The proof theory of classical and constructive inductive definitions.
A 40 year saga, 1968-2008.

Solomon Feferman[1]

1. Pohlers and The Problem. I first met Wolfram Pohlers at a workshop on proof theory organized by Walter Felscher that was held in Tübingen in early April, 1973. Among others at that workshop relevant to the work surveyed here were Kurt Schütte, Wolfram's teacher in Munich, and Wolfram's fellow student Wilfried Buchholz. This is not meant to slight in the least the many other fine logicians who participated there.[2] In Tübingen I gave a couple of survey lectures on results and problems in proof theory that had been occupying much of my attention during the previous decade. The following was the central problem that I emphasized there:

> The need for an ordinally informative, conceptually clear, proof-theoretic reduction of classical theories of iterated arithmetical inductive definitions to corresponding constructive systems.

As will be explained below, meeting that need would be significant for the then ongoing efforts at establishing the constructive foundation for and proof-theoretic ordinal analysis of certain impredicative subsystems of classical analysis. I also spoke in Tübingen about

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[1] This is a somewhat revised text of a lecture that I gave for a general audience at the PohlersFest, Münster, 18 July 2008 in honor of Wolfram Pohlers, on the occasion of his retirement from the Institute for Mathematical Logic at the University of Münster. Wolfram was an invited participant at a conference in my honor at Stanford in 1998, and it was a pleasure, in reciprocation, to help celebrate his great contributions as a researcher, teacher and expositor. In my lecture I took special note of the fact that the culmination of Wolfram's expository work with his long awaited *Proof Theory* text was then in the final stages of production; it has since appeared as Pohlers (2009). In that connection, one should mention the many fine expositions of proof theory that he had previously published, including Pohlers (1987, 1989, 1992, and 1998).

[2] That meeting was organized by Walter Felscher under the sponsorship of the Volkswagen Stiftung; there were no published proceedings. It is Pohlers' recollection that besides him and Felscher, of course, the audience included Wilfried Buchholz, Justus Diller, Ulrich Felgner, Wolfgang Maas, Gert Müller, Helmut Pfeiffer, Kurt Schütte and Helmut Schwichtenberg. By the way, Felscher passed away in the year 2000.

possible methods to tackle the central problem, including both cut-elimination applied to (prima-facie) uncountably infinite derivations and functional interpretation on the one hand, and the use of naturally developed systems of ordinal notation on the other. I recall that my wife and I had driven to Tübingen that morning from Oberwolfach after an unusually short night's sleep, and that I was going on pure adrenalin, so that my lectures were particularly intense. Perhaps this, in addition to the intrinsic interest of the problems that I raised, contributed to Wolfram's excited interest in them. Within a year or so he made the first breakthrough in this area (Pohlers 1975), which was to become the core of his Habilitationsschrift with Professor Schütte (Pohlers 1977). The 1975 breathrough was the start of a five year sustained effort in developing a variety of approaches to the above problem by Wolfram Pohlers, Wilfried Buchholz and my student Wilfried Sieg. The results of that work were jointly reported in the Lecture Notes in Mathematics volume 897, *Iterated Inductive Definitions and Subsystems of Analysis. Recent proof-theoretical studies* (Buchholz et al.1981). In the next section I will give a brief review of what led to posing the above problem in view of several results by Harvey Friedman, William Tait and me at the 1968 Buffalo conference on intuitionism and proof-theory, with some background from a 1963 seminar on the foundations of analysis led by Georg Kreisel at Stanford in which formal theories of "generalized" inductive definitions (i.e., with arithmetical closure conditions) were first formulated.

The goals of proof-theoretic reduction and of proof-theoretic ordinal analysis in one form or another of the relativized Hilbert program (not only for theories of inductive definitions) are here taken at face value, though I have examined both critically; see Feferman (1988, 1993, 2000). In addition to meeting those aims in the problem formulated above are the demands that the solutions be informative and conceptually clear—in short, perspicuous. Granted that these are subjective criteria, nevertheless in practice we are able to make reasonably objective judgments of comparison. For example, we greatly valued Schütte's extension of Gentzen's cut-elimination theorem for the predicate calculus to "semi-formal" systems with infinitary rules of inference, because it exhibited a natural and canonical role for ordinals as lengths of derivations and bounds of cut-rank (cf. Schütte 1977) in the case of arithmetic and its extensions to ramified analysis. To begin with, the Cantor ordinal ε_0 emerged naturally as the upper bound of the lengths of cut-free derivations in the semi-formal system of arithmetic with ω-rule, obtained by eliminating cuts from the (translations into that system of) proofs in Peano Arithmetic PA; by comparison the role of ε_0 in Gentzen's consistency proof of PA still had an *ad hoc* appearance.[3] And the determination by Schütte and me in the mid-1960s of Γ_0 as the upper bound for the ordinal of predicativity simply fell out of his ordinal analysis of the systems of ramified analysis translated into infinitary rules of inference when one added the condition of autonomy. Incidentally, because of the connection with predicativity, these kinds of proof-theoretical methods due to Schütte—of ordinal analysis via cut-elimination theorems for semi-formal systems with countably infinitary rules of inference—have come to be referred to as predicative.

The proof-theoretical work on systems of single and (finitely or transfinitely) iterated arithmetical inductive definitions were the first challenges to obtaining perspicuous ordinal analyses and constructive reductions of impredicative theories. The general problem was both to obtain exact bounds on the provably recursive ordinals and to reduce inductive definitions described "from above" as the least sets satisfying certain arithmetical closure conditions to those constructively generated "from below". In the event, the work on these systems took us only a certain way into the impredicative realm, but the method of local predicativity for semi-formal systems with uncountably infinitary rules of inference that Pohlers developed to deal with them turned out to be of wider application. What I want to emphasize in the following is, first of all, that ordinal analysis and constructive reduction are separable goals and that in various cases, each can be done without the other, and, secondly, that the aim to carry these out in ever more perspicuous ways has led to recurrent methodological innovations. The most recent of these is the application of a version of the method of functional interpretation to theories of inductive definitions by Avigad and Towsner (2008), following a long period in which cut-elimination for various semi-formal systems of uncountably infinitary derivations had been the dominant method, and which itself evolved methodologically with perspicuity as the driving force.

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[3] That role became less mysterious as a result of the work of Buchholz (1997, 2001) explaining Gentzen-style and Takeuti-style reduction steps in infinitary terms.

It is not possible in a survey of this length—and at the level of detail dictated by that—to explain or state results in full; for example, I don't state conservation results that usually accompany theorems on proof-theoretical reduction. Nor is it possible to do justice to all the contributions along the way, let alone all the valuable work on related matters. For example, except for a brief mention in sec. 7 below, I don't go into the extensive proof-theoretical work on iterated fixed point theories. I hope the interested reader will find this survey useful both as an informative overview and as a point of departure to pursue in more detail not only the topics discussed but also those that are only indicated in passing. Finally, this survey offers an opportunity to remind one of open questions and to raise some interesting new ones.

2. From 1968 to 1981, with some prehistory. In my preface, Feferman (1981), to Buchholz et al. (1981), I traced the developments that led up to that work; in this section I'll give a brief summary of that material.

The consideration of formal systems of "generalized" inductive definitions originated with Georg Kreisel (1963) in a seminar that he led on the foundations of analysis held at Stanford in the summer of 1963.[4] Kreisel's aim there was to assess the constructivity of Spector's consistency proof of full second-order analysis (Spector 1962) by means of a functional interpretation in the class of so-called bar recursive functionals. The only candidate for a constructive foundation of those functionals would be the hereditarily continuous functionals given by computable representing functions in the sense of (Kleene 1959) or (Kreisel 1959). So Kreisel asked whether the intuitionistic theory of inductive definitions given by monotonic arithmetical closure conditions, denoted ID_1(mon)[i] below, serves to generate the class of (indices of) representing functions of the bar recursive functionals. Roughly speaking, ID_1(mon), whether classical or intuitionistic, has a predicate P_A for each arithmetic A(P, x) (with a place-holder predicate symbol P) which has been proved to be monotonic in P, together with

[4] The notes for that seminar are assembled in the unpublished volume *Seminar on the Foundations of Analysis, Stanford University 1963. Reports*, of which only a few mimeographed copies were made; one copy is available in the Mathematical Sciences Library of Stanford University.

axioms expressing that P_A is the least predicate definable in the system that satisfies the closure condition $\forall x(A(P, x) \rightarrow P(x))$. In the event, Kreisel showed that the representing functions for bar recursive functionals of types ≤ 2 can be generated in an $ID_1(mon)^i$ but not in general those of type ≥ 3.

Because of this negative result, Kreisel did not personally pursue the study of theories of arithmetical inductive definitions any further, but he did suggest consideration of theories of finitely and transfinitely iterated such definitions as well as special cases involving restrictions on the form of the closure conditions A(P, x). For example, those A in which the predicate symbol P has only positive occurrences are readily established to be monotonic in P. And of special interest among such A are those that correspond to the accessible (i.e., well-founded part) of an arithmetical relation. And, finally, paradigmatic for those are the classes of recursive ordinal number classes O_α introduced in Church and Kleene (1936) and continued in Kleene (1938). The corresponding formal systems for α times iterated inductive definitions (α an ordinal) are denoted (in order of decreasing generality) $ID_\alpha(mon)$, $ID_\alpha(pos)$, $ID_\alpha(acc)$ and $ID_\alpha(O)$ in both classical and intuitionistic logic, where the restriction to the latter is signalled with a superscript 'i'.[5] For limit ordinals λ we shall also be dealing with $ID_{<\lambda}(\text{-})$, the union of the $ID_\alpha(\text{-})$ for $\alpha < \lambda$, of each of these kinds, whether classical or intuitionistic. Finally, when no qualification of ID_α or $ID_{<\lambda}$ is given, it is meant that we are dealing with the corresponding $ID_\alpha(pos)$ or $ID_{<\lambda}(pos)$, since—as will be explained in sec. 5 below—there is a relatively easy reduction of the monotonic case to the positive case. The $ID_\alpha(O)$ theories, or similar ones for constructive tree classes, are of particular interest, because the elements of those classes wear their build-up on their sleeves, i.e. can be retraced constructively; some of the $ID_\alpha(acc)$ classes considered below share that significant feature.

Kreisel's initiative led one to study the relationship between such theories to subsystems of classical analysis considered independently of Spector's approach and as the subject of proof-theoretical investigation in their own right. The first such result was obtained by William Howard some time around 1965, though it was not published until

[5] The positivity requirement has to be modified in the case of intuitionistic systems.

1972. He showed in Howard (1972) that the proof-theoretic ordinal of $ID_1(acc)^i$ is $\varphi\varepsilon_{(\Omega+1)}0$, as measured in the hierarchy of normal functions introduced in Bachmann (1950). Howard's method of proof proceeded via an extension of Gödel's functional interpretation. This was the first ordinally informative characterization of an impredicative system using a system of ordinal notation based on a natural system of ordinal functions. What was left open by Howard's work was whether one could obtain a reduction of the general classical ID_1 to $ID_1(acc)^i$ (and even better to $ID_1(O)^i$) and thus show that the proof-theoretic ordinal is the same, and similarly for the systems of iterated inductive definitions more generally.[6]

Turning now to the 1968 Buffalo Conference on Intuitionism and Proof Theory, here, in brief, is what was done in the three papers I mentioned above.

1. (Friedman 1970) proved that system Σ^1_{n+1}-AC is of the same strength as Δ^1_{n+1}-CA and is conservative over $(\Pi^1_n\text{-CA})_{<\varepsilon(0)}$ for suitable classes of sentences. For n = 1 this tied up with the following two results:

2. (Feferman 1970) gave an interpretation of $(\Pi^1_1\text{-CA})_\alpha$ in ID_α for various α, including $\alpha = \omega$, and of $(\Pi^1_1\text{-CA})_{<\lambda}$ in $ID_{<\lambda}$ for various limit λ, including $\lambda = \varepsilon_0$.[7]

3. (Tait 1970) established the consistency of Σ^1_2-AC via a certain theory of inductive definitions by informally constructive cut-elimination methods applied to uncountably long propositional derivations.

These results and the prior work of Takeuti (1967) containing constructive proofs of consistency of Π^1_1-CA and Π^1_1-CA + BI gave hope that one could obtain a constructive reduction of some of the above second order systems via a reduction of classical theories of iterated inductive definitions to their intuitionistic counterparts.[8] For, among the results of my Buffalo conference article was that the system Π^1_1-CA + BI is proof-theoretically equivalent to ID_ω. What Takeuti had done was to carry out his consistency

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[6] As will be explained in sec. 6, below, Zucker (1971, 1973) showed the ordinals to be the same without a reduction argument and by a method that did not evidently extend to the iterated case.

[7] Actually, the interpretation took one into iterated classical accessibility IDs.

[8] BI is the scheme of Bar Induction, i.e. the implication from well-foundedness to transfinite induction.

proofs by an extension of Gentzen's methods with cut-reduction steps measured in certain partially ordered systems that Takeuti called ordinal diagrams; these are not based on natural systems of ordinal functions such as those in the Bachmann hierarchy. Takeuti proved the well-foundedness of the ordering of ordinal diagrams by constructive arguments that could be formulated in suitable intuitionistic iterated accessible IDs. These methods were later extended to Δ^1_2-CA + BI in Takeuti and Yasugi (1973).

Before proceeding, a few words are necessary about the systems of ordinal functions involved in proof-theoretic ordinal analysis at that time and in subsequent work. Bachmann had extended the classical Veblen hierarchy φ_α (or $\lambda\alpha,\beta.\varphi_\alpha(\beta)$) of critical functions of countable ordinals by use of indices α to certain uncountable ordinals—including those up to the first ε-number greater than Ω—by diagonalizing at α of cofinality Ω, e.g. defining $\varphi\Omega\beta$ to be $\varphi\beta 0$. This method was carried out systematically by Helmut Pfeiffer (1964) by reference to the finite ordinal number classes whose initial ordinals are the Ω_n for $n < \omega$, and then by David Isles (1970) via the number classes up to the first inaccessible ordinal. Each such extension required more and more complicated assignment of fundamental sequences to the ordinals actually drawn from each number class. In 1970, in informal discussions with Peter Aczel, I proposed an alternative method of generating the requisite ordinals and associated functions θ_α in place of the φ_α without any appeal to fundamental sequences and in a uniform way from the function enumerating the initial ordinals Ω_ν of the number classes. Aczel quickly worked out the idea in unpublished notes in a preliminary way; this was then developed systematically by Jane Bridge in her 1972 Oxford dissertation, the results of which were published in Bridge (1975). She showed how to match up the notations obtained in this way with those obtained by the Bachmann-Pfeiffer-Isles procedures, and she initiated work to show that the countable ordinals generated by these means are recursive. The latter verification was carried out systematically and in full in Buchholz (1975); a detailed exposition of the definition and properties of the θ functions was later given in Schütte (1977) in the first sections of Ch. IX. (We'll return below to a much later simplification leading to the ψ functions in Buchholz (1992).)

The first successful results on ordinal analysis for theories of iterated inductive definitions were obtained only on the intuitionistic side by Per Martin-Löf (1971) via normalization theorems for the $ID_n(acc)^i$ systems as formulated in calculi of natural deduction. He conjectured the bounds $\varphi\varepsilon(\Omega_n+1)0$ in the Bachmann-Pfeiffer hierarchies for these and proved that their supremum is the ordinal of $ID_{<\omega}(acc)^i$ by use of Takeuti (1967).

The first breakthrough on the problems of ordinal analysis for the classical systems was made by Pohlers (1975) to give ordinal upper bounds for the finite ID_n also by an adaptation of the methods of Takeuti (1967); this was extended later in his Habilitationsschrift, Pohlers (1977), to arbitrary α, with the result that

$$|ID_\alpha| \leq \theta\varepsilon(\Omega_\alpha+1)0$$

as measured in the modified hierarchies described above. In addition, Buchholz and Pohlers (1978) showed this to be best possible by verification of

$$\theta\varepsilon(\Omega_\alpha+1)0 \leq |ID_\alpha(acc)^i|$$

using a constructive well-ordering proof of each proper initial segment of a natural recursive ordering of order type $\varphi\varepsilon(\Omega_\alpha+1)0$. These results lent further hope to the solution of the reductive problem posed above. Independently of their work, in his Stanford dissertation, Sieg (1977) adapted and extended the method of Tait (1970) followed by a formalization of the cut-elimination argument to reduce ID_α to $ID_{\alpha+1}(O)^i$, and thence $ID_{<\lambda}$ to $ID_{<\lambda}(O)^i$, for limit λ, without requiring any involvement of ordinal bounds.

In view of these results, it was decided to exposit all this work together, with the addition of suitable background material, in a *Lecture Notes in Mathematics* volume. As it turned out, the resulting joint publication Buchholz et al. (1981) contained important new contributions to the basic problems about theories of iterated inductive definitions, and though that volume has been superseded in various respects by later work, it still has much of value and I would recommend it as a starting point to the reader interested in studying this subject in some depth. In particular, my preface (Feferman 1981) to the volume fills out the historical picture to that point. Then the first chapter, Feferman and Sieg (1981a), goes over reductive relationships between various subsystems of Σ^1_2-AC, systems of iterated inductive definitions, and subsystems of the system T_0 of explicit mathematics from Feferman (1975). The second chapter, Feferman and Sieg (1981b) showed how to obtain the reductions of Σ^1_{n+1}-AC to $(\Pi^1_n\text{-CA})_{<\varepsilon(0)}$ by proof-theoretic arguments (based on a method called Herbrand analysis by Sieg), in place of the model-theoretic arguments that had been used by Friedman. Following that, Sieg (1981) presented the work of his thesis in providing the reductions of ID_α to $ID_{\alpha+1}(O)^i$ and of $ID_{<\lambda}$ to $ID_{<\lambda}(O)^i$ for limit λ,without the intervention of ordinal analysis. In the next two chapters Buchholz (1981a, 1981b) introduced uncountably infinitary semi-formal systems making use of a special new $\Omega_{\alpha+1}$-rule in order, in the first of these to obtain the proof-theoretical reduction of the ID_α to suitable $ID_\alpha(acc)^i$ and in the second to reestablish the ordinal bounds previously obtained by Pohlers. Finally, in the last chapter, Pohlers (1981) presented a new approach called the *method of local predicativity*, to accomplish the very same results in a different way. This dispensed with the earlier dependence on the methods of Takeuti's (1967); the more perspicuous method of local predicativity, in its place, utilizes a kind of extension to uncountably branching proof trees of the methods of predicative proof theory. But both Buchholz' and Pohlers' work in the Buchholz et al. (1981) volume required the use of certain syntactically defined collapsing functions, in order to reduce prima-facie uncountable derivations to countable ones in a way that allows one to obtain the recursive ordinal bounds. As will be described in sec. 4, this was superseded a decade later by the work of Buchholz (1992) showing how to obtain the same bounds without the use of such collapsing functions.

3. Admissible proof theory. Insofar as the work in Buchholz et al. (1981) settled the basic problem posed at the beginning, it could be considered the end of the story. But the aim to develop conceptually still clearer methods had already been underway, beginning with the dissertation of Gerhard Jäger (1979), also under Schütte's direction, but in that case with Pohlers' assistance. The novel element there was to embed various of the subsystems of analysis, both predicative and impredicative, in theories of admissible sets, and to carry out the ordinal analysis of the latter by means of a cut-elimination theorem for associated semi-formal systems of ramified set theory. The connection is that one can identify the minimal models of the theories of admissible sets in question as natural initial segments of the constructible hierarchy. This method was further elaborated in Jäger's Habilitationsschrift (1986) (though that relies on the earlier publication for certain proof-theoretic results about ramified set theory).

The systems of admissible set theory considered by Jäger are taken to have a set of urelements interpreted as the set N of natural numbers given with its successor relation. KPN has the usual axioms for Kripke-Platek set theory with urelements (e.g. from Barwise (1975)), including the full induction scheme (IND_N) on the natural numbers and ($IND_\in$) on the membership relation. KPN^w is the system obtained from KPN by replacing the $\in$−induction scheme by the corresponding set induction axiom, KPN^r is obtained by further replacing the N-induction scheme by the corresponding set induction axiom, and, finally, KPN^0 is obtained by completely dropping induction on the membership relation. We may also represent KPN^w as $KPN^r + IND_N$. Also considered are the extensions KPL and KPI of KPN, obtained by adding the axioms that the universe is a limit of admissible sets, and that the universe is an admissible limit of admissible sets, respectively; these are also considered in the 'w', 'r', '0' restricted versions as for KPN.[9] The minimal constructible model L_α of KPI is that for which α is the least recursively inaccessible ordinal.

Among the results of Jäger (1986) is that KPI^0 is a kind of universal theory for systems having Γ_0 as their proof theoretic ordinal, in the sense that all such systems (up to that point) have natural embeddings in KPI^0. Among these is Friedman's theory ATR_0, which also has Γ_0 as a lower bound. The proof theoretic treatement of KPI^0 via ramified set theory takes the place of the earlier proof by Friedman, McAloon and Simpson (1982) of Γ_0 as the ordinal of ATR_0 via model-theoretic arguments. Incidentally, ATR_0 is already embeddable in KPL^0, so KPI^0 is no stronger than that.

Moving on to impredicative systems, ID_1 is embedded in KPN, which was shown to have the Howard ordinal as upper bound in Jäger (1979). The strongest system considered in Jäger (1986) is KPI, and among the further notable results for restricted subsystems of that are:

$$(\Sigma^1_2\text{-AC})_0 \equiv KPI^r, \text{ and } \Sigma^1_2\text{-AC} \equiv KPN^r + IND_N \equiv KPI^r + IND_N = KPI^w,$$

where $\equiv$ is the relation of proof-theoretical equivalence; in both cases, the ordinal analysis

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[9] Jäger (1986) uses KPu, KPl and KPi for what is here denoted by KPN, KPL and KPI, resp. *NB*: the system denoted KPN in Jäger (1979, 1980) is the same as $KPu^r + IND_N$ in the notation of Jäger (1986), and of KPN^w, or alternatively $KPN^r + IND_N$, in the notation used here. KPN is equivalent in strength to the system often denoted as $KP\omega$.

of the set-theoretic side is obtained via cut-elimination via the semi-formal system of ramified set theory. The main upper bound result for the full KPI was obtained in Jäger and Pohlers (1983) using the method of local predicativity to establish the ordinal upper bound, while (as explained below) the lower bound follows from the work of Jäger (1983):

$$\sum^1{}_2\text{-AC} + \text{BI} \equiv \text{KPI and } |\text{KPI}| = \psi_\Omega(\varepsilon_{I+1}),$$

where, for simplicity, I am using the notation introduced later by Buchholz (1992) for the ψ functions in place of the θ functions. For example, the ordinal of ID_α in these terms is $\psi_\Omega\varepsilon(\Omega_{\alpha+1})$ in place of $\theta\varepsilon(\Omega_\alpha+1)0$.

In the survey article Pohlers (1998) it is shown how various subsystems of KPI match up both with subsystems of $\sum^1{}_2$-AC + BI and with theories of iterated inductive definitions, and their proof-theoretic ordinals are identified in terms of the ψ functions; an informative table is given op. cit. p. 333. For example, we have $ID_\omega \equiv \prod^1{}_1$-CA + BI ≡ KPL. Among these are systems lying between $\sum^1{}_2$-AC and $\sum^1{}_2$-AC + BI in strength (alternatively described, between KPI^w and KPI) studied by Michael Rathjen in his dissertation (1988) at Münster under Pohlers' direction, including autonomously iterated theories of inductive definitions and corresponding systems of autonomously iterated $\prod^1{}_1$-CA and of admissible sets; see Pohlers (1998) sec. 3.3.5 for a partial account, since the work of Rathjen(1988) has otherwise not yet been published.

The work on admissible proof theory has also been useful in dealing with systems of explicit mathematics that were formulated and studied in Feferman (1975, 1979). These systems have notions of operations f, g, … and classes (a.k.a. classifications, properties, or [variable] types) A, B,C,…, both objects in a universe V of individuals; relations R, S,… are treated as classes of pairs, using a basic pairing operation on V. Operations are in general partial, but may apply to any element of V, including operations and classes. The strongest system of explicit mathematics dealt with op. cit. in which the operations have an interpretation as partial recursive functions is denoted T_0. For present purposes, I want only to concentrate on one axiom group of T_0, concerning a general operation i of inductive generation. Given any A and (binary) R, i(A, R) is always defined and its value is a class I that satisfies:

$$\forall x \in A[\forall y\ ((y,x) \in R \rightarrow y \in I) \rightarrow x \in I]$$

In addition we have induction on I, which is either taken in the restricted class-induction form

$$\forall x \in A[\forall y\, ((y,x) \in R \to y \in X) \to x \in X] \to I \subseteq X,$$

or as a scheme obtained by substituting for X all formulas of the language of T_0. The system T_0(res-IG) assumes only class-induction, while full T_0 includes the full scheme; the latter does not follow from the former since classes are only assumed to satisfy predicative comprehension in T_0. Informally, i(A, R) is the well-founded part of the relation R, hereditarily in A.

It is easily seen that $ID_{<\varepsilon(0)}(acc)^i$ is contained in $T_0(\text{res-IG})^i$. Moreover, T_0(res-IG) is interpretable in Δ^1_2-CA. So, by the results described in the preceding section we have

$$ID_{<\varepsilon(0)}(acc)^i \equiv T_0(\text{res-IG})^i \equiv T_0(\text{res-IG}) \equiv \sum{}^1_2\text{-AC}$$

Turning next to full T_0, what Jäger showed in his1983 paper was that by use of a primitive recursive ordering $\preccurlyeq$ of order type $\psi_\Omega(\varepsilon_{I+1})$, the well-ordering of each initial segment of the $\preccurlyeq$ relation can be established in T_0^i. I had given an (easy) interpretation of T_0 in

Δ^1_2-CA + BI. So that combined with the (much, much harder) work of Jäger and Pohlers (1983) and Jäger (1983) established

$$T_0^i \equiv T_0 \equiv \sum{}^1_2\text{-AC} + \text{BI}.$$

In analogy to the above, I conjecture that there is a suitable system $ID_{<I}(acc)^i$ in some sense that can be added to the left of these equivalences.

4. A simplified version of local predicativity. That is the title of Buchholz (1992), the next main methodological improvement in this approach. As he writes at the beginning of that paper:

> The method of local predicativity as developed by Pohlers … and extended to subsystems of set theory by Jäger … is a very powerful tool for the ordinal analysis of strong impredicative theories. But up to now it suffers considerably from the fact that it is based on a large amount of very special ordinal theoretic prerequisites. … The purpose of the present paper is to expose a simplified and conceptually improved version of local predicativity which … requires only amazingly little

> ordinal theory. … The most important feature of our new approach however seems to be its conceptual clarity and flexibility, and in particular the fact that its basic concepts (i.e. the infinitary system RS^∞ and the notion of an $\mathcal{H}$-controlled RS^∞ derivation) are in no way related to any system of ordinal notations or collapsing functions. (Buchholz 1992, p.117).

Buchholz there goes on to show how to carry out the ordinal analysis of KPI by this new method in full, absorbable detail. Thenceforth, this simplified method of local predicativity became the gold standard for admissible proof theory. It was continued by Rathjen (1994) in a revised treatment of his 1991 ordinal analysis of KPM, i.e. KP with an axiom saying that the universe is at the level of a Mahlo-admissible ordinal. As he writes (op. cit.) p. 139, KPM is "somewhat at the verge [i.e., upper margin] of admissible proof theory … Roughly speaking the central scheme of KPM falls under the heading of 'Π_2-reflection with constraints'." The first steps in moving beyond admissible proof theory to systems of analysis like Π^1_2-CA, required dealing with Π_n-reflection for arbitrary n, as discussed op. cit., pp. 142ff. For more recent progress—going far beyond our principal concerns here—see Rathjen (2006).

5. Monotone inductive definitions. Though the formal theories of generalized inductive definitions as originally proposed by Kreisel (1963) were of the form $ID_n(mon)^i$, their relationship to the systems $ID_n(acc)^i$ was left unsettled by the work of Buchholz et al. (1981), as was the relationship for the corresponding classical systems.[10] This was first taken up in my paper Feferman (1982a) for the 1981 Brouwer Centenary Symposium. I showed there that, at least on the classical side, $ID_n(mon)$ is a conservative extension of

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[10] At first sight, one could obtain a simple reduction of the ID(mon) theories to the ID(pos) theories (whether classical or intuitionistic) by an application of Lyndon's interpolation theorem to formulas of the form $A(Q,P,x) \wedge \forall u[P(u) \to P'(u)] \to A(Q,P',x)$, derived from prior axiom schemes. This was indeed stated in Sieg (1977); however, Buchholz pointed out to Sieg soon after that there is a gap in the argument, since one should allow both P and P′ to be used together in those schemes. There is no obvious way to get around this obstacle.

$ID_n(O)$ for all n. The method of proof is via an interpretation of $ID_n(mon)$ in a predicative second order extension $ID_n(O)^{(2)}$ which is easily shown to be a conservative extension of $ID_n(O)$. The main work goes into showing that if A(P, x) is an arithmetical formula such that $ID_{n-1}(O)^{(2)}$ proves the monotonicity condition $\forall X \forall Y \forall x[A(X, x) \wedge X \subseteq Y \rightarrow A(Y, x)]$ then one can define a predicate P_A in $ID_n(O)^{(2)}$ to provably satisfy the required closure and induction scheme axioms. In the same paper I also sketched how to generalize these arguments and results to the case of 'α' in place of 'n'. It follows from the work of Buchholz and Pohlers described in sec. 2 that in general $ID_\alpha(mon)$ is proof-theoretically reducible to $ID_\alpha(acc)^i$ and the proof-theoretic ordinals are the same. Incidentally, as noted by Kreisel in 1963, there is no obvious informal argument for the constructivity of $ID_1(mon)^i$ short of quantification over species in the intuitionistic sense.

At the conclusion of Feferman (1982a) I brought attention to the formulation of monotonic inductive definitions in the much more general setting of explicit mathematics. By an operation f from classes to classes, in symbols Cl-Op(f) we mean one such that $\forall X \exists Y (fX = Y)$; then by Mon(f) we mean Cl-Op(f) $\wedge$ $\forall X \forall Y[X \subseteq Y \rightarrow fX \subseteq fY]$. The assertion ELFP(f) that f has a least fixed point is expressed as $\exists X[fX \subseteq X \wedge \forall Y(fY \subseteq Y \rightarrow X \subseteq Y)]$. I suggested adding the following axiom MID for Monotone Inductive Definitions to T_0: $\forall f\,[\mathrm{Mon}(f) \rightarrow \mathrm{ELFP}(f)]$, i.e. the statement that every monotonic operation from classes to classes has a least fixed point. And finally, I raised the question whether T_0 + MID is any stronger than T_0, since as I wrote: "[it] includes all constructive formulations of the iteration of monotone inductive definitions of which I am aware, while T_0 (in its IG axiom) is based squarely on the general iteration of accessibility inductive definitions. Thus it would be of great interest for the present subject to settle the relationship between these theories." At the time I thought that my interpretation of T_0 in Σ^1_2-AC+BI could somehow be extended to one for T_0 + MID, and thus give a general reduction of monotone to accessibility inductive definitions. But as I said loc. cit., I did not succeed in doing this. In fact, it was not obvious how to produce *any* model of T_0 + MID, let alone one bounding its strength by that of T_0.

The first progress on these questions was made by my student Shuzuo Takahashi in his PhD dissertation at Stanford, published as Takahashi (1989). He proved that T_0 + MID is interpretable in $\prod^1_2$-CA + BI; this required a surprisingly difficult model construction, while no lower bound in strength was revealed by Takahashi's work. Meanwhile I had raised the question of the status of a uniform version UMID of the MID axiom, obtained by adding a constant lfp to the language of T_0 with the statement that for any f, if Mon(f) then lfp(f) is a least fixed point of f; the consistency of T_0 + UMID was unsettled by Takahashi's interpretation. These questions of strength were later addressed in a series of papers by Michael Rathjen (1996, 1998, 1999) and a joint one with Thomas Glass and Andreas Schlüter (1997), all surveyed with some further extensions in Rathjen (2002). Here, briefly, are some of the results.

First of all, it was shown in Rathjen (1996) that T_0 + MID is in fact stronger than T_0; in fact T_0(res-IG) + MID proves the existence of a model of T_0. Then in Glass, Rathjen and Schlüter (1997) it was shown that

$$T_0(\text{res-IG}) + \text{MID} \equiv (\textstyle\sum^1_2\text{-AC})^- + (\prod^1_2\text{-CA})^-, \text{ and}$$

$$T_0(\text{res-IG}) + \text{IND}_N + \text{MID} \equiv \textstyle\sum^1_2\text{-AC} + (\prod^1_2\text{-CA})^-,$$

where the minus sign superscript on a scheme indicates that there are no class parameters (i.e. free class variables). Following that, Rathjen (2002) proved that T_0 + MID is bounded in strength by a theory $\mathcal{K}$ that is slightly stronger than $\sum^1_2$-AC + $(\prod^1_2\text{-CA})^-$ + BI.

Rathjen (1999, 2002) also obtained results about the strength of UMID_N (which is the UMID principle relativized to subclasses of N), including the following:

$$T_0(\text{res-IG}) + \text{UMID}_N \equiv (\textstyle\prod^1_2\text{-CA})_0,$$

while

$$\textstyle\prod^1_2\text{-CA} < T_0 + \text{UMID}_N \le \prod^1_2\text{-CA} + \text{BI}.$$

Rathen conjectured (2002), p. 339, that the $\le$ here can be replaced by $\equiv$ and that UMID gives no stronger theory than UMID_N. Finally, it is shown there that

$$T_0 + \text{MID} < T_0 + \text{UMID}_N.$$

All these results are for the systems of explicit mathematics as based on classical logic. About the intuitionistic side of these various theories, Rathjen wrote (loc. cit.) that virtually nothing is known. However, subsequently, Sergei Tupailo (2004) established that the classical and intuitionistic versions of T_0(res-IG) + $UMID_N$ are of the same strength, by an indirect argument via the so-called μ-calculus.[11]

A number of problems about the MID and UMID principles in explicit mathematics are still left open by this work, especially on the intuitionistic side.

6. The method of functional interpretation, 1968-2008. All of the proof-theoretical analyses of classical theories of iterated inductive definitions surveyed above made use of cut-elimination arguments applied to suitable uncountably infinitary sequent-style systems. But for the purely reductive part of the problem, it seemed to me from the beginning that an extension of Gödel's method of functional interpretation could serve to establish the expected results using finite formulas throughout. In an unpublished lecture that I gave at the 1968 Buffalo conference—though circulated in mimeographed notes Feferman (1968)—I obtained a semi-constructive functional interpretation of ID_1 in the classical system $ID_1(T)$, where the set T of constructive countable tree ordinals is a variant of O. The hope was to then reduce $ID_1(T)$ to a suitable $ID_1(acc)^i$ and thereby show that the $|ID_1|$ is the Howard ordinal, but I did not see how to get around the obstacle of essential use of numerical quantification (in its guise as the non-constructive minimum operator μ) in doing so. The next attempts to approach this and the iterated case via functional interpretation were made by my student Jeffery Zucker in his dissertation (1971), the work from which was published in Zucker (1973). Interestingly, Zucker showed that $|ID_1| = |ID_1(acc)^i|$ by application of Howard's majorization technique to my functional interpretation with the μ operator. However, he did not see a way to extend this to the iterated case. What he *was* able to do was give a Kreisel-style modified realizability functional interpretation of $ID_n(acc)^i$ in a theory of constructive tree classes up to level n for each $n < \omega$ and show that they have the same provably recursive

[11] Michael Rathjen has informed me that there is an alternative more direct argument to obtain Tupailo's result via an application of the double negation translation to the operator theory $T_{<\omega}^{OP}$ of Rathjen (1998), which is of the same strength as T_0(resIG)+$UMID_N$; moreover the same method applies to $T_{<\varepsilon(0)}^{OP}$ which is of the same strength as T_0(res-IG)+IND_N+$UMID_N$ and thence of its intuitionistic version.

ordinals; he also sketched how this could be extended to transfinite α.

My notes Feferman (1968) and questions about its approach did not see the general light of day until they were outlined in sec. 9 of my survey with Jeremy Avigad in the *Handbook of Proof Theory* of Gödel's functional interpretation, Avigad and Feferman (1998); I included that section there in the hopes that someone would see how to overcome the obstacle that I had met. To my great satisfaction, that was finally achieved by Avigad with his student Henry Towsner in 2008 by a variant functional interpretation; the fact that this took place in the year of celebration of Wolfram Pohlers' retirement is the reason why I subtitled this piece a forty year long saga. Since this is relatively new and unfamiliar material, I want to sketch how the approach in Avigad and Towsner (2008) proceeds.

As background, let's look briefly at Gödel's original *Dialectica* (or D-) interpretation (1958, 1972) and its consequences; subsequent work follows a broadly similar pattern. Gödel applied the D-interpretation to Heyting Arithmetic HA to reduce it to a quantifier-free theory of primitive recursive functionals of finite type over N that he simply denoted by 'T'. This is carried out via an intermediate translation which sends each formula A of arithmetic into a formula A^D of the form $\exists z\forall x A_D(z, x)$ where z, x are sequences of variables of finite type (possibly empty) and A_D is a quantifier free formula of the language of T. The main theorem was that if HA $\vdash$ A then T $\vdash A_D(t, x)$ for some sequence t of terms of the same type as z; this gives the reduction HA $\leq$ T. A is equivalent to A^D under the assumption of the Axiom of Choice, which in this setting is constructively accepted, plus the non-constructive Markov's Principle and a principle called Independence of Premises. But the interpretation of A by A^D can be applied in combination with the double negation translation of PA into HA to show that these systems have the same provably recursive functions and that, moreover, they are the same as the functions of type 1 generated by the terms of T. For if PA $\vdash \forall x\, \exists y\, R(x, y)$ with R primitive recursive then HA $\vdash\ \forall x\, \neg\neg\exists y\, R(x, y)$ and so by Markov's Principle and the Axiom of Choice we have $\exists z\, \forall x\, R(x, z(x))$; finally, by the D-interpretation, there is a closed term of type 1 such that T $\vdash R(x, t(x))$. The set of functions of type 1 generated by the primitive recursive functionals of finite type is called the 1-section of T. So this result can be summarized by the equations

$$\text{Prov-Rec(PA)} = \text{Prov-Rec(HA)} = \text{1-sec(T)}.$$

Further work must be done if one wants to use this to recapture the result of Kreisel (1952) that the provably recursive functions of PA and HA are just those obtained by recursion on ordinals $\alpha < \varepsilon_0$. This can be obtained via the normalization of the terms of T using an assignment to them of ordinals $< \varepsilon_0$. That was was first carried out by Tait (1965) and later by Howard (1970) in ways akin to the use of ordinals $< \varepsilon_0$ in the cut-elimination arguments for PA by Schütte and Gentzen, respectively.

The details for the functional interpretation of theories of inductive definition are only given in full for ID_1 in Avigad and Towsner (2008) and sketched for arbitrary ID_n in their final section, though they say it can be extended to transfinite iterations. The first step, for a given arithmetical A(P, x), is to translate ID_1 into the classical theory OR_1 of abstract countable tree ordinals extended by axioms (I) for a predicate $I(x, \alpha)$ of natural numbers and (tree) ordinals, interpreted as $x \in I_\alpha$ in the approximations from below to the least fixed point of A. The functional interpretation is then used to obtain a reduction of OR_1 + (I) to an $ID_1(acc)^i$ via a quantifier-free theory T_Ω of primitive recursive functionals of finite type over the tree ordinals and two of its extensions, QT_Ω, which allows quantifiers over all finite type variables, and Q_0T_Ω, which allows only numerical quantification; unless otherwise indicated both are in classical logic. Avigad and Towsner show that $OR_1 + (I) \leq Q_0T_\Omega$ by the Diller-Nahm-Shoenfield variant of the D-interpretation. The problem then is to get rid of Q_0 and pass to intuitionistic logic, which was essentially the obstacle that I and Zucker had met. The novel key step is to establish the reduction $Q_0T_\Omega \leq (QT_\Omega)^i$, using an adaptation of the argument in Sieg (1981) to formalize cut-elimination for a semi-formal version of Q_0T_Ω in $(QT_\Omega)^i$. Finally, the model of T_Ω and thence of $(QT_\Omega)^i$ in the hereditarily recursive operations over the recursive countable tree ordinals may be formalized in $ID_1(O)^i$. Chaining together these successive reductions, Avigad and Towsner obtain:

$$ID_1 \leq ID_1(O)^i,\ |ID_1| = |ID_1(O)^i|, \text{ and}$$

$$\text{Prov-Rec}(ID_1) = \text{Prov-Rec}(ID_1(O)^i) = \text{1-Sec}(T_\Omega).$$

As I said, they assert that the same methods serve to establish $ID_\alpha \le ID_\alpha(O)^i$ and $|ID_\alpha| = |ID_\alpha(O)^i|$ in general; it would be good to see the details of that presented in full. But assuming that is the case, on the basis of present evidence this work of Avigad and Towsner is an improvement on both Sieg (1977, 1981), which only obtained $ID_\alpha \le ID_{\alpha+1}(O)^i$, and Buchholz (1981a), which only obtained $ID_\alpha \le ID_\alpha(acc)^i$. In addition, their functional interpretation has the advantage of giving a mathematical characterization of the provable recursive functions of a given ID theory in terms of the 1-section of a natural class of functionals. Of course, one would need to use something like the methods of local predicativity *with* ordinal analysis in order to further describe those functions in terms of suitable ordinal recursions.

7. Conclusion. All the work surveyed here illustrates how the initial aim to use the constructive reduction and ordinal analysis of theories of iterated inductive definitions for the extension of Hilbert's program to impredicative systems of analysis became transmuted into a subject of interest in its own right. In addition, the continuing desire for conceptually clear arguments led to successive methodological improvements, which in turn proved useful in other applications. Though the proof theory of iterated inductive definitions as first order systems falls far short of serving to deal with the next level of impredicative systems of analysis such as Π^1_2-CA, the work described in sec. 5 on classical and constructive theories of monotonic inductive definitions suggests that suitable second order theories of such may be useful for that purpose.

To conclude, here are some questions suggested by the work that has been surveyed above.

1. One does not have to be a devotee of purity of method to ask whether an alternative, more purely functional interpretation approach might be possible to arrive at the reduction $ID_\alpha \le ID_\alpha(O)^i$ in general. Recall that Zucker (1973) showed that the proof theoretic ordinals of ID_1 and T_Ω are the same by applying the majorization argument of Howard (1973) to the semi-constructive functional interpretation of my 1968 notes. For me, this is reminiscent of the use by Kohlenbach (1992) of his method of monotone functional interpretation to eliminate numerical quantification in the reduction of the system WKL to PRA.

So the question is whether the appeal to cut-elimination in the final step of the Avigad and Towsner work both for ID_1 and in general for ID_α can be avoided by an application of the monotone functional interpretation or one of its variants, such as the bounded functional interpretation of Ferreira and Oliva (2005). Incidentally, I was misled by the work of Avigad and Towsner (2008) into thinking that they had somehow refined Sieg's argument to replace 'α+1' by 'α' in the target system. But it seems that that was only possible in combination with their use of functional interpretation. So if a purely functional interpretation approach does not succeed to obtain a proof-theoretic reduction of ID_α to $ID_\alpha(O)$[i], it is still a question whether a refinement of Sieg's arguments using cut-elimination can achieve the same result.

2. What part of mathematics can be carried out in ID_1? A recent interesting case study is provided by Avigad and Towsner (2009) (cf. also Avigad (2009) sec. 5): a version of the structure theorem in combinatorial ergodic theory due to Furstenberg (1977) can be formalized in ID_1, via the interpretation in Q_0T_Ω+(I) described in the preceding section. That theorem was used by Furstenberg to prove by conceptually high level means the famous theorem of Szemerédi (1975), whose original combinatorial proof was very difficult. The work of Beleznay and Foreman (1996) suggests that the full Furstenberg structure theorem is equivalent to the Π^1_1 comprehension axiom. But the work of Avigad and Towsner shows that the full strength of the structure theorem is far from necessary for the ergodic-theoretic proof of the Szemerédi theorem. As this example shows, it may be that the pursuit of what other mathematics can be formalized in ID_1 is more conveniently examined in proof-theoretically equivalent systems in which ordinals play an explicit role, such as the theory OR_1 + (I) or its functional interpretations in the preceding section.

3. What about what can be done in iterated IDs?

4. Ordinal analysis only tells us something about the provably countable ordinals of a theory. In the case of the ID_αs, it would seem to make sense to talk about their

provably uncountable ordinals. How would that be defined, and what can be established about them?

5. ID_1 is similar to Peano Arithmetic in various respects. In Feferman (1996) I introduced the general notion of an open-ended schematic axiom system and its unfolding, to explain the idea of what we ought to accept if we have accepted given notions and given principles concerning them. In Feferman and Strahm (2000) we showed that the full unfolding of a very basic schematic system NFA for non-finitist arithmetic is proof-theoretically equivalent to predicative analysis. There is a natural formulation of a basic schematic system NFI which stands to ID_1 as NFA stands to PA. What is its unfolding?

6. A side development of the work on theories of iterated inductive definitions is that on theories of iterated fixed point theories ID_α^, whose basic axiom for a given A takes the form $\forall x\, [A(P_A, x) \leftrightarrow P_A(x)]$. Building on work of Aczel characterizing the strength of ID_1^, I showed in Feferman (1982b) that the union of the finitely iterated fixed point theories is equivalent in strength to predicative analysis. That work was continued into the transfinite by Jäger, Kahle, Setzer and Strahm (1999) who showed that even though one thereby goes beyond predicativity in strength, the methods of predicative proof theory can still be applied. They thus introduced the term *metapredicativity* for the study of systems that can be treated by such means. In unpublished work by Jäger and Strahm, that even goes beyond ID_1. One should try to characterize the domain of metapredicativity in terms analogous to those used at the outset to characterize predicativity as the limit of the autonomous progression of ramified systems. Assuming that, I would conjecture that the full unfolding of the schematic system NFI suggested above is proof-theoretically equivalent to the union of the metapredicative systems.

7. The set-theoretical treatment of least fixed points of monotonic operator apply to operators on subsets of arbitrary sets M. Are there reasonable theories of IDs over other sets than the natural numbers, e.g. the real numbers? What can be said about their strength?

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References

P. Aczel, H. Simmons and S. Wainer, eds. (1992), *Proof Theory*, Cambridge University Press, Cambridge.

J. Avigad (2009), The metamathematics of ergodic theory, *Annals of Pure and Applied Logic* 157, 64-76.

J. Avigad and S. Feferman (1998), Gödel's functional ("Dialectica") interpretation, in Buss (1998), 337-405.

J. Avigad and H. Towsner (2008), Functional interpretation and inductive definitions, http://arxiv.org/abs/0802.1938; to appear in *J. Symbolic Logic*.

____________________ (2009), Metastability in the Furstenberg-Zimmer tower, http://arxiv.org/abs/0902.0356 .

H. Bachmann (1950), Die Normalfunktionen und das Problem der ausgezeichneten Folgen von Ordnungzahlen, *Vierteljahresschr. Nat. Ges., Zürich* 95, 5-37.

J. Barwise (1975), *Admissible Sets and Structures*, Springer-Verlag, Berlin.

F. Beleznay and M. Foreman (1996), The complexity of the collection of measure-distal transformations, *Ergodic Theory and Dynamical Systems* 16, 929-962.

J. Bridge (1975), A simplification of the Bachmann method for generating large countable ordinals, *J. Symbolic Logic* 40, 171-185.

W. Buchholz (1975), Normalfunktionen und konstruktiven Systeme von Ordinalzahlen, in *Proof Theory Symposion, Kiel 1974,* Lecture Notes in Mathematics 500, 4-25.

__________ (1981a), The $\Omega_{\mu+1}$-rule, in Buchholz et al. (1981), 188-233.

__________ (1981b), Ordinal analysis of ID_ν, in Buchholz et al. (1981), 234-260.

__________ (1992), A simplified version of local predicativity, in Aczel et al. (1992), 115-147.

__________ (1997) Explaining Gentzen's consistency proof within infinitary proof theory, in G. Gottlob, A. Leitsch and D. Mundici (eds.), *Computational Logic and Proof Theory. KGC '97*, Lecture Notes in Computer Science 1298.

__________ (2001) Explaining the Gentzen-Takeuti reduction steps: A second-order system, *Archive for Mathematical Logic* 40,

W. Buchholz, S. Feferman, W. Pohlers and W. Sieg (1981), *Iterated Inductive Definitions and Subsystems of Analysis: Recent proof-theoretical studies*, Lecture Notes in Mathematics 897.

W. Buchholz and W. Pohlers (1978), Provable well-orderings of formal theories for transfinitely iterated inductive definitions, *J. Symbolic Logic* 43, 118-125.

S. R. Buss, ed. (1998), *Handbook of Proof Theory*, Elsevier, Amsterdam.

A. Church and S. C. Kleene (1936), Formal definitions in the theory of ordinal numbers, *Fundamenta Mathematicae* 28, 11-21.

S. Feferman (1968), Ordinals associated with theories for one inductively defined set. (Unpublished notes.)

__________ (1970) Formal theories for transfinite iteration of generalized inductive definitions and some subsystems of analysis, in Kino et al. (1970), 303-326.

__________ (1975), A language and axioms for explicit mathematics, in J. N. Crossley (ed.) *Algebra and Logic*, Lecture Notes in Mahtematics 450, 87-139.

__________ (1979), Constructive theories of functions and classes, in M. Boffa et al. (eds.) *Logic Colloquium '78*, North-Holland, Amsterdam, 159-224.

__________ (1981), How we got from there to here, in Buchholz et al. (1981), 1-15.

__________ (1982a), Iterated inductive fixed-point theories: application to Hancock's conjecture, in G. Metakides (ed.), *Patras Logic Symposion*, North-Holland, Amsterdam, 171-196.

__________ (1982b), Monotone inductive definitions, in A. S. Troelstra and D. van Dalen (eds.), *The L.E.J. Brouwer Centenary Symposium,* North-Holland, Amsterdam, 77-89.

__________ (1988), Hilbert's program relativized: proof-theoretical and foundational reductions, *J. Symbolic Logic* 53, 364-384.

__________ (1993), What rests on what? The proof-theoretical and foundational analysis of mathematics, in J. Czermak (ed.), *Philosophy of Mathematics, Vol.I,* Verlag-Hölder-Pichler-Tempsky, Vienna, 147-171. Reprinted in Feferman 1998, 187-208.

__________ (1996), Gödel's program for new axioms: Why, where, how and what?, in P. Hájek (ed.), *Gödel '96*, Lecture Notes in Logic 6, 3-22.

__________ (1998), *In the Light of Logic*, Oxford University Press, New York.

__________ (2000) Does reductive proof theory have a viable rationale?, *Erkenntnis* 53, 63-96.

S. Feferman and W. Sieg (1981a), Iterated inductive definitions and subsystems of analysis, in Buchholz et al. (1981), 16-77.

____________________ (1981b) Proof theoretic equivalences between classical and constructive theories for analysis, in Buchholz et al. (1981) 78-142.

S. Feferman and T. Strahm (2000), The unfolding of non-finitist arithmetic, *Annals of Pure and Applied Logic* 104, 75—96.

F. Ferreira and P. Oliva (2005), Bounded functional interpretation, *Annals of Pure and Applied Logic* 135, 73-112.

H. M. Friedman (1970) Iterated inductive definitions and $\sum^1{}_2$-AC, in Kino, Myhill and Vesley 1970, 435-442.

H. M. Friedman, K. McAloon, and S. G. Simpson (1982), A finite combinatorial principle which is equivalent to the 1-consistency of predicative analysis, in G. Metakides (ed.), *Patras Logic Symposion*, North-Holland, Amsterdam, 197-230.

H. Furstenberg (1977), Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, *J. d'Analyse Mathematique* 31, 204-256.

T. Glass, M. Rathjen and A. Schlüter (1997), The strength of monotone inductive definitions in explicit mathematics, *Annals of Pure and Applied Logic* 85, 1-46.

K. Gödel (1958), Über eine bisher noch nicht benüzte Erweiterung des finiten Standpunktes, *Dialectica* 12, 280-287. Reproduced with English translation in Gödel (1990), 241-251.

_______ (1972), On an extension of finitary methods which has not yet been used, in Gödel (1990), 271-280.

_______ (1990), *Collected Works,Vol. II. Publications 1938-1974* (S. Feferman, et al., eds.), Oxford University Press, New York.

A. Heyting, ed. (1959), *Constructivity in Mathematics*, North-Holland, Amsterdam.

W. A. Howard (1970), Assignment of ordinals to terms for primitive recursive functionals of finite type, in Kino et al. (1970), 453-468.

____________ (1972), A system of abstract constructive ordinals, *J. Symbolic Logic* 37, 355-374.

____________ (1973), Hereditarily majorizable functionals of finite type, in Troelstra (1973), 454-461.

D. Isles (1970), Regular ordinals and normal forms, in Kino et al. (1970), 339-361.

G. Jäger (1979), *Die Konstruktible Hierarchie als Hilfsmittel zur beweistheoretischen Untersuchung von Teilsystemen der Mengenlehre und Analysis,* Dissertation, Ludwig-Maximilians-Universität, Munich.

______ (1980), Beweistheorie von KPN, *Archiv für Mathematische Logik und Grundlagenforschung* 20,

______ (1983), A well-ordering proof for Feferman's theoryT_0, *Archive for Mathematical Logic* 23, 65-77.

______ (1986), *Theories for Admissible Sets. A unifying approach to proof theory*, Bibliopolis, Naples.

G. Jäger, R. Kahle, A. Setzer and T. Strahm (1999), The proof-theoretic analysis of transfinitely iterated fixed point theories, *J. Symbolic Logic* 64, pp. 53–67.

G. Jäger and W. Pohlers (1983), Eine beweistheoretische Untersuchung von (Δ^1_2-CA) + (BI) und verwandter Systeme, *Bayerische Akademie der Wissenschaften*, Sitzungsberichte 1982, 1-28.

A. Kino, J. Myhill and R. Vesley, eds. (1970), *Intuitionism and Proof Theory*, North-Holland, Amsterdam.

S. C. Kleene (1938), On notation for ordinal numbers, *J. Symbolic Logic* 3, 150-155.

__________ (1959), Countable functionals, in Heyting (1959), 81-100.

U. Kohlenbach (1992), Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization, *J. Symbolic Logic* 57, 1239-1273.

G. Kreisel (1952), On the interpretation of non-finitist proofs, part II: interpretation of number theory, applications, *J. Symbolic Logic*17, 43-58.

________ (1959) Interpretation of analysis by means of constructive functionals of finite type, in Heyting (1959), 101-128.

________ (1963), Generalized inductive definitions, in *Seminar on the Foundations of Analysis, Stanford 1963. Reports* (mimeographed), Mathematical Sciences Library, Stanford University, 3.1-3.25.

P. Martin-Löf (1971), Hauptsatz for the intuitionistic theory of iterated inductive definitions, in J. E. Fenstad (ed.), *Proceedings of the Second Scandinavian Logic Symposium,* North-Holland, Amsterdam, 179-216.

H. Pfeiffer (1964), *Ausgezeichnete Folgen für gewisse Abschnitte der zweiten und weiteren Zahlklassen*, Dissertation, Technische Hochschule Hannover.

W. Pohlers (1975), An upper bound for the provability of transfinite induction in systems with N-times iterated inductive definitions, in J. Diller and G. H. Müller (eds.), *ISILC Proof Theory Symposium*, Lecture Notes in Mathematics 500, 271-289.

_________ (1977), *Beweistheorie der iterierten induktiven Definitionen,* Habilitationsschrift, Ludwig-Maximilians-Universität, Munich.

_________ (1981) Proof-theoretical analysis of ID_ν by the method of local predicativity, in Buchholz et al. (1981), 261-357.

_________ (1987), Contributions of the Schütte school in Munich to proof theory, in Takeuti (1987), 406-431.

_________ (1989) *Proof Theory. An introduction*, Lecture Notes in Mathematics, v. 1407.

_________ (1992) A short course in ordinal analysis, in Aczel et al. (1992), 27-78.

_________ (1998) Subsystems of set theory and second order number theory, in Buss (1998), 209-335.

_________ (2009) *Proof Theory. The first step into impredicativity*, Springer-Verlag, Berlin.

M. Rathjen (1988), *Untersuchungen zu Teilsystemen der Zahlentheorie zweiter Stufe und der Mengenlehre mit einer zwischen* Δ^1_2-CA *und* Δ^1_2-CA+BI *liegenden Beweisstärke*, Doctoral thesis, University of Münster.

_________ (1994), Admissible proof theory and beyond, in D. Prawitz, B. Skyrms, and D. Westerstahl (eds.), *Logic, Methodology and Philosophy of Science IX*, Elsevier, Amsterdam.

_________ (1996), Monotone inductive definitions in explicit mathematics, *J. Symbolic Logic* 61, 125-146

_________ (1998), Explicit mathematics with the monotone fixed point principle, *J. Symbolic Logic* 63, 509-542.

_________ (1999), Explicit mathematics with the monotone fixed point principle. II: Models, *J. Symbolic Logic* 64, 517-550.

_________ (2002), Explicit mathematics with monotone inductive definitions: A survey, in Sieg, et al. (2002), 329-346.

_________ (2006), The art of ordinal analysis, in *Proceedings of the International Congress of Mathematicians*, Vol. II, European Mathematical Society, 45-69.

K. Schütte (1977), *Proof Theory*, Springer-Verlag, Berlin.

W. Sieg (1977), *Trees in Metamathematics. Theories of inductive definitions and subsystems of analysis,* PhD Dissertation, Stanford University.

_______ (1981), Inductive definitions, constructive ordinals, and normal derivations, in Buchholz et al. (1981), 143-187.

W. Sieg, R. Sommer and C. Talcott, eds. (2002), *Reflections on the Foundations of Mathematics. Essays in honor of Solomon Feferman,* Lecture Notes in Logic 15, Assoc. for Symbolic Logic, A.K. Peters, Ltd., Natick.

C. Spector (1962), Provably recursive functions of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics, in J. C. E. Dekker (ed.), *Recursive Function Theory. Proc. Symposia in Pure Mathematics* 5, AMS, Rhode Island, 1-27.

E. Szemerédi (1975), On sets of integers containing no k elements in arithmetic progression, *Acta Arithmetica* 27, 199-245.

W. W. Tait (1965), Infinitely long terms of transfinite type, in J. N. Crossley and M. A. E. Dummett (eds.), *Formal Systems and Recursive Functions*, North-Holland, Amsterdam, 176-185.

_________ (1970) Applications of the cut elimination theorem to some subsystems of classical analysis, in Kino et al. (1970), 475-488.

S. Takahashi (1989), Monotone inductive definitions in a constructive theory of functions and classes, *Annals of Pure and Applied Logic* 42, 255-279.

G. Takeuti (1967), Consistency proofs of subsystems of classical analysis, *Annals of Mathematics* 86, 299-348.

_________ (1987), *Proof Theory*, North-Holland, Amsterdam, 2nd edn, (with appendices by G. Kreisel, W. Pohlers, S. Simpson and S. Feferman).

G. Takeuti and M. Yasugi (1973), The ordinals of the system of second order arithmetic with the provably Δ^1_2-comprehension axiom and with the Δ^1_2-comprehension axiom respectively, *Japanese J. of Mathematics* 41, 1-67.

A. S. Troelstra, ed. (1973), *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis,* Lecture Notes in Mathematics 344;

____________, ed. (1993), Second corrected edition of Troelstra (1973), Institute for Logic, Language and Computation, Amsterdam, Technical Notes X-1993-05.

S. Tupailo (2004), On the intuitionistic strength of monotone inductive definitions, *J. Symbolic Logic* 69, 790-798.

J. I. Zucker (1971), *Proof Theoretic Studies of Systems of Iterated Inductive Definitions and Subsystems of Analysis,* PhD Dissertation, Stanford University.

_________(1973), Iterated inductive definitions, trees and ordinals, in Troelstra (1973) 392-453; minor additions and corrections in Troelstra (1993).

Stanford University

feferman@stanford.edu