ARIZONA WINTER SCHOOL NOTES

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The interplay among "arithmetic", "topology", and "geometry" has been a central theme in algebraic geometry since long before the Weil conjectures. This course is intended to give a taste of a number of related questions where ideas in one area directly imply ideas in another, or (more subtly) suggest through metaphor statements one should hope/believe/expect/prove to be true.

1. THE GROTHENDIECK RING OF VARIETIES

A central player in this series of lectures will be the Grothendieck ring of varieties, over a given field k. To set terms, let $\boxed{V_k}$ be the set of finite type schemes over a field k, or (it will quickly not matter) varieties over k. You are welcome to imagine k as your favorite field, which depending on the person may be $\mathbb{Q}$ or $\mathbb{C}$ or $\mathbb{F}_p$, but we will want to be flexible about what k is.

Define the **Grothendieck ring of varieties** $\boxed{R = R_k}$ (nonstandard notation!) as follows.

- As an additive group, R is generated by symbols of the form $[X]$, where X is a variety (up to isomorphism).
- The additive structure of R is generated by the following. If $Z \subset X$ is a closed embedding/subset, with complement U, then we say $[X] = [U] + [Z]$. (Exercise: show that $[X] = [X^{\text{red}}]$.)
- Multiplication in R is defined by the relation if W and X are varieties, then $[W][X] = [W \times X]$.

This turns R into a commutative ring.

1.1. *Examples.* $0 = [\varnothing]$, $1 = [\text{pt}]$. For convenience, define $\mathbb{L} = [\mathbb{A}^1]$ (an important definition for us!). The $[\mathbb{P}^2] = \mathbb{L}^2 + \mathbb{L} + 1$.

1.A. EXERCISE. if $W \to X$ is a $\mathbb{P}^n$-bundle (in the Zariski topology), show that $[W] = [X][\mathbb{P}^n]$.

For motivation: one of the central ideas of mathematics is turning general phenonena into linear phenomena, which we can calculate with. A first example is calculus, which turns differentiable things into linear things. Another example is "cohomology" which

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turns complicated things into groups or rings, from which much information can be extracted. This is some very blunt way of turning the category of varieties into a ring.

1.2. Example: "classical" geometry/topology.

Let's get a feel for this by applying this in an analogous case you might have some intuition for, that of "nice topological spaces" (say, finite CW-complexes).

Then $[\mathbb{R}] = [R^{<0}] + [0] + [\mathbb{R}^{>0}]$, so $[\mathbb{R}] = 2[\mathbb{R}] + 1$, from which $[\mathbb{R}] = -1$ and $[\mathbb{R}^k] = (-1)^k$. Then by decomposing our spaces into such cells, we get that $[X]$ is some integer — and it turns out that $[X] = \chi_c(X)$, the Euler characteristic of cohomology with compact supports. (Important note: if X is smooth and compact, then this is just cohomology with compact supports agrees with usual cohomology: $h^i(X) = h^i_c(X)$, so $\chi(X) = \chi_c(X)$: $\sum(-1)^i h^i_c(X) = \sum(-1)^i h^i(X)$.)

It seems that there is not much here. The Grothendieck ring is only $\mathbb{Z}$. Lots of information is lost. For example,

$$[S^1] = [S^1 \times S^1] = [S^1 \coprod S^1] = [\varnothing] = 0.$$

But in fact, even here there is something going on. How do we know that $1 \neq 0$?! How do we know that 1 is torsion-free? (Maybe $R = \mathbb{Z}/17$...) Making this work requires having some understanding of cohomology with compact supports.

And furthermore, once you have done this, you have a number of nontrivial results (such as, for example, that if you "cut up a genus g compact oriented surface into contractible pieces, with the "map" having v vertices, e edges, and f faces, then $v - e + f = 2 - 2g$ — Euler's formula.

This example drives home the fact that you should think of the map $V \to R$ from varieties to the Grothendieck ring. The Grothendieck ring should be seen as some sort of "universal Euler characteristic", at least for anything that you should think of as "cohomology with finite supports". For this reason, this should be seen as a baby version of a motive (encompassing only certain kinds of cohomology-like things), the ring R is sometimes called the ring of baby motives. But unlike the theory of motives, the definition is short. Such a map $V \to R$ is sometimes called a "motivic measure".

So let's see what we get when we move into objects that are algebraic as well as geometric. It is hard to figure out what the ring is, but we can try to understand it through various quotients — in other words, maps from V to some ring A that descends to a map from R to that ring A.

1.3. Example: $k = \mathbb{F}_q$.

First, suppose k is a finite field $\mathbb{F}_q$. Then each variety has a finite number of rational points — finitely many $\mathbb{F}_q$-valued points — so we have a "point-counting" map $\# : R_{\mathbb{F}_q} \to$

$\mathbb{Z}$. Clearly this respects the addition and multiplication relations. So we have already shown that R is not the 0-ring.

So at this point we have a clue that counting points should be thought of as some sort of Euler characteristic. I will come back to this in the next lecture.

1.4. Example: $k = \mathbb{C}$.

Next suppore k is $\mathbb{C}$, a seemingly quite different situation. Certainly every complex variety is a finite type CW-complex, so we have a map $\chi_c(X, \mathbb{Z}) : R \to \mathbb{Z}$. This is already useful. For example, as $\mathbb{CP}^2$ is smooth and complete (complete = proper over k; over $\mathbb{C}$, complete = compact), we can immediately see that $\chi(\mathbb{CP}^2) = 3$, as $\chi_c(\mathbb{L}) = 1$.

But there is more — the fact that a geometric object is *algebraic* imposes more structure than you might hope. If X is smooth and complete (a compact complex algebraic manifold), then *we recover each* $h^i_c(X, \mathbb{Q})$*, not just their alternating sum*. For example, for $\mathbb{CP}^2$, we can recover from $[\mathbb{CP}^2] = \mathbb{L}^2 + \mathbb{L} + 1$, we can recover that

$$h^0(\mathbb{CP}^2) = 1, \quad h^1(\mathbb{CP}^2) = 0, \quad h^2(\mathbb{CP}^2) = 1, \quad h^3(\mathbb{CP}^2) = 0, \quad h^4(\mathbb{CP}^2) = 1.$$

This is an incredibly deep, important, and amazing fact, and comes from Deligne's theory of weights. (We will see how to prove this shortly.)

If you haven't worked extensively with cohomology, I want to point out how striking this is. By considering h^0, we see that the number of components comes from this. In other words, if you take a bunch of smooth compact varieties, cut them into pieces, and rearrange them in a different way into a different bunch of smooth compact varieties, the number appearing will be the same. The dimension of the biggest component is also fixed. (The dimension of the smaller components are not determined, though — can you think of an example? One will come up soon — see (1).)

1.B. EXERCISE. find the cohomology groups of products of projective spaces, for example $\mathbb{P}^2 \times \mathbb{P}^1$. Find the cohomology groups of a Grassmannian, say the Grassmannian $G(2, 4)$ of two-dimensional subspaces of $\mathbb{C}^4$.

Even without smoothness or compactness, you can still make predictions. Rather than making this precise, I will just give an examples.

From $[\mathbb{C}^*] = \mathbb{L} - 1$, we can interpret the $\mathbb{L}$ as $h^0(\mathbb{C}^*) = 1$ and $h^1(\mathbb{C}^*) = -1$.

You can get even more information on the cohomology groups. In particular, if X is smooth and complete, then you can recover the *Hodge structures* on the cohomology groups. If you don't know what that is, there is no need to worry; just be aware that it is an addition structure on the cohomology groups with $\mathbb{C}$-coefficients, taking into account the $\mathbb{Q}$-coefficients.

And even if X is not smooth and complete, you still get strong Hodge-theoretic information, by way of the notion of "mixed Hodge structures". (As with the case of $\mathbb{C}^*$, this allows you to make predicitons. This idea has been very fruitful — see for example [HN] and [HVR].)

1.5. How to get at this ring in general: Bittner's presentation in characteristic 0.

The fact that we can such different bits of information out of the ring, and in particular consequences of Deligne's theory of weights, forces us to ask: what other information is hiding in this ring? What other Euler characteristics are out there? What *is* this ring? We really have very little idea about this question, but what ideas we have are very interesting.

Note that $R = R(V_k)$ is generated as an abelian group by classes of smooth affine varieties. This means that it is generated as an abelian group by classes of smooth projective varieties.

If $Z \hookrightarrow X$ is a closed embedding of a smooth projective variety in another smooth projective variety, then if $\mathrm{Bl}_Z X$ is the blow-up of X along Z with exceptional divisor $E_Z X$, then clearly

$$[Bl_Z X] - [E_Y X] = [X] - [Z] \tag{1}$$

1.6. *Theorem (Bittner,* [Bi]*).* — R is isomorphic to the free abelian group on clsases of smooth irreducible projective varieties, modulo such relations.

The proof is very short, and uses the (very important) weak factorization theorem of Abramovich, Karu, Matsuki, and Wlodarcyzk [AKMW], which states that any birational map between smooth varieties can be factored into a sequences of blow-ups and blow-downs along smooth center.

1.7. Applications of Bittner's presentation.

Because Bittner's presentation involves only smooth projective varieties, it allows us to determine new motivic measures, by defining them on smooth projective varieties, and showing that they satisfy this one single relation (1).

For example, there is a motivic measure $a_0 : V \to Z$, which is the "virtual number of components": write any variety X as a combination of smooth varieties. Write each of these smooth varieties in terms of smooth *projective* varieties. Then the number of such varieties (counted appropriately) depends only on X!

1.C. EXERCISE. Work out this invariant for a nodal cubic curve in $\mathbb{A}^2$.

1.D. EXERCISE. Show that there is motivic measure $a_i : V \to Z$ defined by $a_i(X) = h^i(X, \mathcal{O}_X)$ for $i > 0$ for smooth projective X.

1.E. EXERCISE. If you know a little about the properties of Hodge structures, and how they change by blowing up, show that the Hodge structure on h^i is also a motivic measure.

1.8. *Application: Bittner duality on* $R_{\mathbb{L}}$.

Here is a weird application.

1.9. *Theorem (Bittner duality). — There is a duality on* $R_{\mathbb{L}}$ *sending* X *to* $L^{-\dim X}X$ *for smooth irreducible projective varieties.*

In particular, $\mathbb{L} \mapsto \mathbb{L}^{-1}$, $[\mathbb{P}^1] = \mathbb{L} + 1 \mapsto 1 + 1/\mathbb{L}$.

This is really a duality, and somehow has to do with Poincare duality! Notice what it does to $\mathbb{P}^1$.

I see this as one of the many ways in which nature is telling us to invert $\mathbb{L}$. But if you invert $\mathbb{L}$, you might lose information: Remember that $R_{\mathbb{L}} = R[1/\mathbb{L}] = R[x]/(x\mathbb{L} - 1)$. Localizations are not always injective; $R \to R_{\mathbb{L}}$ will kill any class α such that $\alpha\mathbb{L} = 0$.

For this reason there has long been a question/speculation/conjecture (Denef-Loeser [DL] and many others):

1.10. *Conjecture. —* $\mathbb{L}$ *is not a* 0*-divisor. In other words,* $R \to R_{\mathbb{L}}$ *is an injection.*

This has recently been shown to be false in a dramatic work of Lev Borisov [Bo]; more on this later.

1.11. *Proof of Bittner duality.* Suppose Z is codimension c in X.

$$\begin{aligned} [E_Z X] &= [\mathbb{P}^{c-1}][Z] \\ (\mathbb{L}-1)[E_Z X] &= (\mathbb{L}^c - 1)[Z] \\ \text{(subtracting (1))} \quad [Bl_Z X] - \mathbb{L}[E_Z X] &= [X] - \mathbb{L}^c[Z] \end{aligned}$$

Dividing by $\mathbb{L}^{\dim X}$, we get

$$[Bl_Z X]/\mathbb{L}^{\dim Bl_Z X} - [E_Z X]/\mathbb{L}^{\dim E_Z X} = [X]/\mathbb{L}^{\dim X} - [Z]/\mathbb{L}^{\dim Z}$$

as desired. □

1.12. *Application: connection to stable birational equivalence classes.*

Recall that two irreducible varieties X and Y are birational if they have "isomorphic open subsets". They are said to be *stably birational* if $X \times \mathbb{A}^m$ is birational to $Y \times \mathbb{A}^n$ for some m and n. Even taking into account the difference in dimension, stable birationality is a strictly weaker equivalence relation than birationality. (It is an amazing fact due to Beauville, Colliot-Thelene, Sansuc, and Swinnerton-Dyer [BCTSSD] that there are nonrational varieties that are nonethless stably rational.)

Stable birational equivalence classes of irreducible varieties form a commutative semigroup $\mathbb{Z}[SB]$.

1.13. *Theorem (Larsen-Lunts,* [LL]*). — There is a unique ring homomorphism* $R \to \mathbb{Z}[SB]$ *sending* X *to* X *for smooth projective varieties.* (But not for other varieties!)

Proof. Clearly there is at most one such map. But this map respects Bittner's relation (1) □

So we have a new motivic measure! What information have we lost? From the fact that $\mathbb{P}^1$ and a point map to the same thing (they are both rational), $\mathbb{L}$ maps to 0, so any multiple of $\mathbb{L}$ maps to 0. In fact we lose "nothing else".

1.14. *Theorem (Larsen-Lunts,* [LL]*). — The kernel is* $(\mathbb{L})$.

This is less easy.

1.F. EXERCISE (REALITY CHECK). Show that $(\mathbb{L})$ is contained in the kernel.

1.G. EXERCISE. If you have some topological experience, try out some examples that are smooth but not projective, or projective but not smooth, or not smooth or projective.

2. SYMMETRIC POWERS, AND THE MOTIVIC ZETA FUNCTION

We now come to a central player in our discussions: the *motivic zeta function*. This is a notion which was formally defined remarkly late — in an unpublished manuscript of Kapranov in 2000 — and should have been defined far earlier. (I think Grothendieck had discussed this idea in a letter to Serre, but I can't remember the reference.)

2.1. Definition (Kapranov, [K]**).** The motivic zeta function of X is defined as

$$Z_X(t) = \sum [\mathrm{Sym}^n X] t^n \in R[[t]].$$

Let me try to motivate this definition — or more precisely, try to get across how incredibly well-motivated this is.

But first, note that if $X = U \coprod Z$ where Z is closed and U is open, then $Z_X(t) = Z_U(t) Z_Y(t)$.

2.A. EXERCISE. Prove this.

Thus we can really define $Z_X(t)$ for any X in the Grothendieck ring R. We have a map of *groups*

$$Z : R \to (1 + tR[[t]]) \subset R[[t]]^\times,$$

turning $+$ into $\times$.

2.2. Example: "classical" geometry/topology.

To get a feel for this, let's start topologically, thinking about "usual geometric spaces" (say finite CW-complexes). For example, Sym^n of a point is just a point, so $Z_{pt}(t) = 1 + t + t^2 + \cdots = 1/(1-t)$.

We already know that the (analogue of the) Grothendieck ring contains no more information than the Euler characteristic with compact supports.

2.3. *Theorem (Macdonald, 1962,* [M]*).* — $Z_X(t) = \sum \chi_c(\mathrm{Sym}^n X)t^n = \frac{1}{(1-t)^{\chi_c(X)}}$

So the Euler characteristic of $\mathrm{Sym}^n X$ is

$$(-1)^n \binom{-\chi_c(X)}{n} = (-1)^n \frac{(-\chi_c(X))(-\chi_c(X)-1)\cdots(-\chi_c(X)-n+1)}{n!}$$

— it depends only on $\chi_c(X)$! If X is a compact manifold, then you can erase the c's — I won't say why.

This is a great theorem! Better yet, it has a one-line proof.

Proof. As $Z_X(pt) = 1/(1-t)$, and $[X] = \chi_c(X)[pt]$, and $R \to R[[t]]$ sends addition to multiplication, the result follows. □

2.4. Example: $k = \mathbb{F}_q$**.**

As before, we now try this out in the case of a finite field $\mathbb{F}_q$. Here we have the point-counting function $\# : R \to \mathbb{Z}$. Applying this to the motivic zeta function, we get the Weil zeta function:

2.B. Exercise. Show that $\#(Z_X(t)) = \zeta_X(t)$.

Recall that the Weil zeta function is usually defined as follows. Take X, and count points over all extensinos of $\mathbb{F}_q$. Put these in a generating function in an appropriate way, and massage the function a little.

What this exercise is saying is that you could instead just count $\mathbb{F}_q$-points of the symmetric powers — in a geometer's mind, a much more direct and short definition. I like this definition better.

Weil's reason for introducing his zeta function was because of his thoughts around the Weil conjectures connecting number theory and topology and algebraic geometry, the famous "Rosetta stone" he described in a letter to his sister Simone Weil. In particular, the first part of the Weil conjectures is that the Weil zeta function is always *rational*. (This was first proved by Dwork in 1960, well before the rest.)

So notice now that our two examples of motivic zeta functions (or at least zeta functions coming from motivic measures) are rational, for very different reasons (Macdonald, and Weil/Dwork). This leads to a natural question, that should have been asked in the 1960's, but was first asked (to my knowledge) in 2000.

2.5. *Question (Kapranov, unpublished* [K]*).* Is $Z_X(t)$ rational?

Here k can be any field. I want to try to convince you that this is an important question, and that you should really believe that the answer should be yes.

Let me first start with the evidence.

If X is a curve the answer is yes. (With a rational point: use Weil's proof of rationality of his zeta function. Without a rational point: needs more care; shown by Daniel Litt.)

If $k = \mathbb{C}$, then it is true for "Hodge structures" (Cheah's thesis 1994 — more soon).

If $k = \mathbb{F}_q$, then it is true for point-counting by the rationality of the Weil zeta function $\zeta_X(t)$ (Dwork, 1960).

So that is a lot of evidence.

But even better: if the conjecture is true, it would *imply* the rationality of the Weil zeta function, thereby by giving a direct and *very geometric* proof of that part of the Weil conjectures, and would give a serious attack on most of the rest of them! (Dwork's proof was very different, and was definitely a "number theorist's proof" rather than a "geometer's proof" — more on this later.)

And then this would even have strong consequences in characteristic 0. If I tell you that a power series is actual rational, say $f(t) = g(t)/h(t)$ (f a power series, g and t polynomials), then from $f(t)h(t) = g(t)$ will give you a recursive way to compute the coefficients of f.

So if we knew the first few $\mathrm{Sym}^n X$, we would know them all, by "cutting-and-pasting". For each X, there would be an explicit recipe!

Thus we have a statement which is short, beautiful, with lots of evidence over lots of fields, and which would have dramatic consequences. So you have to believe that this result is true. But:

2.6. *Theorem (Larsen-Lunts, 2003-4,* [LL]*).* — The answer is "no"!!

This is a really amazing result. We do not know what the Grothendieck ring R is, and we understand it only through "motivic measures", which are sorts of Euler characteristics.

In every case we knew, it looked polynomial. So they had to come up with a dramatically different kind of Euler characteristic. (More on this soon.)

2.7. *Big philosophical question.* Where in between R and $\mathbb{Z}$ does the zeta function start being rational? What makes the zeta function rational?

Larsen and Lunts' new Euler characteristic takes the value 0 on $\mathbb{L}$. Thus: we do not know the answer if we work in the localization $R_{\mathbb{L}}$. And if the answer was yes here, it would explain all the consequences we know.

2.8. *Question.* Is the zeta function rational over $R_{\mathbb{L}}$? (Note: this is not ended to be an exercise for the evening sessions...)

So at this point we are disappointed and frustrated; we wanted to get at the Weil conjectures, and we did not succeed. But for now, we look for inspiration back over $\mathbb{C}$.

2.9. Example: $k = \mathbb{C}$**.**

Recall Macdonald's Theorem, which was the topological analogue of rationality:

2.10. *Theorem (Macdonald, 1962,* [M]*).* —

$$Z_X(t) = \sum \chi_c(\mathrm{Sym}^n X)t^n = \frac{(1-t)^{h^1}(1-t)^{h^3}\cdots}{(1-t)^{h^0}(1-t)^{h^2}\cdots}$$

When cohomology has additional structure, then we could hope to extend the theorem to this case. For example, Hodge structure is this kind of additional structure.

2.11. *Theorem (Cheah* [C]*).* — *If* X *is a complex variety, then*

$$\sum_{n\geq 0} \left(h^{p,q}(H^r_c(\mathrm{Sym}^n X))x^p y^q (-z)^r\right) t^n = \prod_{p,q,r\geq 0} \left(\frac{1}{1-x^p y^q z^r t}\right)^{(-1)^r h^{p,q}(H^r_c(X))}$$

Here, $H^r_c(\mathrm{Sym}^n X) = H^r_c(\mathrm{Sym}^n X, \mathbb{C})$ is just rth cohomology with compact support, with coefficients in $\mathbb{C}$. This comes with a Hodge structure (technically, a "mixed Hodge structure", defined by Deligne). Although I haven't defined it, it comes with pieces of dmiension $h^{p,q}$ for various integers p and q. More preicsely, the vector space H^r is filtered by vector spaces of these dimensions. All you need to know is that this is some number, and $\sum h^{p,q}(H^r) = h^r$. So this is an enriched version of Macdonald, which now takes into account the extra we have on the cohomology, thanks to the lucky fact that we are working in algebraic geometry, and not plain old topology.

We try to replace numbers with vector spaces. I don't just want the size of the cohomology groups — I want the cohomology groups, along with whatever information they may have!

Let's try to make this work. I want to state a theorem, and then figure out how to define things.

"Hodge structures" are cohomology groups with some more information, so we want to remember more than vector spaces. So "cohomology" H^r takes as input from Varieties over $k = \mathbb{C}$ — let me call this category $\boxed{V_k}$. It maps to mapping to $\boxed{VS^+}$, finite-dimensional vector spaces over some field, with some "additional structure", which respects the vector space structure (so we can have kernels, cokernels, etc. — it is an abelian category). There is no reason to have the vector space be over k, and we will see soon that we will want it to be characteristic 0, even if k is not. We want the additional structure to respect things like $\oplus$, $\otimes$, $\bigwedge$, etc.

Cohomology is functorial, so we want contravariant functors $H^i : V_k \to VS^+$.

We want to talk about Euler characteristics $\sum(-1)^i H^i(X) \in K(VS^+)$, so to have this sum make sense, we require *Axiom 1 of 3:* For each X, $H^i(X) = 0$ for $i \gg 0$. . Then we have an Euler characteristic

$$\boxed{\chi(X) := \sum(-1)^i H^i(X) \in K(VS^+)}$$

("virtual vector spaces with additional structure" — the Grothendieck group of vector spaces with additional structure).

2.12. *Theorem (??!!).* —

$$\sum_{i=0}^{\infty} \chi(\operatorname{Sym}^i X) t^i = \frac{1}{(1-t)^{\chi(X)}}$$

What does this mean?! The left side lies in $K(VS^+)[[t]]$, so the right side should too.

And when we apply the "dimension" function $K(VS^+) \to \mathbb{Z}$, we should recover Macdonald's Theorem.

For $V \in VS^+$, define

$$\boxed{(1-t)^V := \sum_{i=1}^{\infty} (-1)^i (\wedge^i V) t^i.}$$

Note that $(1-t)^V (1-t)^W = (1-t)^{V \oplus W}$ by properties of $\bigwedge$ and $\oplus$.

2.C. EXERCISE (IMPORTANT FUN FACT). Show that $1/(1-t)^V$ is the generating function for symmetric powers $\operatorname{Sym}^n V$.

So now we have a theorem which makes sense. Next, we prove the theorem, which will force us to realize that additional axioms are necessary.

I will just tell you what they are.

Axiom 2 (the Kunneth formula). $H^n(X \times Y) = \oplus_{i=0}^n H^i(X) \otimes H^{n-i}(Y)$.

If T is a variety and G is a finite group of automorphisms of T, then define $\phi : T \to T/G$. Then (as cohomology is a functor) we have $\phi^* : H^i(T/G) \to H^i(T)$. *Axiom 3* ϕ^* identifies $H^i(T/G)$ with $H^i(T)^G$. This is where you could imagnine that we want characteristic 0. This fact was proved in cohomology in Grothendieck's Tohoku paper [Gr]. So it is very reasonable, and reasonable to ask of any "cohomology-like" functor.

Given these axioms, I will now prove the theorem!

2.13. *Proof of Theorem 2.12.*

We have $H^*(\mathrm{Sym}^n X) \cong H^*(X^n)^{S_n}$ by the second axiom. But all of the cohomology groups of X^n can be formally written in terms of the cohomology groups of X, in some universal way, by the third axiom. Then by the second axiom, (with the action of the symmetric group) is given to us (again, by functoriality of cohomology.

So now to check

$$\sum_{i=0}^{\infty} \chi(\mathrm{Sym}^i X)t^i = \frac{1}{(1-t)^{\chi(X)}}$$

we just have to check something completely formal, where we just expand both sides in terms of $H^i(X)$ for $i = 1$ through ∞. You should believe that this "must" be true, and can then go ahead and check this algebra fact.

2.D. EXERCISE (NOT SO EASY, BUT CAN BE DONE WITH FINESSE). Prove this fact.

$\square$

Applying this to Hodge structures, we find:

2.14. *Corollary.* — Cheah's Theorem 2.11 is true.

(Why did it take until 1994 to know this?)

As another application: counting rational points on varieties over $\mathbb{F}_q$ can be interpreted in this way. We have étale cohomology $H^r_{et}(X, \mathbb{Q}_\ell)$. It is not hard to prove that the number of $\mathbb{F}_q$ points is $\sum_i (-1)^i \mathrm{tr}(F|_{H^r(X,\mathbb{Q}_\ell)})$ where F is the action of Frobenius. So we take VS^+ to be the category of $\mathbb{Q}_\ell$ vector spaces, along with a representation of Frobenius. We also have to check three axioms. Once we have done that, we have:

2.15. *Corollary.* — The Weil zeta function is rational!

So now we see good reasons for zeta functions to be rational, if they come for varieties. But how would we show nonrationality of the motivic zeta function? We need some motivic measure. Here is Larsen and Lunts' answer. As usual, we need to describe it for smooth projective varieties, and then show that it satisfies Bitter's relation (1). For X

smooth projective they define

$$\mu(X) = 1 + h^0(\Omega^1)t + h^0(\Omega^2)t^2 + \cdots + h^0(\Omega^{\dim X})t^{\dim X} \in 1 + t\mathbb{Z}[[t]].$$

Notice: $\mu(\mathbb{L}) = 0$. Reason: $\mu(\mathbb{P}^1) = \mu(1) = 1$.

2.E. EXERCISE. Prove that μ satisfies Bittner's relation (1), and thus show that it is a motivic measure.

3. MORE CONJECTURES/QUESTIONS/SPECULATIONS ABOUT THE GROTHENDIECK RING OF VARIETIES $R(V_k)$

The conjecture that the motivic zeta function was rational was false, but it took us to some very interesting places. This is why conjectures/questions/speculations about the Grothendieck ring of varieties R are so interesting — they can only be proved *or* disproved in an interesting way. Different people have different opinions about these conjectures, but my general opinion is that every such conjecture/question/speculation is wrong, but wrong for interesting reasons. I will now describe other examples.

3.1. The motivic stabilization of symmetric powers conjecture.

Given the Weil conjectures (or even our proof of rationality, modulo those axioms), the Weil zeta function of X (proper geometrically irreducible of dimension d) looks like this:

$$\zeta_X = \frac{p_1(t)p_3(t)\cdots p_{2d-1}(t)}{p_0(t)p_2(t)\cdots p_{2d}(t)} = \sum a_i t^i$$

where $p_i(t)$ is a polynomial of degree $h^i_{et}(X)$, and $p_0(t) = 1 - t$ and $p_{2d} = (1 - q^d t^{2d})$.

How do these coefficients grow? As motivation, we consider a question from calculus class. Consider a rational function whose coefficients are in $\mathbb{C}$, expanded as a generating series (around 0). How do its coefficients grow? You take the root in the denominator of smallest size — say for convenience that there is only 1 — then the coefficients will grow like powers of the inverse of that root.

3.A. EXERCISE. Prove this, perhaps using partial fractions.

In our case, the Weil conjectures include implication of the sizes of the roots. The biggest inverse root is q^d, coming from the top cohomology class. This implies that the t^i coefficient of the Weil zeta function $\zeta_X(t)$ grows like $(q^d)^i$. This is precisely the Hasse-Weil estimates, which are much easier than the Weil conjectures. In other words, we (as a species) knew this rate of growth (and it is very very useful) without knowing rationality. In the case of the motivic zeta function, perhaps the analogue of this might be true, despite the lack of rationality.

3.2. *Motivic Stabilization Conjecture/Question/Speculation (V-Wood). — Suppose* X *is geometrically irreducible. Then* $\mathrm{Sym}^n X \mathbb{L}^{n \dim X} \in \hat{R}_{\mathbb{L}}$ *converges.*

We need to make sense of this completion on the right. Things on the left are in $R_{\mathbb{L}}$, but the denominators will get bigger and bigger. So we need to complete in some sense. We do this by filtering R (and hence $R_{\mathbb{L}}$ by dimension); I will skip the details.

So motivated by the Hasse-Weil estimates, we wonder if this is true.

Here are other reasons to hope that it is true.

We know it is true when you specialize to Hodge structures, because we have rationality of the zeta function with that "motivic measure"

We know the analogue is true when you specialize to point-counting, because we know the rationality of the Weil zeta function.

I had mentioned that the motivic zeta function is rational when X is a curve (and the case when X has a point you can do as an exercise).

It is true for stably rational varieties, even though these motivic zeta functions may not be rational (V-Wood). (But note: we have inverted $\mathbb{L}$, so Larsen and Lunts' proof doesn't work — maybe this motivic zeta function with $\mathbb{L}$ inverted *is* rational!)

All of these feel like they are number theoretic. But there are very topological reasons for this.

If X is a reasonable topological space, with a point p, then we have maps $\mathrm{Sym}^n X \to \mathrm{Sym}^{n+1} X$ (by just "adding a copy of p"). Then long before we arithmetic and algebraic geometers got into this, the topologists knew that as n gets big this "stabilizes". To first approximation, and most relevant for us: for any i, once you have a big enough n, then, this map becomes an isomorphism on H^i. But better yet: the homotopy stabilizes up to "codimension i". (Arnav Tripathy can explain exactly how high you have to go for each i. Ask him — he has a very pretty proof.) So we have some "limiting homotopy type" $\mathrm{Sym}^\infty X$. And we even know what it is — this is the Dold-Thom theorem, which gives the answer in terms of the cohomology groups of X — it doesn't depend on the homotopy type of X, just the cohomology groups.

This is an example of a stabilization result in topology, and such results long predate our analogous understanding in arithmetic and algebraic geometry. Harer stability is another example.

Another remark: we had shown, modulo an axiom or two, that the cohomology groups of Sym^n of a topological space depend only on the cohomology groups of the topological space, and the analogue was arithmetically powerful, taking us to the Weil conjectures. In topology, more is true: Sym^n works "up to homotopy" — Sym^n is a functor from homotopy types to homotopy types. This is actually a surprising statement, and leads to interesting facts on the arithmetic/algebraic side.

I want to mention *two results in this vein*.

Algebraically, we would want X and $X \times \mathbb{A}^1$ to be "homotopic" to each other. But $\mathrm{Sym}^n(X)$ and $\mathrm{Sym}^n(X \times \mathbb{A}^1)$ can be very different. For example, $\mathrm{Sym}^n(\mathbb{A}^m)$ is singular.

3.3. *Theorem (Totaro's observation, about 2001,* [Gö]*). — In the Grothendieck ring of varieties,* $[\mathcal{S}y\mathfrak{m}^n(X \times \mathbb{A}^1)] = [(\mathrm{Sym}^n X)] \times [\mathbb{A}^n]$.

(He did the case where X is a point, but his argument works in general.) He had a very clever one-line argument using Hilbert 90 — can you find it? To give you a more precise question:

3.B. EXERCISE. Show that you can cut-and-paste $\mathrm{Sym}^n \mathbb{A}^m$ into $\mathbb{A}^{nm}$. Example: $\mathrm{Sym}^n \mathbb{A}^1 = \mathbb{A}^n$, even without cut-and-pasting.

Before Totaro's idea, it had been a big deal to show that $\mathrm{Sym}^n \mathbb{A}^m$ was even rational! And his proof is nonconstructive (just as Hilbert 90 can be) — and after his argument, an actual cut-and-pasting was found. This should really have been known in the 19th century.

Much harder, and related to this winter school:

3.4. *Theorem (Tripathy, 2015* [T]*). —* "Sym" and "étale realization" commute.

(If you know something about étale homotopy theory, you will realize that this opens the door to a lot of great things. As just one example, it leads to an "étale Dold-Thom Theorem".)

3.5. *Back to the Motivic Stabilization of Symmetric Powers Conjecture 3.2.*

So this conjecture is simultaneously motivated by the Weil conjectures in arithmetic geometry, and the Dold-Thom theorem in topology. This combination would potentially lead to something very useful. Recall that when we proved the rationality of various zeta functions by a "universal method", we needed cohomology functors. It isn't clear that we have such things in the Grothendieck ring R. More generally, one of Grothendieck's standard conjectures (which is clearly motivated by this point of view) is that there exist such functors for motives (not baby motives — the real thing).

But in Dold-Thom, the functor $X \mapsto \mathrm{Sym}^n X$ takes something and breaks it up into something that depends only on its cohomology groups — so if the Motivic stabilization conjecture 3.2 is true, it might give us a place in which we could break up a baby motive into cohomology groups, and thus do many many things. (Again, I should repeat that I do not believe the conjecture is true... But at least it makes clear that the conjecture is interesting...)

3.6. The cut-and-paste conjecture.