FOUNDATIONS OF CATEGORY THEORY: WHAT REMAINS TO BE DONE

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ASL 2011 Annual Meet

UC Berkeley, March 27, 2011

Feferman(1977), "Categorical Foundations and Foundations of Category Theory" (CF&FCT)

- Session on CF&FCT proposed by E. Landry; participants: G. Hellman, E. Landry, J.-P. Marquis and C. McLarty
- Quick review of CF&FCT (two parts)
- Part I critiqued the CF view (Lawvere 1966, MacLane, Awodey, others) that CT provides an autonomous framework for the foundations of mathematics

But Mac Lane led the way in FCT

- Concerns with foundations of CT from the beginning of the subject (Eilenberg, Mac Lane 1945)
- Pursuit through various systems of set theory (1961, 1969, etc.) ending with "one-universe" theory. Taken as basis of Working text (1971)
- Systematic use of "small" and "locally small" categories
- Now universally accepted that such distinctions are essential for statements of many theorems
- Prime example: Freyd's AFT ("not baroque")

Why still belief in autonomous CF?

- Despite such detailed evidence "on the ground" to the contrary, case for autonomous CF still being made for philosophical and/or ideological reasons
- One argument: ZFC can be replaced by Lawvere's ETCS (Mac Lane 1986, 1998, McLarty 2004, etc.)
- My view: Deceptive ideological shell game
- No time today to continue my response to CF

Set-theoretical Foundations of CT

- Granted some such (NB!) foundations are necessary, began Part II of CF&FCT with review of Mac Lane, Grothendieck approaches, and then pointed out defects w.r.t. two requirements (R1) and (R2) below.
- Proposed another approach in Part II that turned out to have only limited advantages, and certain disadvantages; discontinued work on it.
- But had started to work on other approaches some years before. And will talk about new progress on one of those today.

The Requirements for FCT

- Within an axiomatic system S as basic framework:
- (R1) Should be able to form the category of all structures of a given kind, e.g. **Grp, Top, Cat**
- (R2) Should be able to form the category of all functors from A to B, for any categories A, B
- (R3) Should be able to prove existence of basic math structures and carry out usual set-theoretical operations
- (R4) Finally, consistency of S should be established relative to some currently accepted system of set theory.

How do standard approaches fare?

- 1. S = BGC (Eilenberg-Mac Lane '45, Mac Lane '61) small = set, large = proper class. (R1) and (R2) not met (as in the following standard and semi-standard approaches). (R3) in full. (R4) BGC is conservative over ZFC.
- 2. S = ZFC + "there exists a universe" (Mac Lane '69, '71). For a given universe U, small = set in U, large = proper subset of U, super-large = set in V. (R4) S equivalent in strength to ZFC + "there exists a strongly inaccessible cardinal"

Standard to semi-standard approaches

- 3. S = ZFC + "there exist arbitrarily large universes" (Grothendieck c. '69). Relativize 'small', 'large', 'super-large' to any universe U.
  (R4) strength: ZFC + "there exist arbitrarily large strong inaccessibles"
- 4. S = ZFC/s (Fef '69). s is an added symbol with axioms for $(s, \in) \preccurlyeq (V, \in)$. small = set in s, etc.
  (R4) Conservative over ZFC (Montague, Vaught).

Standard and semi-standard approaches (cont'd)

- 5. S = ZFC/s + "s is a universe" (Fef '69). (R4) Requires Mahlo hierarchy of strongly inaccessible cardinals for consistency proof
- M. A. Shulman, "Set theory for category theory" (arXiv 2008) gives a useful survey of 1-5 and advantages/disadvantages in applications

A Non-standard approach

- S = S* (Feferman 1974, 2006, here). S* is an extension of MKC theory of sets and classes by NFU+P , an enriched *stratified* theory of classes.
- (R4) S* is consistent relative to ZFC + "there exist two strongly inaccessible cardinals".

Stratified systems
Background: NF and NFU

- NF (Quine 1937) -- Variables A, B, C,...,X, Y, Z, basic relations =, $\in$. A formula is stratified if it comes from a formula of Simple Type Theory by suppressing types.
- NF axioms: Extensionality (Ext) and Stratified Comprehension Axiom scheme (SCA), i.e. $(\exists A)(\forall X)[X \in A \leftrightarrow \varphi]$ for all stratified φ (no 'A').
- Consistency of NF is a long open problem.

NF and NFU (Cont'd)

- NFU replaces (Ext) by (Ext)′, Weakened Extensionality, allowing Urelements
- Jensen (1969) proved consistency of NFU (+infinity, choice) rel. to PA (Z, ZC), using methods of Specker and Ehrenfeucht/ Mostowski; also stronger extensions.
- NFU proves closure under unordered pair, unions, intersections, power class, complement. Also have a universal class V. So, $V \in V$.

The problem of Pairing; NFU + P

- Usual definition of pairing, $(X,Y) = \{\{X\},\{X,Y\}\}$ works but at the cost of going 2 up in type levels.
- Solution: NFU + P adds a binary operation symbol P with Pairing Axiom,
  $P(X,Y) = P(Z,W) \rightarrow X=Z \wedge Y=W$.
- For SCA in NFU+P, expand definition of stratification so "type level" of P(s,t) same as for both s,t.

Relations, Structures and CT in NFU+P

- NFU+P proved consistent by a simple modification of Jensen's methods
- Tuples obtained by iterated pairing, then relations and functions as usual. Can prove closure under $X \times Y$ and $X \to Y$ for any X,Y.
- First-order structures are tuples (A,...,R,...,F...). Use SCA to form the classes Grp, Top, Cat, etc., of <u>all</u> groups, topological spaces, categories, etc.

(R1) and (R2) for NFU+P

- (R1) is met in full. In particular, can form the structure **Cat** = (Cat, Funct,...), "the category of all categories" and prove **Cat** $\in$ Cat.
- (R2) is met in full. Can prove that if A, B are in Cat then (A$\rightarrow$B) is in Cat. More generally, can also prove that the 2-category (Cat, Funct, Nat,...) is in Cat.

(R3) for NFU + P

- (R3) has <u>two specific problems</u> in NFU+P:
- (Prob 1) to go from an equivalence relation to the class of equivalence classes
- (Prob 2) to form unrestricted Cartesian products.

Adding Universal Choice to take care of Problem 1

- Form the extension NFU+(P, C) obtained by adding a constant symbol C with axiom
  (UC) $\exists X(X \in A) \to C(A) \in A$.
- The stratification condition for SCA is now that C(t) is always assigned type level one less than that for t.
- UC allows us to define equivalence classes--given (A, E)--as $X/E = C(\{Y|(X,Y)\in E\})$. Then the map $F(X)=X/E$ from A to A/E exists by extended SCA.
- Consistency (R4) still holds; cf. below.

A partial solution to Problem 2

- Don't know of any way to get full Cartesian prods in an enriched stratified theory of classes.
- But can form $\Pi F(x)[x \in I]$ for any class I of <u>sets</u> and F from I to classes, in the system S* of sets and classes of Fef ('74, '06); sets may be assigned any type level in the stratification conditions.
- Let S† be the extension of S* by C with UC and its stratification conditions. Consistency of S† is proved by a modification of that for S* in Fef('74);sets are the constructibles up to a strongly inaccessible κ .

What remains to be done?

- The advantages and disadvantages of working in S‡ need to be tested by working with specific cases (e.g.,Freyd AFT, Yoneda Lemma, Kan Extension Theorem, etc.)
- The ecumenical point of view about the appropriate framework for FCT in Fef('77) was that that should be carried out via some sort of theory of operations and collections. One should still pursue alternatives.
- My candidates: systems of Explicit Mathematics and Operational Set Theory.

Fef Refs

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