SCHLESSINGER'S CRITERIA

RAVI VAKIL

I'm tired of writing this on the board repeatedly, so here it is in the form of a handout.

Fix our functor $F : \mathcal{C} \to \text{Sets}$.

Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{1}$$

(1) F has a hull iff F has properties H1–H3:

H1. (You can glue.) (1) is a surjection whenever $A'' \to A$ is a small extension. Equivalently whenever $A'' \to A$ is *any* surjection.
H2. (Uniqueness of gluing on $k[\epsilon]/\epsilon^2$.) (1) is a bijection when $A = k$, $A'' = k[\epsilon]/\epsilon^2$. Equivalently, $A'' = k[V]$. Then by previous lemma, t_F is a k-vector space.
H3. (finite-dimensional tangent space) $\dim_k(t_F) < \infty$.

(2) F is pro-representable if and only if F has the additional property

H4. (bijection for gluing a small extension to itself)

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A'). \tag{2}$$

is a *bijection* for any small extension $A' \to A$.

Important comment. Assume F satisfies H1–H3. Now given a fairly small extension $p : A' \to A$. Given any $a \in F(A)$, i.e. family over A, the set of lifts to $F(A')$ has a transitive action by the group $t_F \otimes I$. H4 is precisely the condition that this set is a principal homogeneous space under $t_F \otimes I$.

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