjective. This states that multiplication doesn't kill any more than necessary.)

Using the above, we see that first-order deformations of $X$ are precisely given by infinitesimals extensions of $X$ by $\mathcal{O}_X$, completing the proof of the theorem.

## 2. Artin rings

**Motivation.** Show that $\mathcal{M}_g$ is nonsingular. By way of: $\mathcal{M}_g$ is nonsingular at a point $[C] \in \mathcal{M}_g$. Infinitesimal lifting property (and finite type) gives it to us.

More generally, $\overline{\mathcal{M}}_g$ is nonsingular, and boundary divisors intersect transversely. We'll show this by understanding the "deformation space of a node". (Sketch.)

**Definition.** An *Artin ring* is a ring satisfying the ascending chain condition. For rings over a field $k$, this precisely those rings that are finite-dimensional vector spaces. (Draw picture.) A *local Artin ring* is an Artin ring with only one maximal ideal, e.g. $k[x, y, z]/(x^2, y^3, z^4 - x - y)$.

*Example:* The $n$th-order formal neighborhood of a $k$-valued point of a scheme $X$. Locally, it looks like $(A, \mathfrak{m})$; the neighborhood is $A/\mathfrak{m}^{n+1}$.

Let $\mathcal{C}$ be the category of local Artin rings over $k$, with residue field $k$. In other words, the objects are $(A, \mathfrak{m})$ with residue field $k$, and morphisms induce isomorphim of the residue field.

*Non-example.* We've lost some Artin rings. For example, the second-order formal neighborhood of $(p)$ in $\operatorname{Spec} \mathbb{Z}$ is (Spec of ) $\mathbb{Z}/p^2$, which is not a $\mathbb{Z}/p$-algebra.

*Universal example.* These are precisely the $n$th order formal neighbourhoods of schemes over $k$ that are locally of finite type (or even locally of finite presentation).

Let $\hat{\mathcal{C}}$ be the category of *complete* Noetherian local $k$-algebras, with residue field $k$, for which $A/\mathfrak{m}^n$ is in $\mathcal{C}$ for all $n$. Notice that $\mathcal{C}$ is a full subcategory of $\mathcal{C}$.

*Example.* A formal neighborhood of a $k$-valued point of a scheme over $k$, i.e. the inverse limit of its $n$th order rings. Usually denoted Spf rather than Spec, to remind you of the limit, and the topology involved.

Denote $t_A^*$ by $\mathfrak{m}/\mathfrak{m}^2$; the Zariski cotangent space of $\operatorname{Spec} A$.

Here are some basic facts about Artin rings.

I forgot to mention (but will next day): *Algebra exercise.* A morphism $B \to A$ in $\mathcal{C}$ is surjective if and only if the induced map $t_B^* \to t_A^*$ is surjective. $\mathcal{C}$ replaced by $\hat{\mathcal{C}}$.

We can also check when a morphism $\mathcal{C}$ is (formally) smooth. (I've put formally in brackets, as quasicompactness is automatic.)

**Definition.** Suppose $G \to F$, in $\hat{\mathcal{C}}$. Then we need to check if

$$\begin{array}{ccc} \operatorname{Spf} A & \to & \operatorname{Spf} F \\ \downarrow & \overset{?}{\nearrow} & \downarrow \\ \operatorname{Spf} B & \to & \operatorname{Spf} G \end{array}$$

where $B$ is an extension of $A$ by a square-zero ideal, and $B$ and $A$ are in $\mathcal{C}$. Then we say $\operatorname{Spec} F \to \operatorname{Spec} G$ is *smooth*.

Don't be scared by Spf; just do this on rings. I've written Spf so as to keep the arrows going in the geometric direction.

*Remove square-zero! Replace by surjection $B \to A$, again, to check this, just need to check on tangent spaces.*

Similarly, you can define *etale* (exists exactly one) and *unramified* (at most one).

Thus $\operatorname{Spf} F \to \operatorname{Spf} G$ in $\hat{\mathcal{C}}$ is smooth (etale, unr) if for all $\operatorname{Spf} A \to \operatorname{Spf} B$ in $\mathcal{C}$, where $B \to A$ is surjective,

$$\operatorname{Hom}(F, B) \to \operatorname{Hom}(F, A) \times_{\operatorname{Hom}_{G,A}} \operatorname{Hom}(G, B)$$

as sets, is surjective, (bijective, injective).

*Exercise.* It is equivalent to require that $B$ and $A$ are in $\hat{\mathcal{C}}$. (Sketch why.)

*Exercise.* (a) $\operatorname{Spf} F \to \operatorname{Spf} G$ is smooth iff $F$ is a power series ring over $G$. (Etale: isomorphism; unramified: closed immersion.)

(b) A composition of smooth morphisms is smooth. (Etale, unr.)

(c) If $u : \operatorname{Spf} F \to \operatorname{Spf} G$ and $v : \operatorname{Spf} G \to \operatorname{Spf} H$ and $u$ is surjective and $vu$ is smooth, then $v$ is smooth.

(d) Smoothness is preserved by base change. (Etale, unr.)

We talked at length about Spf.