Notes on Operational Set Theory, I. Generalization of "small" large cardinals in classical and admissible set theory.

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"Small" large cardinal notions in the language of ZFC are those large cardinal notions consistent with $V = L$.

We have the original (1) and analogues (2–7) of small large cardinal notions in:

1. Classical set theory (ZFC)
2. Admissible set theory (KP)
3. Admissible recursion theory
4. Constructive set theory (CST)
5. Explicit mathematics (T_0)
6. Constructive type theory (Martin-Löf)
7. Recursive ordinal notation systems

"The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features." (E. H. Moore, 1910, re linear algebra and linear analysis)

Main aim here: to develop a common language for small large cardinal notions to include 1–7. This is a program in progress. Show how this could be done for 1–3. Should be reasonably adaptable to 4–5.

Approach: expand language of set theory to allow us to talk about general set theoretical operations (possibly partial); formulate the large cardinal notions in question in terms of operational closure conditions. This is a partial adaptation of Explicit Mathematics notions to the set-theoretical framework. The large cardinal notions treated here are for inaccessibles,

[1]These notes are expansions of material originally intended for the first part of a lecture in the section on Proof Theory and the Foundations of Mathematics, meeting of the Amer. Math. Soc., Columbus, OH, 21–23 Sept. 2001. The second part (to be prepared) concerns applications of operational set theory to the foundations of category theory.

Mahlo cardinals and weakly compact cardinals. A general reflection principle is formulated below from which these should all follow.

An early version of the present approach was presented in [Feferman 1996]. For analogues of large cardinals up to Mahlo in the Explicit Mathematics setting, see [Jäger t.a.]. (More historical notes and references at the end).

OST ("Operational Set Theory")

L_{OST}, the language of OST, extends the language $L_{ZF}(=, \in)$ by a three place relation symbol $\mathrm{App}(x, y, z)$ and various constants (to be seen). The interpretation of $\mathrm{App}(x, y, z)$ is that x represents an operation which when applied to y is defined and has value z.

A further extension of L_{OST} allows us to introduce application terms

$$s, t, \ldots := \text{ variable} \mid \text{constant} \mid st$$

The Axioms of OST divide into four parts:

A. Applicative axioms

B. Logical operations

C. Extensionality and foundation

D. Set existence axioms

We take up notation for the A axioms first.
Abbreviations:

$$\begin{aligned}
&t \simeq x \text{ for } t = x \text{ when } t \text{ is a var. or const.} \\
&st \simeq z \text{ for } \exists x, y[s \simeq x \wedge t \simeq y \wedge \ \mathrm{App}(x, y, z)] \\
&t \downarrow \ \text{ for } \exists x(t \simeq x) \\
&s = t \text{ for } \exists x, y(s \simeq x \wedge t \simeq y \wedge x = y) \\
&s \simeq t \text{ for } (s \downarrow \vee t \downarrow \rightarrow s = t)
\end{aligned}$$

Remark: this is the original (1975) approach to the applicative language for systems of Explicit Mathematics. Since then we have included application terms in the basic language with Beeson's Logic of Partial Terms (LPT), featuring $\forall x \varphi(x) \wedge t \downarrow \rightarrow \varphi(t)$ (and the dual for $\exists$) in the underlying logic. Here there are reasons for hewing to the basic language of App with only vars. and consts. as terms.

NB. Operations are regarded as *intensional objects*, or *representations* in the universe V of all sets, of extensional operations on V. In the axioms A, we have two constants $\mathbf{k}$, $\mathbf{s}$ for the (partial) combinators. As usual, in $s\ t_1 \ldots t_n$ association is to the left.

A. *Applicative axioms of OST.*

(i) $xy \simeq z \wedge xy \simeq w \to z = w$

(ii) $\mathbf{k}xy = x$

(iii) $\mathbf{s}xyz \simeq xz(yz)$

As usual from (i) – (iii) we can introduce for each term t a term $\lambda x.t$ whose variables are those of t other than x and such that

$$\lambda x.t \downarrow \wedge (\lambda x.t)y \simeq t(y/x),$$

and then a *recursor* **rec** with

$$\mathbf{rec} f \downarrow \wedge [\mathbf{rec} f = g \to gx \simeq fgx]$$

In the axiom groups B and D we write

$$\begin{array}{l} t \in b \text{ for } \exists x(t = x \wedge x \in b) \\ f : a \to b \text{ for } \forall x(x \in a \to fx \in b) \\ f : a^2 \to b \text{ for } \forall x, y(x, y \in a \to fxy \in b) \end{array}$$

The sets a and/or b may be replaced by the "class" V [classes are not officially part of the systems OST, and V is not a separate constant]. So, for example, $f : a \to V$, means f is total on a, and $f : V \to b$ means f maps all sets into b, and $f : V \to V$ means f is a total operation. Similarly for V^2 in place of V, etc.

Note that under our definition, if $f : a \to b$ and $a' \subseteq a$ then $f : a' \to b$.

In the axioms B we have two further constants $\mathbf{t}$ and $\mathbf{f}$, for Truth and Falsity; axioms D will guarantee the existence of a set $\{\mathbf{t}, \mathbf{f}\}$. When $f : a \to \{\mathbf{t}, \mathbf{f}\}$, we may regard f as a *definite predicate* on the set a; similarly with V, V^2 in place of a.

n-ary Boolean operations are $f : \{\mathbf{t}, \mathbf{f}\}^n \to \{\mathbf{t}, \mathbf{f}\}$. In this group we also have constants **el, cnj, neg, all,** respectively for the predicate of *elementhood*, the Boolean operations of *conjunction* and *negation*, and the operation of <u>*bounded*</u> *universal quantification*; bounded quantifiers $\forall x \in a(\ldots)$ and $\exists x \in a(\ldots)$ are explained as usual.

B. *Logical operations*

(i) $\mathbf{t} \neq \mathbf{f}$

(ii) $\mathbf{el} : V^2 \to \{\mathbf{t}, \mathbf{f}\} \wedge \forall x, y[\mathbf{el}xy = \mathbf{t} \leftrightarrow x \in y]$

(iii) $\mathbf{cnj} : \{\mathbf{t}, \mathbf{f}\}^2 \to \{\mathbf{t}, \mathbf{f}\} \wedge \forall x, y[\mathbf{cnj}xy = \mathbf{t} \leftrightarrow x = \mathbf{t} \wedge y = \mathbf{t}]$

(iv) $\mathbf{neg} : \{\mathbf{t}, \mathbf{f}\} \to \{\mathbf{t}, \mathbf{f}\} \wedge \forall x[x = \mathbf{t} \vee x = \mathbf{f} \to \mathbf{neg}x \neq x]$

(v) $(f : a \to \{\mathbf{t}, \mathbf{f}\}) \to \mathbf{all}fa \in \{\mathbf{t}, \mathbf{f}\} \wedge [\mathbf{all}fa = \mathbf{t} \leftrightarrow \forall x \in a(fx = \mathbf{t})]$

C. *General set axioms.*

(i) Extensionality as usual

(ii) $\in$-Induction, $\mathrm{Ind}_{\in}$,

$$\forall x(\forall y \in x\, \varphi(y) \to \varphi(x)) \to \forall x \varphi(x)$$

for *all* formulas $\varphi(x, \ldots)$ of the language.

Using the operator **rec** we can define operations by recursion on sets, resp. ordinals, and use the scheme $\mathrm{Ind}_{\in}$ to prove that they are total on V, resp. ORD.

The set existence axioms D make use of three new (functional) operation constants, **S** for *separation*, **R** for *replacement* (or *range*) and **C** for *choice*. In addition, classical OST makes use of the constant $\mathcal{P}$ for the *power set operation.*

D. *Set existence axioms*

(i) Empty set, unordered pair, union and infinity as usual

(ii) Separation for definite properties

$$(f : a \to \{\mathbf{t}, \mathbf{f}\}) \to \mathbf{S} f a \downarrow \wedge \forall x[x \in \mathbf{S} f a \leftrightarrow x \in a \wedge f x = \mathbf{t}]$$

(iii) Replacement (or range)

$$(f : a \to V) \to \mathbf{R} f a \downarrow \wedge \forall y[y \in \mathbf{R} f a \leftrightarrow \exists x \in a(y = fx)]$$

(iv) Choice

$$\exists x(fx = \mathbf{t}) \to \mathbf{C} f \downarrow \wedge f(\mathbf{C} f) = \mathbf{t}.$$

(v) Power set axiom (only for classical OST)

$$\mathcal{P} : V \to V \wedge \forall a, x[x \in \mathcal{P} a \leftrightarrow x \subseteq a]$$

We denote the optional axiom (v) by $(\mathcal{P})$, so the systems we are considering are OST$\pm(\mathcal{P})$.

Definition. The ess-$\sum(\mathrm{App}^{+})$ formulas $\psi, \chi, \ldots$ are generated as follows:

$$\begin{aligned} \psi := (x = y) | \neg(x = y) | (x \in y) | \neg(x \in y) | \mathrm{App}(x, y, z) | \\ | \psi \wedge \chi | \psi \vee \chi | \forall y \in x \psi | \exists y \in x \psi | \exists y \psi \end{aligned}$$

The Δ_0 formulas are generated without App and unrestricted $\exists$.

In the following $\psi(\underline{x})$ indicates a formula with free variables contained in $\underline{x} = x_1, \ldots, x_n$, and $t\underline{x}$ is written for $tx_1 \ldots x_n$

Lemma 1.

(i) With each Δ_0 formula $\psi(\underline{x})$ is associated a closed term t_ψ such that

$$t_\psi \downarrow \wedge (t_\psi : V^n \to \{\mathbf{t}, \mathbf{f}\}) \wedge \forall \underline{x}[\psi(\underline{x}) \leftrightarrow t_\psi \underline{x} = \mathbf{t}]$$

(ii) With each ess-$\sum(\text{App}^+)$ formulas $\psi(\underline{x})$ is associated a closed term t_ψ such that

$$t_\psi \downarrow \wedge \forall \underline{x}[\psi(\underline{x}) \leftrightarrow t_\psi \underline{x} = \mathbf{t}]$$

Proof (idea). First define a characteristic function **eq** using the logical operations in B, since $x = y \leftrightarrow x \subseteq y \wedge y \subseteq x$. Then the rest of (i) follows by B. For (ii), first use **k** and **s** to define **ap** with $\mathbf{ap}xy \simeq xy$. Then $\text{App}(x, y, z) \leftrightarrow \mathbf{eq}(\mathbf{ap}xy)z = \mathbf{t}$. The only new thing that has to be considered in (ii) is unrestricted $\exists$. Given $\psi(\underline{x}) = \exists y \chi(\underline{x}, y)$ and t_χ for $\chi(\underline{x}, y)$; then we can take $t_\psi = \lambda \underline{x}.t_\chi \underline{x}(\mathbf{C}(\lambda y t_\chi \underline{x} y))$, using the general choice operator $\mathbf{C}$.

Corollary 2. We have closed terms 0 for the empty set, ω for the first infinite ordinal, **p** for unordered pair, and $\bigcup$ for union.
Proof. Each is given by an axiom of the form $\exists y \psi$ where ψ is in Δ_0 form, and where y is the unique set specified in terms of the parameters of ψ. Then apply $\mathbf{C}$ to choose that y.

Theorem 3. (Strength of OST)

(i) $\text{KP}\omega +\ \text{AC}\ \subseteq\ \text{OST}$

(ii) OST is interpretable in $\text{KP}\omega + \text{V} = \text{L}$.

Proof (idea) (i) is direct from Lemma 1 and the set axioms C, D. For (ii) we interpret the applicative structure in the codes for $\sum_1$ definable functions, obtained by uniformizing the $\sum_1$ predicates.

We thus get conservation of OST over KPω for absolute formulas.

Theorem 4. (Strength of OST + $(\mathcal{P})$)

(i) ZFC$\subseteq$ OST + $(\mathcal{P})$

(ii) OST + $(\mathcal{P})$ is interpretable in ZFC+ V = L.

Proof (idea). This time, for (ii), interpret the applicative structure in the codes for functions $\sum_1$ definable in terms of $\mathcal{P}$.

For (i) some work must first be done. Use the remark following axioms C to define the cumulative hierarchy V_α by recursion on α, and then show that every set belongs to some V_α. Then use $\mathbf{C}$ to define Skolem functions (from the inside out) for all formulas in the language of ZF. Then prove reflection in the form that for each such $\varphi(\underline{x})$

$$\forall a \exists b[a \subseteq b \wedge \text{Trans}(b) \wedge \forall \underline{x} \in a(\varphi(\underline{x}) \leftrightarrow \varphi^{(b)}(\underline{x}))].$$

We can take b to be some V_β by the usual argument of closing under Skolem functions and then up the next level in the cumulative hierarchy. From this we get full Separation (Replacement) from Separation (Replacement) for Δ_0 formulas.

Again we have conservation over ZF for absolute formulas.

Next are a few useful observations.

Define, as usual, $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$, and $\mathrm{dom}(a), \mathrm{rng}(a)$ for the set a restricted to its subset of pairs, considered in the set-theoretical sense as a binary relation. We write $\mathrm{Fun}(a)$ if $\forall x \in \mathrm{dom}(a)\exists! y\langle x, y\rangle \in a$, and if this holds, $a(x)$ for the unique y such that $\langle x, y\rangle \in a$.

Lemma 5. We have a closed term **prod** such that for each $a, b, \mathbf{prod}ab\downarrow$ and $\mathbf{prod}ab = a\times b$.
Proof. Let f be such that for each $x, y, fxy = \langle x, y\rangle$, and let $f_x = \lambda y.fxy$. Then for each $x \in a$, $f_x : b \to \{x\} \times b$, and $\mathbf{R}f_x = \{x\} \times b$.

The operation $g = \lambda x.\mathbf{R}(\lambda y.fxy)$ thus has
$\mathbf{R}g = \{\{x\} \times b|x \in a\}$ and so $a \times b = \bigcup(\mathbf{R}g)$.

Lemma 6.

(i) We have closed terms $\mathbf{p}_0$ and $\mathbf{p}_1$ such that for each x, y

$$\mathbf{p}_0\langle x, y\rangle = x \wedge \mathbf{p}_1\langle x, y\rangle = y.$$

(ii) We have closed terms **dom** and **rng** such that for each a,

$$\mathbf{dom}a = dom(a) \text{ and } \mathbf{rng}a = rng(a).$$

(iii) We have a closed term **op** such that for each a, $\mathbf{op}a\downarrow$ and if $\mathrm{Fun}(a)$ and $f = \mathbf{op}a$ then for each $x \in dom(a), fx = a(x)$.

Proof. By Lemma 1, Cor. 2 and the choice operator $\mathbf{C}$. (ii) is obtained from the fact that $\mathrm{dom}(a)$ and $\mathrm{rng}(a)$ are $\subseteq$ the double union of a, and we then apply the separation operator $\mathbf{S}$. For (iii) we can take

$$\mathbf{op}a = \lambda x.\mathbf{C}(\lambda y.t(x, y)) \text{ where } t(x, y) = \mathbf{t} \leftrightarrow \langle x, y\rangle \in a.$$

Note that by (iii) every function in the set-theoretical sense is represented by an operation (in a uniform way). The following gives a partial converse, namely that the restriction of an operation to a set is extensionally equivalent to such a function.

Lemma 7. There is a closed term **fun** such that for each f, a, if $f : a \to V$ then $\mathbf{fun}fa\downarrow$ and if $c = \mathbf{fun}fa$ then $\mathrm{Fun}(c)$ and for each $x \in \mathrm{dom}(c)$, $c(x) = fx$.
Proof. Let $b = \mathbf{R}fa$, so $f : a \to b$. We want $c = \{\langle x, y\rangle|x \in a \wedge y \in b \wedge fx \simeq y\}$. This is given by $c = \{z|z \in a \times b \wedge \mathbf{eq}(\mathbf{ap}fx)y = \mathbf{t}\}$, which is constructed via $\mathbf{S}$ and **prod**.

We're now ready to consider the operational formulation in the language of ZFC of some (small) large cardinal axioms.[2] In the following we use l.c. Greek letters $\alpha, \beta, \ldots, \kappa, \lambda, \ldots, \xi, \eta, \zeta$ for ordinals, defined as usual.

[2]These can just as well be formulated as (small) large universe axioms as has been done in Explicit Mathematics and Constructive Set Theory and Constructive Type Theory.

Definition

(i) $\mathrm{Reg}(\kappa) :\leftrightarrow \kappa > 0 \wedge \forall\alpha, f[\alpha < \kappa \wedge (f : \alpha \to \kappa) \to \exists\beta < \kappa(f : \alpha \to \beta)]$

(ii) $\mathrm{Inacc}(\kappa) :\leftrightarrow \mathrm{Reg}(\kappa) \wedge \forall\alpha < \kappa\exists\beta < \kappa[\mathrm{Reg}(\beta) \wedge \alpha < \beta]$

(iii) $\mathrm{Reg}_1(\kappa) :\leftrightarrow \kappa > 0 \wedge \forall f[(f : \kappa \to \kappa) \to \exists\alpha < \kappa(0 < \alpha \wedge f : \alpha \to \alpha)]$

(iv) $\mathrm{Mahlo}(\kappa) :\leftrightarrow \kappa > 0 \wedge \forall f[(f : \kappa \to \kappa) \to \forall\xi < \kappa\exists\alpha < \kappa\,(\xi < \alpha \wedge \mathrm{Reg}(\alpha) \wedge f : \alpha \to \alpha)]$

(v) The statements Reg, Inacc, Reg_1 and Mahlo are obtained by replacing κ by the class ORD of all ordinals.

By Lemmas 6, and 7, the meanings of Reg, Reg_1, Inacc and Mahlo are the same whether the 'f' variables are interpreted in the intensional operational sense or in the extensional set-theoretical sense.

Lemma 8.

(i) $\mathrm{Reg}_1(\kappa) \leftrightarrow \mathrm{Reg}(\kappa) \wedge \kappa > \omega$

(ii) $\mathrm{Reg}_1 \leftrightarrow \mathrm{Reg}$

Proof idea for (ii). Define normality for operations as usual, show that every such operation has arbitrarily large ω-cofinal fixed points, and show that every f is majorized by a normal g (and such that $g0$ is arbitrarily large). Then to show Reg$\to$ Reg_1, given f: ORD$\to$ ORD, using such g, find $\alpha > 0$ with $g\alpha = \alpha$, so that then $g : \alpha \to \alpha$, hence also $f : \alpha \to \alpha$. Conversely, given $\alpha > 0 \wedge (f : \alpha \to \mathrm{ORD})$, choose normal g majorizing f with $g0 = \alpha$, and find $\beta > 0$ with $g : \beta \to \beta$. Then $\alpha < \beta$ and so $f : \alpha \to \beta$. The proof of (i) relativizes the argument to κ.

The statement corresponding to Lemma 8(i) in ZF, with functions in the set-theoretical sense instead of operations as here was stated in [Aczel and Richter 1972]. This was used by them to motivate a definition of Reg_2, again with set-theoretical functions. Here we do the same with operations instead of functions.

Definition. $(f \equiv g) :\leftrightarrow \forall x(fx \simeq gx)$

$(f|\alpha \equiv g|\alpha) :\leftrightarrow \forall\xi < \alpha(f\xi \simeq g\xi)$

Definition. Write $f \in \kappa^\kappa$ if $f : \kappa \to \kappa$, and $F : \kappa^\kappa \to \kappa^\kappa$ if

$$\forall f \in \kappa^\kappa(Ff \in \kappa^\kappa) \wedge \forall f, g \in \kappa^\kappa[f \equiv g \to Ff \equiv Fg].$$

We say F is κ-*bounded* if

$$\forall f \in \kappa^\kappa\forall\xi < \kappa\exists\gamma < \kappa\forall g \in \kappa^\kappa[f|\gamma \equiv g|\gamma \to Ff\xi = Fg\xi].$$

α is a κ-*witness* for F if

$$0 < \alpha < \kappa \wedge \forall f \in \kappa^\kappa[f \in \alpha^\alpha \to Ff \in \alpha^\alpha].$$

Similarly, define $f \in \mathrm{ORD}^{\mathrm{ORD}}$, $F : \mathrm{ORD}^{\mathrm{ORD}} \to \mathrm{ORD}^{\mathrm{ORD}}$, F is *bounded*, and α is a *witness* for F, by replacing κ by ORD throughout.

Definition.

$$\begin{aligned}
&\mathrm{Reg}_2(\kappa) :\leftrightarrow \forall F[F \ \kappa\text{-bounded} \to \exists\alpha(\alpha \text{ is a } \kappa\text{-witness for } F)] \\
&\mathrm{Reg}_2 :\leftrightarrow \forall F[F \text{ bounded } \to \exists\alpha(\alpha \text{ is a witness for } F)].
\end{aligned}$$

[Aczel and Richter 1972] state, and [Richter and Aczel 1974] prove, that if we use set-theoretic functions in place of operations, then in ZFC, κ is Reg_2 iff κ is weakly compact. By Lemmas 6 and 7, the set-theoretical interpretation of $\mathrm{Reg}_2(\kappa)$ is equivalent to its definition above, since κ^κ can be replaced by the set of all functions from κ to κ in the set-theoretical sense, and then F can be replaced by a function on that set to itself. On the other hand, it is not clear if the operational sentence Reg_2 has a set-theoretical interpretation.

The two Aczel and Richter papers also give an analogue formulation of these notions in terms of recursion theory on admissible sets. If κ is an admissible ordinal and we interpret $fx \simeq y$ as $\{f\}(x) \simeq y$ in the sense of the $\sum_1$ recursion theory on κ (or L_κ) then each statement φ translates into a statement φ^{Ad} which gives the analogue notion. In the case of Reg_2 the analogue notion is proved by them in [Richter and Aczel 1974] to be equivalent to $\prod_3$-reflection. Formalizing the arguments of Aczel and Richter, one should have the following (I have not checked the details):

Theorem 9

(i) OST + (Inacc) is interpretable in KPi + V = L.

(ii) OST + (Mahlo) is interpretable in KPM +V = L.

(iii) OST + (Reg_2) is interpretable in $\mathrm{KP}\omega$ + ($\prod_3$-reflection) + V = L.

In each case we interpret the theory on the left in the theory on the right using the translation of φ as φ^{Ad}. While it is not obvious that the theories on the right are contained in those on the left, it is hard to believe that they are any stronger. In terms of the relation $\equiv$ of proof-theoretical equivalence, I thus make the following

Conjectures.

(i) OST + (Inacc)$\equiv$ KPi

(ii) OST + (Mahlo)$\equiv$ KPM

(iii) OST + (Reg_2) $\equiv$ $\mathrm{KP}\omega$ + ($\prod_3$-reflection).

It should be noted that in the interpretation $(\mathrm{Reg}_2)^{\mathrm{Ad}}$ the assumption that the functional F is bounded can be dropped, since F is represented by an effective operation, and so is continuous by a generalization of the Myhill-Shepherdson theorem.

In [Aczel and Richter 1972] a generalization called n-regularity of Reg_2 is indicated for each $n \geq 2$, which we write here as Reg_n, and they state that the interpretation of Reg_n on an admissible κ is equivalent to $\prod_{n+1}$-reflection. Unfortunately, the indicated definition is not supplied, and they did not pursue the matter further in the 1974 paper. The idea is to carry the notion of boundedness, which is a form of continuity, to higher types. One way this might be done is to generalize to recursion theory on L_κ the notion of hereditarily continuous (or countable) functional due, independently, to Kleene and Kreisel (cf. [Kreisel 1959]), but the details don't look simple. In the admissible interpretation by effective operations of higher type, the boundedness or continuity condition can be dropped, but that is not suitable for an abstract formulation. Alternatively, one could simply adjoin the assumption that all functionals of finite type are suitably bounded and work from there. Finally, it would be of interest to generalize these ideas to transfinite types to get operational equivalents in the admissible setting to $\prod_\alpha$-reflection.

The last thing I want to do here is describe an abstract reflection principle covering both classical and admissible set theory, from which the above "small large cardinal" principles and others follow, and which has a certain plausibility. For this, it is convenient to expand the language of OST further by *Op-variable* $f, g, h, \ldots$ with or without subscripts. These still range over V, but in formulas they will only occur in special positions.

Definition. An *Op-formula* is one generated from the atomic formulas $x = y$, $x \in y$, $\mathrm{App}(f, x, y)$, where x, y are set variables and f is an Op-variable, by the Boolean operations and quantification with respect to both set variables and Op-variables.

By a $\forall$-*Op formula* we mean one in which all quantified occurrences of Op-variables are in positive $\forall$ form. If φ is an Op-formula, then $\varphi^{(a)}$ is the result of relativizing all set quantifiers $\forall x(\ldots)$ and $\exists x(\ldots)$ in φ to a, i.e. as $\forall x \in a(\ldots)$ and $\exists x \in a(\ldots)$, resp.
NB. Under this process of relativization, Op-quantified variables are not changed.

Op-Reflection Principle. For each Op-formula $\varphi(\underline{x}, \underline{f})$, we have

$$\varphi(\underline{x}, \underline{f}) \rightarrow \exists a[\ \mathrm{Trans}(a) \wedge \underline{x} \in a \wedge \varphi^{(a)}(\underline{x}, \underline{f})]$$

By the $\forall$-Op-Reflection Principle we mean the same restricted to $\forall$-Op formulas φ.

Claim. Inacc and Mahlo follow from the $\forall$-Op-Reflection Principle. The same holds for obvious generalizations of Mahlo (hyper-Mahlo, hyper-hyper-Mahlo, etc.) The argument is that, first, Reg is a consequence of OST. Then use the reflection principle to prove that there are arbitrarily large κ with $\mathrm{Reg}(\kappa)$; thus Inacc holds. To prove Mahlo, given f, let $\varphi(f)$ be $(f : \mathrm{ORD} \rightarrow \mathrm{ORD}) \wedge$ Reg. Then if we get an a such that $\varphi(f)^{(a)}$ and $\kappa = a \cap \mathrm{ORD}$ then $(f : \kappa \rightarrow \kappa) \wedge \mathrm{Reg}(\kappa)$. (So on for hyper-Mahlo, etc.)

On the other hand, it is not obvious whether Reg_2 is a consequence of the Op-Reflection Principle, let alone its $\forall$ form. For the very formulation of Reg_2 in terms of functionals is not given as an Op formula. But there should be an equivalent reformulation of Reg_2 by means of an Op formula, by replacing the functionals by associated "neighborhood functions". This has to be checked out.

Conjecture. We can prove in ZFC that OST + ($\forall$-Op-Reflection Principle) is consistent.

Question. Is OST (Op-Reflection Principle) consistent?

Historical notes and acknowledgments.

The axiomatization by [von Neumann 1925] of a theory of sets and functions is a precursor in spirit of OST. Von Neumann's functions are of type 1 over the universe of sets and are closed under combinatory and logical axioms; it would be of interest to re-examine that work in the light of OST.

[Jäger and Strahm 1998] presented a form of Reg_2 in the Explicit Mathematics setting at a conference in Castiglioncello. I did not take in at that time their motivation from the work of Aczel and Richter. I owe it to Michael Rathjen for bringing the form Reg_2 of weak compactness and the work of Aczel and Richter to my attention at the Mittag-Leffler Institute in Sweden in the spring of 2001.

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