MODULI SPACES AND DEFORMATION THEORY, CLASS 5

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Contents

Initial remarks. Classes 3 and 4 aren't yet posted on the web, but I still hope to write them up. Due to time constraints, I don't think I'll have time to write up notes that are as clean as the ones I posted last year in the introductory course. But I figure it would be better to have some notes of reasonable quality rather than none of great quality.

In case you're wondering about the exercises, none of are to be handed in (yet). I'll give out some problems in a handout at some point, but not for another week at least. But I still encourage you to try the exercises, which I found quite enlightening.

Today: I'd like to define a category fibered in groupoids over another category, and in particular a groupoid over the category of schemes. I could have just put this definition on the board a few classes ago, but I wanted to motivate why this is a good definition, and why we are being so anal-retenttive with definitions: because things go wrong if we don't.

Then I'll define a morphism between such things, or more precisely, a functor. Then I'll describe fiber products.

Then I'll return to a couple of motivating examples.

Finally, I'll motivate and define Grothendieck topologies, and in particular the etale topology.

Remark: Much of what I say is due to comments given by many people in this class.

Some initial facts: Exercise. If you have a "connected groupoid", where each object has associated group, or "automorphism group", G, then the number of arrows

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from one object to another can be given in bijection with G. Exercise: Interesting comment from Brian on a connected groupoid with abelian automorphism group. The identification of automorphism groups is by conjugation; so when the group is abelian, the automorphism groups are *canonically* isomorphic. Which means that there are two groupoids of the following form (describe it).

1. Groupoids over {Sch}

We're interested in "moduli spaces".

Initially: contravariant functor from {Sch} to sets. Things that worked well: representable functors. Certain moduli functors, such as the Hilbert functor.

When we had automorphisms, we had major issues. If the objects in question have no automorphisms, there's no problem. If you have a genus 1 curve with level 3 structure, and I do, then they are isomorphic, or they're not; if they are, they are isomorphic in only one way. So we can really get away with considering them the "same". But if you have nontrivial automorphisms, there is more than one way that we can identify them.

So tentatively: moduli functor is a contravariant functor from {Sch} to groupoids. I should mention that groupoids form a category. I will how try to convince you that something wonky is going on.

1.1. **Equivalence of groupoids.**

If we're going to consider moduli functors in this abstract way, where families are different, then the following problem comes up, which I briefly mentioned last time.

Hilbert functor take 1: closed subschemes flat over the base. Groupoid: one-element set.

Hilbert functor take 2: closed immersions, flat over the base. Groupoid: huge connected groupoid, with trivial automorphism group.

These two are different groupoids, but we want to say they are essentially the same. Well, they are equivalent categories. Recall again that two morphisms of categories $F : A \to B$ and $G : B \to A$ give an equivalence of categories if $F \circ G$ and $G \circ F$ are equivalent to the identiy B and A respectively.

Exercise. Show that those two groupoids are equivalent.

This equivalence (which Sharon says topologists consider to be "homotopies") will get rid of any conusion over definition of moduli functors.

Moral. Just as "being in love means never having to say your sorry", "being an algebraic geometer means never saying two objects are the same" — just perhaps canonically isomorphic. We will deliberately try to avoid every saying same now.

Possible patch: moduli functors are contravariant functors from schemes to "groupoids modulo equivalence". Uh-oh: is this a category? What's going on? This gets hairy, so I'll give a better patch later on.

Problem 1: Pullbacks are only defined up to unique isomorphism

This moral gives us problems with describing moduli functors as being contravariant functors from schemes to groupoids.

Problem 2: Deepee's explicit example.

Moral: fiber products of groupoids don't respect equivalence of categories. This indicates that the category of groupoids isn't what we want.

Here's another problem, which again motivates thinking about things slightly more generally. Recall the exercise I gave before: show that the following diagram (of functors to groupoids) is a fiber square.

$$\begin{array}{ccc} \tilde{S} & \rightarrow & pt \\ \downarrow & & \downarrow \\ S & \rightarrow & pt/2 \end{array}$$

In fact it isn't. I didn't tell you how to take fiber products of groupoids, but there's an obvious candidate. (I haven't checked if this really is a fiber product, and it won't matter.)

Deepee suggested looking at what happens when S is a point, and evaluating these functors at a point. (Say this clearly.) Here are the groupoids. **Picture.** But here's the actual fiber product. **Picture.**

Now let me replace one of the points by an equivalent category. (Do it, show they are equivalent.) Do the fiber product, get the different answer.)

Here's the patch. Don't think of a fiber product in the category of groupoids. (In any case, I haven't checked if fiber products should exist! If they do, this is it.)

In fact, you can define something analogous to fiber product for categories (and groupoids are, of course, categories). I'll also call it a fiber product. The key thing is that you should remember that you are no longer allowed to say things are equal.

Set-up:

$$\begin{array}{ccc} A \times_C B & \rightarrow & B \\ \downarrow & & \downarrow G \\ A & \xrightarrow{F} & C \end{array}$$

The objects of the upper corner are $(a \in A, b \in B, i : F(a) \rightarrow G(b))$. What are morphisms $(a, b, i) \rightarrow (a', b', i')$?

$$\begin{array}{ccc} F(a) & \xrightarrow{i} & G(b) \\ \downarrow & & \downarrow \\ F(a') & \xrightarrow{i'} & G(b') \end{array}$$

commutes.

Example: We get the right answer in Deepee's example!

Fact. Fiber products for categories respects equivalence. (Exercise, if you'd like.)

2. A category fibered in groupoids, or a groupoid over $\{\text{Sch}\}$

So here is the right definition.

Definition. A *groupoid over* $\{\text{Sch}\}$ is a category F together with a *covariant* functor $p_F : F \to \{\text{Sch}\}$ such that:

(i) If $f : X \to Y$ is an arrow in $\{\text{Sch}\}$ and y is an object of F with $p_F(y) = Y$, then there exists an arrow $g : x \to y$ such that $p_F(g) = f$.

A moduli functor has this property. By (ii) below, this "pullback" is defined up to canonical isomorphism.

(ii) (Say it.)

Exercise. Use (ii), substituting $Y = X$, to show that this "pullback" is defined up to canonical isomorphism.

For each scheme $X \in \{\text{Sch}\}$, the "preimage of X" is a category. Precisely: take the objects upstairs mapping to X, and the morphisms upstairs mapping to the identity.

Exercise. Show that this is a groupoid.

Hence to every X, again, we have a groupoid. However, pullback isn't required to be "canonical". **Definition.** We say this groupoid are the "sections over X" (explanation later), and write it $F(X)$.

Exercise. Show that a contravariant functor from $\{\text{Sch}\}$ to groupoids can be interpreted as a groupoid over $\{\text{Sch}\}$.

So usually we define a moduli groupoid by defining its sections over X, and the pullback-type maps are assumed to be obvious. For example, the moduli groupoid $\mathcal{M}_g$ is still: $X \mapsto$ groupoid of proper flat smooth morphisms $C \to X$ such that the geometric fibers are connected genus g curves.

We really think of equivalent categories as the same.

We can define fibered products of these categories as before; we can work out the fibered product of the categories over each $X \in \{\text{Sch}\}$.

Big exercise. With these redefinitions, show that the following square is a fibered product of categories.

$$\begin{array}{ccc} \tilde{S} & \rightarrow & pt \\ \downarrow & & \downarrow \\ S & \rightarrow & pt/2 \end{array}$$

(To talk about pt, you may prefer to replace {Sch} with {Sch}/ $\operatorname{Spec} \overline{k}$.)

Exercise for the algebraic geometry experts. Show that $F_2 \rightarrow F_2^{Hilb}$ is a morphisms of groupoids over {Sch}.

Another big exercise. Assuming the previous exercise, compute the fibred product:

$$\begin{array}{cccc} ? & \rightarrow & pt & \\ \downarrow & & \downarrow & \\ F_2 & \rightarrow & F_2^{Hilb} & = \mathbb{P}^2 \end{array}$$

(where now = means equivalent!). There are two cases. If pt doesn't map to the discriminant locus, then $? = pt$. If pt maps to the discriminant locus, then $? = pt/2$.

Hence we get the following geometric picture. (Draw it. Isomorphism away from the discriminant locus. Half-points along the discriminant locus.)

I should mention that this groupoid is called $(\mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}/2)$. (Say a bit about that; quotients by finite groups.)

3. Grothendieck topologies

Moduli groupoids have nicer structure than this, which they share with schemes. To describe that, I'll tell you about Grothendieck topologies, including the etale topology.

Let me first recast the classical topology in categorical language. As I describe it, you should also think of the Zariski topology.

Suppose we have a topological space X. Then imagine a category A where the objects are the open sets of X, and the morphisms are inclusions. There's a final object X. Notice that $U \cap V$ is the fiber product in this category. Also, unions are categorical sums. What's a presheaf F of sets? Precisely a contravariant functor from the category A to the category of sets. To make it a sheaf, we have an additional two properties. A key ingredient is the idea of a *covering*; a covering of an open set U is, of course, a union of open sets $\cup_i U_i = U$. A sheaf is a presheaf (i.e. contravariant functor) such that for all coverings,

$$F(U) \rightarrow \prod_i F(U_i) \rightarrow\rightarrow \prod_{i,j} F(U_i \cap U_j)$$

is "exact" (interpret). [How to get to rightarrows on top of each other?] Identity and gluability.

Global sections: $F(X)$.

We can talk about sheaves of abelian groups, rings, etc. We have cohomology of sheaves in 2 ways.

(a) In "good situations", Cech cohomology.

(b) We have a left-exact functor, global sections. The category of abelian sheves can be shown to be an abelian category with "lots" of injective objects. Hence higher derived functors exist.

In this example, the objects are sets, in fact subsets of X; but no mention was made of this in the definitions.

Here are the definition/axioms for a Grothendieck topology T; along side the definition, I'll give an example, which is the etale topology of a scheme X.

It's a category A whose objects are called open sets.

(Etale maps $U \to X$. Morphisms are $U_1 \to U_2$ over X; notice that these have to be etale too. (Exercise — Ezra sketched an argument why.))

And a set of "coverings". A covering is a set of morphisms in A, where the morphisms have the same image. ($U_i \to U$, where the union of the images is all of U)

a. Fibred products exist. (Clear.) **(Daniel pointed out to me after class that we need the fact that any morphisms in the category are etale, which was mentioned above.)**

b. All isomorphisms are coverings. (Clear.) Coverings of coverings are coverings. I.e.: If $\{U_i \stackrel{p_i}{\to} U\}$ is a covering, and if for all i, $\{U_{ij} \stackrel{p_i \circ q_{ij}}{\to} U\alpha\}$ is a covering, then the whole collection $\{U_{ij} \stackrel{p_i \circ q_{ij}}{\to} U\}$ is a covering. (Clear.)

c. Coverings pullback. I.e. If $\{U_\alpha \stackrel{p_i}{\to} U\}$ is a covering, and $V \to U$ is any morphism, then $V \times_U V_\alpha \stackrel{q_\alpha}{\to} V$ is a covering. (Clear.)

d. A has a final object. We'll want to remove this later on. (Clear.)

Another example, where we'd not want a final object: we can put an etale topolgy on the category of all schemes. (For the experts: this is the difference between the big and small etale sites. But if you know that, you're probably not in this course.)

A sheaf of sets on T is a contravariant functor F (that's the presheaf) such that the following diagram of sets is exact:

$$F(U) \to \prod_i F(U_i) \to\to \prod_{i,j} F(U_i \cap U_j).$$

Again, we can define cohomology groups.

Example/exercise: The “structure sheaf” (define) is a presheaf. (It is in fact a sheaf.)

Coming soon. Thinking about the etale topology in a hands-on way. Definition of a stack. Definition of a Deligne-Mumford stack.