MODULI SPACES AND DEFORMATION THEORY, CLASS 18

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Contents

Where we are: We've proved Schlessinger's criteria. Jason applied this to the Quot functor last day. Today, I'll apply it to the Picard functor and the Deformation functor.

Recall Schlessinger's criteria for existence of universal deformations and hulls (miniversal deformations). (Have your handouts handy!)

Fix our functor $F : \mathcal{C} \to \text{Sets}$.

Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{1}$$

Schlessinger's Theorem.

(1) F has a hull iff F has properties H1–H3:

- H1. (You can glue.) (1) is a surjection whenever $A'' \to A$ is a small extension. Equivalently whenever $A'' \to A$ is *any* surjection.
- H2. (Uniqueness of gluing on $k[\epsilon]/\epsilon^2$.) (1) is a bijection when $A = k$, $A'' = k[\epsilon]/\epsilon^2$. Equivalently, $A'' = k[V]$. Then by previous lemma, t_F is a k-vector space.
- H3. (finite-dimensional tangent space) $\dim_k(t_F) < \infty$.

(2) F is pro-representable if and only if F has the additional property

- H4. (bijection for gluing a small extension to itself)

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A'). \tag{2}$$

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is a *bijection* for any small extension $A' \to A$.

Recall from earlier. Assume F satisfies H1–H3. Now given a fairly small extension $p : A' \to A$. Given any $a \in F(A)$, i.e. family over A, the set of lifts to $F(A')$ has a transitive action by the group $t_F \otimes I$. H4 is precisely the condition that this set is a principal homogeneous space under $t_F \otimes I$. (Say more here.)

1. Statement: Prorepresentability of the Picard functor

Here's the Picard functor I want to consider. Fix a scheme X. Recall the Picard group $\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*)$ (Cech cohomology). In fact this is a group scheme.

For convenience, let $X_A = X \times_k \operatorname{Spec} A$. Fix $\mathcal{L}_0 \in \operatorname{Pic} X$. We will study deformations of this line bundle.

Let $P(A)$ be

$$\left\{ \begin{array}{rccc} & \mathcal{L} & \to & \mathcal{L}_0 \\ & | & & | \\ X \times \operatorname{Spec} A = & X_A & \to & X \end{array} \right\} / \text{isom.}$$

That's not quite right; more precisely, line bundles on $\mathcal{L}_A$ that restrict to something isomorphic to $\mathcal{L}_0$ on X.

Those are the families; morphisms are pullback diagrams.

Theorem. Assume $H^0(X, \mathcal{O}_X) = k$, and $h^1(X, \mathcal{O}_X)$ is finite (e.g. if X is proper and connected). Then P is prorepresentable and $t_p \cong H^1(X, \mathcal{O}_X)$.

1.1. **Important remark: Relative version.**

This result is more interesting in a relative situation, if X is deforming at the same time. Fix

$$\begin{array}{ccc} X_0 & \to & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \to & \operatorname{Spf} \Lambda \end{array} \quad \text{flat}$$

where Λ is a complete local Noetherian ring. (For example, $k[[x_1, \dots, x_n]]$; or else, $k = \mathbb{F}_p$, $\Lambda = \mathbb{Z}_p$.) The category $\mathcal{C}$ becomes the category of Artinian local Λ-algebras having residue field k, i.e. there is a structure morphism $\Lambda \to A$ or $\operatorname{Spf} A \to \operatorname{Spf} \Lambda$ inducing a trivial extension of residue fields.

Schlessinger's theorem applies in this relative setting; the proof is actually identical to the one I've given to the more limited version. (I decided to just state the simpler version to keep notation to a minimum, as it's already pretty hairy.)

Theorem. If $H^0(X_0, \mathcal{O}_{X_0}) = k$, and $h^1(X_0, \mathcal{O}_{X_0})$ is finite, then P is prorepresentable (and $t_P = H^1(X_0, \mathcal{O}_{X_0})$).

What this means: $\operatorname{Spf} R \to \operatorname{Spf} \Lambda$.

Also, the first hypothesis turns out to imply $A \to H^0(X_A, \mathcal{O}_A)$ is an isomorphism for all $A \in \mathcal{C}$, using flatness. Theme: to get isomorphism over A, need only check over closed fiber.

The proof to this is also identical to the proof I'm going to give to the more limited case.

2. Recall flatness results

Flatness Lemma.

The bottom square is

$$\begin{array}{ccc} B := A' \times_A A'' & \to & A'' \\ \downarrow & & \downarrow \quad \text{surjection} \\ A' & \to & A \end{array}$$

(in $\mathcal{C}$).

Over it, you have

$$\begin{array}{ccc} ? & \to & M'' \\ \downarrow & & \downarrow \quad u'' \\ M' & \xrightarrow{u'} & M \end{array}$$

Each of these three squares (incident to A in the cube) are tensor products, i.e. $M \cong M' \otimes_A A'$ (via u'), and $M \cong M'' \otimes_A A''$ (via u'').

Also, M' is a flat A'-module and M'' is a flat A''-module, and M is a flat A-module.

It's easy to fill in the upper corner with N: $N = M' \times_M M''$, which is indeed a B-algebra:

$$\begin{array}{ccc} N = M' \times_M M'' & \to & M'' \\ \downarrow & & \downarrow \\ M' & \to & M \end{array}$$

Then: N is a flat B-module! And the remaining two squares are pullback squares, i.e. $N \times_B A' \xrightarrow{\sim} M'$ and $N \times_B A'' \xrightarrow{\sim} M''$.

In terms of pseudo-geometry: we have this glued together family, and it has the properties we want: it is a flat family, and these two squares are pullback squares.

Flatness Corollary. With the same notation as above, let L be a B-module in a commutative diagram

$$\begin{array}{ccccc} & L & \stackrel{q''}{\to} & M'' & \\ q' & \downarrow & & \downarrow & u'' \\ & M' & \stackrel{u'}{\to} & M & \end{array}$$

such that q' induces an *iso*morphism $L \times_B A' \to M'$. Then the canonical morphism $q' \times q'' : L \to M' \times_M M'' = N$ is an isomorphism.

3. Proof of prorepresentability of the Picard functor

We check H1–H4.

We can check H1, H2, H4 all at once. Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map (1)

$$P(A' \times_A A'') \to P(A') \times_{P(A)} P(A''). \tag{3}$$

We will show that this is a bijection when $A'' \to A$ is a small extension. So assume this is the case for the rest of the proof.

Let $u' : (A', L') \to (A, L)$, $u'' : (A'', L'') \to (A, L)$ be morphisms of couples.

Let $Y = X_A$, $X' = X_{A'}$, $X'' = X_{A''}$.

On the topological space $X = X_k = X_0$, we have morphisms $\mathcal{O}_{X'} \to \mathcal{O}_Y$ and $\mathcal{O}_{X''} \to \mathcal{O}_Y$ of sheaves of algebras on this topological space, and sheaves L', L, L'' where we have isomorphisms $L' \otimes_{A'} A \xrightarrow{\sim} L$, $L'' \otimes_{A''} A \xrightarrow{\sim} L$.

Let $B = A' \times_A A''$, $Z = X_B$. Then we have a commutative diagram of sheaves

$$\begin{array}{ccc} \mathcal{O}_Z & \to & \mathcal{O}_{X''} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X'} & \to & \mathcal{O}_Y \end{array}$$

of sheaves on X_0.

There's another sheaf of B-algebras on this topological space: $\mathcal{O}_{X'} \times_{\mathcal{O}_Y} \mathcal{O}_{X''}$, whose sections are

$$\mathcal{O}_{X'} \times_{\mathcal{O}_Y} \mathcal{O}_{X''}(U) = \mathcal{O}_{X'}(U) \times_{\mathcal{O}_Y(U)} \mathcal{O}_{X''}(U).$$

We have a canonical isomorphism $\mathcal{O}_Z \xrightarrow{\sim} \mathcal{O}_{X'} \times_{\mathcal{O}_Y} \mathcal{O}_{X''}$, by the Flatness Corollary.

Similarly, construct a sheaf $N = L' \times_L L''$ as a sheaf on the space; it is an invertible sheaf on Z (i.e. first it is a sheaf of B-modules). And the projections of N on L and L'' induce isomorphisms $N \otimes_B A' \xrightarrow{\sim} L'$, $N \otimes_B A'' \xrightarrow{\sim} L''$.

Reality check: we've just shown that (1) is always a surjection. We now need to check that it is an injection.

Suppose M is another invertible sheaf on Z for which there exist isomorphisms $M \otimes A' \xrightarrow{\sim} L'$, $M \otimes A'' \xrightarrow{\sim} L''$.

Note!: We don't assume that this induces an isomorphism when we restrict both of these to X_0! More precisely:

We have morphisms $q' : M \to L'$, $q'' : M \to L''$ which induce these isomorphisms. Hence we have a commutative diagram:

$$\begin{array}{ccccccc} & & & M & & & \\ & & \stackrel{q'}{\swarrow} & & \stackrel{q''}{\searrow} & & \\ & L' & & & & L'' & \\ & u' \searrow & & & & \swarrow u'' & \\ & & L & \stackrel{\theta}{\to} & L & & \end{array}$$

Here θ is given precisely by

$$L \to L' \otimes_{A'} A \to M \otimes_B A \to L''' \otimes_{A''} A \to L.$$

By hypothesis 2, θ is multiplication by some $a \in A^*$. Lifting a back to A'', we can change q'' to $a''q''$; thus we may assume that $u'q' = u''q''$. It follows from the Flatness Corollary that $M \xrightarrow{\sim} N$. Thus (1) is a bijection for *any* surjection $A'' \to A$ in $\mathcal{C}$.

That proves H1, H2, H4.

Let's now prove H3, i.e. $\dim_k(t_F) < \infty$.

We need to consider $X_{k[\epsilon]}$; note that

$$\mathcal{O}_{X_{k[\epsilon]}} = \mathcal{O}_X \oplus \epsilon \mathcal{O}_X.$$

Thus there's a split exact sequence

$$0 \to \mathcal{O}_X \stackrel{f \mapsto 1+\epsilon f}{\to} \mathcal{O}^*_{X_{k[\epsilon]}} \to \mathcal{O}^*_X \to 1.$$

Note that this first arrow is really the *exp* map.

These are all sheaves on the same space of course.

The fact that it's split means that the long exact sequence can be taken to start at H^1, from which

$$F(k[\epsilon]) = \ker\{H^1(X, \mathcal{O}^*_Y) \to H^1(X_0, \mathcal{O}^*_{X_0})\} \cong H^1(X_0, \mathcal{O}_{X_0}).$$

This has finite dimension by hypothesis, completing the proof of the theorem.

4. The deformation functor has a hull (in good situations)

Recall the deformation functor. Fix a scheme X over k. The deformation functor associates to each $A \in \mathcal{C}$ the set of diagrams

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \quad \text{flat} \\ \operatorname{Spec} K & \hookrightarrow & \operatorname{Spec} A \end{array}$$

inducing $X \xrightarrow{\sim} Y \times_A k$, up to isomorphism (say what this is).

Small but important and useful fact. There's a fact about flatness that will come in handy later in the course. This isomorphism just requires an isomorphism over the closed points $\operatorname{Spec} k$. Reason: *Fun lemma.* Let A be a ring, J a nilpotent ideal in A (e.g. $A \in \mathcal{C}$, $J \neq (1)$), and $u : M \to N$ a homomorphism of A modules, N flat over A. If $\overline{u} : M/JM \to N/JN$ is an isomorphism, then u is an isomorphism too.

Call this deformation functor D or D_X.

Theorem. If X is either (a) proper over k, or (b) affine with isolated singularities (e.g. node), then D has a hull.

In case (b), I don't think X need be finite type, as I think I said to someone incorrectly after class!

The method is quite similar to the Picard functor case. We'll prove this next time.