MODULI SPACES AND DEFORMATION THEORY, CLASS 19

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Contents

Recall Schlessinger's criteria for existence of universal deformations and hulls (miniversal deformations). (Have your handouts handy!)

Fix our functor $F : \mathcal{C} \to \text{Sets}$.

Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{1}$$

Schlessinger's Theorem.

(1) F has a hull iff F has properties H1–H3:

- H1. (You can glue.) (1) is a surjection whenever $A'' \to A$ is a small extension. Equivalently whenever $A'' \to A$ is *any* surjection.
- H2. (Uniqueness of gluing on $k[\epsilon]/\epsilon^2$.) (1) is a bijection when $A = k$, $A'' = k[\epsilon]/\epsilon^2$. Equivalently, $A'' = k[V]$. Then by previous lemma, t_F is a k-vector space.
- H3. (finite-dimensional tangent space) $\dim_k(t_F) < \infty$.

(2) F is pro-representable if and only if F has the additional property

- H4. (bijection for gluing a small extension to itself)

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A'). \tag{2}$$

is a *bijection* for any small extension $A' \to A$.

Recall from earlier. Assume F satisfies H1–H3. Now given a fairly small extension $p : A' \to A$. Given any $a \in F(A)$, i.e. family over A, the set of lifts to

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$F(A')$ has a transitive action by the group $t_F \otimes I$. H4 is precisely the condition that this set is a principal homogeneous space under $t_F \otimes I$. (Say more here.)

1. The deformation functor has a hull (in good situations)

Recall the deformation functor. Fix a scheme X over k. The deformation functor associates to each $A \in \mathcal{C}$ the set of diagrams

$$\begin{array}{ccccc} X & \hookrightarrow & Y & \\ \downarrow & & \downarrow & \text{flat} \\ \operatorname{Spec} K & \hookrightarrow & \operatorname{Spec} A & \end{array}$$

inducing $X \stackrel{\sim}{\to} Y \times_A k$, up to isomorphism (say what this is).

Mentioned last day: **Small but important and useful fact.** There's a fact about flatness that will come in handy later in the course. This isomorphism just requires an isomorphism over the closed points $\operatorname{Spec} k$. Reason: *Fun lemma.* Let A be a ring, J a nilpotent ideal in A (e.g. $A \in \mathcal{C}$, $J \neq (1)$), and $u : M \to N$ a homomorphism of A modules, N flat over A. If $\overline{u} : M/JM \to N/JN$ is an isomorphism, then u is an isomorphism too.

Call this deformation functor D or D_X.

Theorem. If X is either (a) proper over k, or (b) affine with isolated singularities (e.g. node), then D has a hull.

In case (b), I think we need X to be finite type; let's see what happens.

Once again, we look at (1):

$$D(A' \times_A A'') \to D(A') \times_{D(A)} D(A''). \tag{3}$$

where $A'' \to A$ is a small extension. We prove H1 and H2 first, i.e. that this is always a surjection, and that in the special case $k[\epsilon] \to k$ it is a bijection.

Fix $(A', X') \to (A, X)$, $(A'', X'') \to (A, X)$. We have a diagram of deformations

$$\begin{array}{ccc} & & X'' \\ & & \uparrow \quad u'' \\ X' & \stackrel{u'}{\leftarrow} & Y. \end{array}$$

These can all be considered sheaves of algebras on the same topological space $|X|$. We construct the sheaf $\mathcal{O}_{X'} \times_{\mathcal{O}_Y} \mathcal{O}_{X''}$ of $A' \times_A A''$ algebras. Then this actually gives a scheme: $(|X|, \mathcal{O}_{X'} \times_{\mathcal{O}_Y} \mathcal{O}_{X''})$; call it Z. In fact it is the cofiber product or sum in the category of preschemes under Y, homeomorphic to Y.

By our flatness lemma, it is flat over $A' \times_A A''$, and is an element of $D(A' \times_A A'')$.

Hence (1) is surjective; thus H1 is checked.

Next for H2, that (1) is a bijection in the case $A'' = k[\epsilon]$, $A = k$. Suppose now that W is another deformation over B, inducing the deformations

$$\begin{array}{ccccc} & & W & & \\ & \stackrel{q'}{\nearrow} & & \stackrel{q''}{\nwarrow} & \\ X' & & & & X'' \\ u' \nwarrow & & & & \nearrow u'' \\ & Y & \stackrel{\theta}{\rightarrow} & Y & \end{array}$$

If θ were the identity, then W would be Z by our flatness corollary, proving H2. But it might not in general.

But in H2, we have $A = k$, i.e. $Y = X$, so θ is an isomorphism! So we win. H2 is done.

All that remains: H3.

We've already shown that if X is smooth over k, then $t_D \cong H^1(X, T_X)$, so t_D has finite dimension if X is smooth and proper over k; this completes the proof of part (a) in the case of X smooth.

For (a) in general (proper), (b), affine with isolated singularities, I'll invoke some machinery.

For any scheme X locally of finite type over k, there is an exact sequence

$$0 \to H^1(X, T^0) \to t_D \to H^0(X, T^1) \to H^2(X, T^0)$$

where T^0 is the sheaf of derivations of $\mathcal{O}_X$, T^1 is a (coherent) sheaf isomorphic to the sheaf of germs of deformations of X/k to $k[\epsilon]$.

Remarks on this: (i) For example, if X is smooth over k, then $T^0 = T$, $T^1 = 0$. (ii) It will not surprise many of you to hear that this comes from a spectral sequence.

So in (a), if X is proper, we win. In (b), we also win (as t_D sits between a space of dimension 0 and one of finite-dimension).

Corollary. The hull in this case prorepresents D iff for each small extension $A' \to A$ and each deformation Y' of X/k to A', every automorphism of the deformation $Y' \otimes_{A'} A$ is induced by an automorphism of Y'.

This follows from our general comments after the statement of Schlessinger's criteria.

Again, the key example where we don't have prorepresentability is a deformation of a node connecting two rational curves.

2. Deformations of local complete intersections

Suppose X is a local complete intersection scheme over k. We'll study deformations of X. Then the first-order deformations are given by $\operatorname{Ext}^1(\Omega_X, \mathcal{O}_X)$. In the case when X is nonsingular, we get $H^1(\mathcal{O}_X, T_X)$, which we've already proved.

Here's the plan. Local complete intersections are locally cut out by the expected number of equations. In other words, given any point p, some neigborhood U is isomorphic to a closed subscheme of some $\mathbb{A}^n$, for some n, and is cut out by an expected number of equations.

3. Embedded deformations

Define complete intersection, local complete intersection. We'll see that the morphism of functors from embedded deformations to deformations is formally smooth. This will let us find hulls to deformations. Gluing these ideas together, we'll understand deformations of local complete intersections.

References:

- Vistoli's *Deformation of complete intersections* (eprints),
- M. Artin's *Deformations of singularities* (Tata notes)
- M. Artin (tall guy in building 2)

Facts. Fix $X \hookrightarrow \mathbb{A}^n$, affine over k, not required to be complete intersection. Let $X_A \to \operatorname{Spec} A$ (flat) be a deformation of X, $A \in \mathcal{C}$.

Lemma. X_A is affine; in fact, $X_A \hookrightarrow \mathbb{A}^n \times \operatorname{Spec} A$.

Caution: I've typed this in quickly, and it hasn't been done very well!

Proof. It suffices to prove:

$$0 \to J \to A' \to A \to 0$$

$(J^2 = 0)$,

$$\begin{array}{ccc} \mathbb{A}^n_A & \hookrightarrow & \mathbb{A}^n_{A'} \\ \uparrow & & \uparrow ? \\ X_A & \hookrightarrow X_{A'} & \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \hookrightarrow \operatorname{Spec} A' & \end{array} .$$

On level of algebras:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & I & & \\ & & \downarrow & & \\ A'[x_1,\dots,x_n] & \stackrel{?}{\to} & \mathcal{O}(X_{A'}) & & \\ \downarrow & & \downarrow A[x_1,\dots,x_n] & \stackrel{?}{\to} & \mathcal{O}(X_A) \end{array}$$

Here I is an ideal sheaf. $I^2 = 0$ as $J^2 = 0$. Hence I is an $\mathcal{O}_{X_A}$-module. $H^1(X_A, I) = 0$.

Thus

$$0 = rightarrow H^0(X_A, I) \to H^0(X_{A'}, \mathcal{O}_{X_{A'}}) \to H^0(X_A, \mathcal{O}_{X_A}) \to 0.$$

So lift $x_i \in H^0(X_A, \mathcal{O}_{X_A})$ to $x_i' \in H^0(X_{A'}, \mathcal{O}_{X_{A'}})$.

Claim. $A'[x_1', \dots, x_n'] \to \mathcal{O}(X_{A'})$.

Proof. This is the half of the fun lemma not requiring flatness. $M' \to N'$ A'-modules; $M' \otimes_{A'} A \to N' \otimes_{A'} A$ is surjective. Then $M' \to N'$ is also surjective.

Actually, we're not done the proof of the lemma yet! We don't yet know that $X_{A'}$ is affine! See next day for a bit more.

3.1. **The functor of embedded deformations.** Define a functor $\mathrm{Def}_{X \hookrightarrow \mathbb{A}^n}$ on $\mathcal{C}$, that sends A to the closed subschemes X_A of $\mathbb{A}^n_A$ that are flat over A, and that restrict to X over k.

We have a morphism of functors $\mathrm{Def}_{X \hookrightarrow \mathbb{A}^n} \to \mathrm{Def}_X$. We've now proved that this is formally smooth. Here's how.

Coming up: Flatness and relations.

Idea: $X \subset \mathbb{A}^n$ cut out by m equations, dimension $n - m$. Then flat embedded deformations correspond precisely to jiggling the equations.