MODULI SPACES AND DEFORMATION THEORY, CLASS 9

RAVI VAKIL

Contents

1. Intersection theory on nonsingular proper DM-stacks

Remark: A proper nonsingular stack has a good intersection theory.

Points have degrees.

Want to check transversality.

Example: $\Delta/12 = \omega$ on stack of elliptic curves. (This means up to torsion, but in fact $\omega \equiv \Delta/12$.) This corresponds to the fact that there's a weight 12 modular form Δ. I'm a little confused, because I thought sections of ω were weight 2, but Julia mentioned that there were differing weight conventions

2. Studying moduli stacks formally locally

What's a stack? Locally, a scheme. What does it look like? A scheme! So it isn't scary. We will recover formal neighbourhoods. Not of the coarse moduli space, but of the stack. For example, in the elliptic curve example, we'll get...

Philosophy: construct the neighbourhood by knowing what's going on in the central fiber.

Plan: Schlessinger.

First question: how to see the space is smooth? We will map local Artin schemes in. First-order neighborhood: Zariski tangent space. Intuitively, the space is smooth if we can extend freely.

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Let's play with this. Let's work over a fixed algebraically closed field $\overline{k}$. We can assume the target is an affine scheme $\operatorname{Spec} A$. Let's map some scheme into it $\operatorname{Spec} B$, which we think of as a fat point. We want to extend it. A "bigger scheme" is $\operatorname{Spec} B'$, where $B = B'/I$, i.e.

$$0 \to I \to B' \to B \to 0$$

as B-modules. Let's take $I^2 = 0$.

The Infinitesimal Lifting Property. Suppose $\operatorname{Spec} A$ is a nonsingular variety over $\overline{k}$. Given a ($\overline{k}$-algebra) homomorphism $f : A \to B$, then there is a morphism $g : A \to B'$ lifting it (draw diagram). *Keep on board.*

In fact, in general, this is essentially a good definition of smoothness. More precisely:

Definition. Suppose $I \subset A$ is an ideal such that $I^2 = 0$, and $\psi : \operatorname{Spec} A/I \to X$, $\operatorname{Spec} A \to Y$.

(i) If some $\tilde{\psi} : \operatorname{Spec} A \to X$ exists (for all such I, A) we say $X \to Y$ is *formally smooth.*

(ii) If at most one exists, we say it is *formally unramified.* (EGA: this is $\Omega^1_{X/Y} = 0$.)

(iii) If exactly one exists, then *formally etale.*

unr = formally unr + lfpr; etale = formally etale + lfpr = (nontrivially) flat + unr; smooth = formally sm + lfpr = (nontrivially) lfpr + flat + smooth fibers.

In case of ft over $\operatorname{Spec} \overline{k}$, unramified: pull back an ideal, get maximal ideal.

Example of something formally unramified but not unramified: $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$. Example of something formally etale but not etale: Spec of a local ring, mapping to a variety.

I'll begin the proof of the Infinitesimal Lifting Property in a moment. But first, a fun digression that I show my honors calculus students.

The dual numbers are $\mathbb{C}[\epsilon]/\epsilon^2$. Consider the following ring morphism from the ring of rational functions on $\mathbb{C}$ to functions of the form $f(x)+\epsilon g(x)$: $f(x) \mapsto f(x+\epsilon)$. Thus to every function $g(x)$, we attach another function $g(x)$. In fact it is $f'(x)$. The Leibnitz rule comes out for free (do it).

More generally, suppose we have a ring R (replacing $\mathbb{C}(x)$, the set of functions $f(x)$), and an R-module M (replacing the set of functions $g(x)$, which is also $\mathbb{C}(x)$). We can endow the R-module $R \oplus M$ with a ring structure, such that $M^2 = 0$. In other words, $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. What are the ring homomorphisms $R \to R \oplus M$, that are the identity on the first element? They are given by

$r \mapsto (r, g(r))$, where g satisfies:

$$(r_1, g(r_1))(r_2, g(r_2)) = (r_1 r_2, g(r_1 r_2))$$

This is by definition the derivations of R into M, i.e. $\mathrm{Der}(R, M) = \mathrm{Hom}(\Omega_R, M)$. We are a couple of short verifications of showing that such homomorphisms are parametrized by $\mathrm{Der}(R, M)$. That wasn't so bad, and that's the level at which derivations come up in these sorts of arguments.

Proof of the Infinitesimal Lifting Property. We'll see how nonsingularity comes into it. For the first bit, nonsingularity doesn't come into it.

First, suppose you have two liftings g, g'.

Proposition. Their difference is an element of $\mathrm{Hom}_A(\Omega_{A/k}, I)$, i.e. a $\overline{k}$-derivation of A into I. If you've not used to this notation, this will give you a chance to figure it out!

Interpreting the statement. First, note that I is an A-module, as it is a B-module. A derivation means an A-module map $A \to I$ that satisfies the Leibnitz rule: $\theta(ab) = a\theta(b) + b\theta(a)$.

Check Leibnitz: $\theta(ab) = g(a)g(b) - g'(a)g'(b)$, so

$$a\theta(b) + b\theta(a) = g(a)(g(b) - g(b')) + g'(b)(g(a) - g'(a)).$$

(Careful in explaining this!) □

Conversely, suppose we have one lifting g, and a derivation θ of A into I. Then $g + \theta$ is another extension. Proof: $g' : A \to B'$. It is additive clearly. It is multiplicative:

$$\begin{aligned} g'(ab) - g'(a)g'(b) &= (g(ab) + \theta(ab)) - (g(a) + \theta(a))(g(b) + \theta(b)) \\ &= \theta(ab) - g(b)\theta(a) - g(a)\theta(b) = 0. \end{aligned}$$

□

Hence the choices of extension is either empty, or an "affine $\mathrm{Hom}_A(\Omega_{A/k}, I)$-space".

Illustrative example (which people didn't find that helpful). Suppose A is $\overline{k}[x_1, \dots, x_a]$, and suppose $B = \overline{k}$, $B' = \overline{k}[y_1, \dots, y_b]/(y_1, \dots, y_n)^2$. Take the map $A \to B$ by $x_1, \dots, x_a$ (draw picture). Then there's no problem extending this to $A \to B'$, and $x_i \mapsto \sum c_{ij} y_j$. We see that the choices are precisely in correspondence with $\mathrm{Hom}(\Omega_A, I)$.

Suppose next that A is as before, and now our new B is our old B': $\overline{k}[y_1, \dots, y_b]/(y_1, \dots, y_n)^2$, and take $B' = \overline{k}[y_1, \dots, y_b]/(y_1, \dots, y_n)^3$. Suppose we have a map $A \to B$ given by $x_i \mapsto \sum c_{ij} y_j$. Then what are the extensions to $A \to B'$? Precisely $x_i \mapsto$

$\sum c_{ij} y_j + \sum_{j \leq k} d_{ijk} y_j y_k$. Again, the choices are precisely in correspondence with $\mathrm{Hom}(\Omega_A, I)$.

Cleary we're not ever going to get any obstructions, and we can extend these as far as we wish to go, to a map $\overline{k}[x_1, \dots, x_a] \to \overline{k}[[y_1, \dots, y_b]]$.

The proof of the infinitesimal lifting property will come next lecture.

Here's an application, that will come up in deforming nonsingular varieties

Theorem. Let X be a nonsingular variety over $\overline{k}$, and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a bijeciton between the set of infinitesimal extensions of X by $\mathcal{F}$ up to isomorphism, and the group $H^1(X, \mathcal{F} \otimes \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf of X.

(Describe this algebraically, and geometrically. Explain what this means.)

In order to do this, we do the affine case, and then patch. In the affine case, there is no higher cohomology. The full proof will come next day; for now, we'll content ourselves with the affine case.

Lemma. Suppose in addition that X is affine, $X = \operatorname{Spec} A$, $\mathcal{F} = \tilde{M}$. Then any extension is isomorphic to the trivial one.

Precisely, the trivial one is the morphism $A \oplus M$ (recall the ring structure). Suppose you have some other $0 \to M \to \tilde{M} \to A \to 0$; we want to show that $\tilde{M} \cong M \oplus A$, such that the projections to A agree. So this is just an algebra question. I think it's hard as an algebra question! But we use the previous question, and it becomes easy.

Consider $0 \to M \to \tilde{M} \to A \to 0$, and map A isomorphically to A; then there is a lifting to $\tilde{M}$. (Draw it in, and note that it is a morphism of rings.) Then it is quick to check that $\tilde{M} = A \oplus M$ (as $\tilde{M}$-modules), and that the ring structures agree. $\square$

Remark. For future reference, note that we have some choices of the lifting, i.e. choice of expression of $\tilde{M}$ as $A \oplus M$. How many? Answer: $\mathrm{Hom}_A(\Omega_A/\overline{k}, M) = H^0(\operatorname{Spec} A, \mathcal{F} \otimes \mathcal{T}_A)$. Intuition: H^0 of this sheaf parametrizes "automorphisms", and H^1 will paremetrize deformations.

3. Coming next

I'll start by giving some (better) illustrative examples of the infinitesimal lifting property. Then I'll finish the proof of theorem about infinitesimal extensions of X. And then I'll prove the infinitesimal lifting property. Then we'll start deforming...