MODULI SPACES AND DEFORMATION THEORY, CLASS 20

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Contents

1. Special lecture by Ragni Piene

Most of this class was a special lecture by Ragni Piene, visiting from Oslo.

2. Embedded deformations (of $X \hookrightarrow \mathbb{A}^n$)

(I mentioned an observation of Brian that a family of smooth affine varieties is formally locally trivial, even when the family is not trivial!)

We were proving:

Proposition. Suppose we have a fiber diagram

$$\begin{array}{ccc} X & \hookrightarrow & X_A \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} A \end{array}$$

where $A \in \mathcal{C}$. Suppose $X \hookrightarrow \mathbb{A}^n$. Then there is a closed immersion $X_A \hookrightarrow \mathbb{A}^n_A := \mathbb{A}^n \times \operatorname{Spec} A$. (Translation.)

This was reduced to:

Proposition. Suppose $A' \to A$ is a square-zero extension in $\mathcal{C}$, with kernel J. Suppose we have a fiber diagram

$$\begin{array}{ccc} X_A & \hookrightarrow & X_{A'} \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \hookrightarrow & \operatorname{Spec} A'. \end{array}$$

Suppose $X \hookrightarrow \mathbb{A}^n_A$. Then there is a closed immersion $X_{A'} \hookrightarrow \mathbb{A}^n_{A'}$. Brian loves this sort of question.

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Let's use the notation $\mathcal{O}(X)$ to be the ring of global sections of the structure sheaf of a scheme X.

The argument went as follows. Let $x_1, \ldots, x_n \in \mathcal{O}(X_A)$ be the functions on X_A mapping it to $\mathbb{A}^n_A$.

The first step is to lift them to functions of $\mathcal{O}(X_{A'})$; this will give a map $X_{A'} \to \mathbb{A}^n_{A'}$, restricting to the map over A.

Here's how that happened. Let $|X|$ be the underlying topological space of X_A and $X_{A'}$. Let $\mathcal{I}$ be the ideal sheaf of X_A in $X_{A'}$, so

$$0 \to \mathcal{I} \to \mathcal{O}_{X_{A'}} \to \mathcal{O}_{X_A} \to 0.$$

Then $J^2 = 0$ meant $\mathcal{I}^2 = 0$. Thus $\mathcal{I}$ is actually a coherent sheaf of $\mathcal{O}_{X_A}$ modules. X_A is affine, so $H^1(X_A, \mathcal{I}) = H^1(X, \mathcal{I}) = 0$. Thus the long exact sequence begins:

$$0 \to H^0(X, \mathcal{I}) \to \mathcal{O}(X_{A'}) \to \mathcal{O}(X_A) \to 0.$$

Thus we can lift $x_1, \ldots, x_n$.

So now we have a map $X_{A'} \to \mathbb{A}^n_{A'}$, and we wish to show that this is a closed immersion. On the level of topological spaces, it's already an inclusion of $X_{A'}$ as a closed subset (as on the level of topological spaces, this is the same as $X_A \hookrightarrow \mathbb{A}^n_A$).

Hence we only need to check that this separates tangent vectors. This is left as an exercise.

The upshot of this is that the natural morphism of functors (Emb def. X) $\to$ (Def X) is formally smooth.

3. Relations Criterion

So our fundamental question is this.

Suppose we have $X \hookrightarrow \mathbb{A}^n$. Suppose we have a fiber diagram

$$\begin{array}{ccc} X & \hookrightarrow & X_A \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} A \end{array}$$

where $A \in \mathcal{C}$. Suppose $X_A \hookrightarrow \mathbb{A}^n_A$, restricting to $X \hookrightarrow \mathbb{A}^n$.

How can we tell when X_A is flat over A?

(Some description here.)

An A-module M is flat if $N \to M \otimes_A N$ is exact. It's easy to show that this is equivalent to $\operatorname{Tor}_1^A(M, N) = 0$ for all finitely-generated N; this doesn't use any property of $\mathcal{C}$.

However, an A-module that's finitely generated can be filtered so that successive quotients are all k, so M is flat if $\mathrm{Tor}_1(M, k) = 0$. (**Warning:** Here we are using $A \in \mathcal{C}$!)

Now suppose we have a presentation for the ideal of definition (call it I_A) of X_A in $\mathbb{A}^n_A$:

$$A[x_1, \ldots, x_n]^l \to A[x_1, \ldots, x_n]^m \to A[x_1, \ldots, x_n] \to \mathcal{O}(X_A) \to 0.$$

(Explain.)

$$A[x_1, \ldots, x_n]^l \to A[x_1, \ldots, x_n]^m \to I_A \to 0,$$
$$0 \to I_A \to A[x_1, \ldots, x_n] \to \mathcal{O}(X_A) \to 0.$$

Lemma. $\mathcal{O}(X_A)$ is A-flat iff the above presentation for I_A, tensored by k, is a presentation for I.

Proof. Break it up into two pieces, using I_A (see above). (Short check.)

We also note that flatness is the same as $I_A \otimes_A k = I$.

The following is a fundamental useful fact. (I didn't really state it during class — although Jason effectively did. I'll state it next time.)

Proposition (Relations criterion for flatness). Let $\mathcal{O}(X) = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, and $\mathcal{O}(X_A) = A[x_1, \ldots, x_n]/(f'_1, ..., f'_m)$ where f'_i are liftings of f_i. Then X_A is flat over A iff every relation among $(f_1, \ldots, f_m)$ lifts to a relation among $(f'_1, \ldots, f'_m)$.

I'll now prove the Relation criterion. Later, we'll use it to prove stuff.

For the rest of this discussion, suppose we are given $\mathcal{O}(X) = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and liftings f'_i of f_i to $A[X]$.

This translates to: We have an exact sequence

$$A[x_1, \ldots, x_n]^m \to A[x_1, \ldots, x_n] \to \mathcal{O}(X_A) \to 0,$$

which, when tensored with k, gives

$$k[x_1, \ldots, x_n]^m \to k[x_1, \ldots, x_n] \to \mathcal{O}(X) \to 0.$$

Note that this need not imply that $I_A \otimes k = I$ (where I_A, and I, are as defined earlier). If it did, we'd already have flatness!

A bit more explanation: Suppose we have a complete set of relations for the f_i's. In other words, an exact sequence

$$k[x_1, \ldots, x_n]^l \to k[x_1, \ldots, x_n]^m \to k[x_1, \ldots, x_n] \to \mathcal{O}(X) \to 0.$$

Then giving a *lifting* of these relations means giving

$$A[x_1, \ldots, x_n]^l \to A[x_1, \ldots, x_n]^m \to A[x_1, \ldots, x_n] \to \mathcal{O}(X_A) \to 0,$$

which is exact except at $A[x_1, \dots, x_n]^m$, where it is only assumed to be a *complex*. In other words, there may be more relations! We're requiring that the relations lift, not that there aren't any more relations upstairs.