Structure near the cusps

1. The setup

We fix a positive integer N and for any $z \in \mathbf{P}^1(\mathbf{Q}) \cup \mathfrak{h}$ we let the point $[z] \in X_0(N)$ denote the $\Gamma_0(N)$-orbit of z. In case $[z]$ is a cusp, we want to analyze the nature of isomorphism classes of pairs (E, C_N) which are classified by points on $Y_0(N)$ which are "near" $[z]$. Let's first consider what happens for $z = \infty$, a punctured neighborhood of which is represented by

$$\{z \in \mathfrak{h} \,|\, \mathrm{Im}(z) > M\}/(z \sim z+1).$$

Via $e^{2\pi i(\cdot)}$ this is analytically identified with the punctured disc $\{q \in \mathbf{C}^\times \,|\, 0 < |q| < e^{-2\pi M}\}$.

We know that distinct points in such a punctured disc correspond to distinct pairs (E, C_N) up to isomorphism, but these can be described in a manner which is analytically "nice" in q. Namely, the exponential map $e^{2\pi i(\cdot)}$ identifies $\mathbf{C}/[1, z]$ with $\mathbf{C}^\times/q^{\mathbf{Z}}$ and this carries the subgroup $\langle 1/N \rangle$ over to the subgroup μ_N of Nth roots of unity in $\mathbf{C}^\times$. For $0 < |q| \ll 1$, we conclude that

$$(\mathbf{C}^\times/q^{\mathbf{Z}}, \mu_N)$$

provides a "nice" formula which describes isomorphism class representatives for the structures classified on $Y_0(N)$ by points near ∞.

In order to make this description more precise, it is convenient at this point to introduce the language of relative Tate curves.

Lemma 1.1. *Let $\Delta^* = \{q \in \mathbf{C}^\times \,|\, 0 < |q| < 1\}$. The natural map of analytic Δ^*-groups*

$$\mathbf{Z} \times \Delta^* \hookrightarrow \mathbf{C}^\times \times \Delta^*$$

defined by $(n, q) \mapsto (q^n, q)$ is a closed immersion for which the quotient map

$$\mathbf{C}^\times \times \Delta^* \twoheadrightarrow (\mathbf{C}^\times \times \Delta^*)/(\mathbf{Z} \times \Delta^*)$$

is a covering map. The quotient is proper over Δ^* and admits a unique analytic structure with respect to which the quotient map is analytic. As such, this analytic quotient is an analytic Δ^*-group whose fiber over $q \in \Delta^*$ is $\mathbf{C}^\times/q^{\mathbf{Z}}$ with its usual analytic group structure.

Proof. This situation is essentially that obtained by applying $e^{2\pi i(\cdot)}$ to the more classical Weierstrass picture of

$$\mathbf{Z}^2 \times \mathfrak{h} \hookrightarrow \mathbf{C} \times \mathfrak{h}$$

defined by $(m, n; z) \mapsto (mz + n; z)$. The C^∞-isomorphism $\mathbf{R}^2 \times \mathfrak{h} \simeq \mathbf{C} \times \mathfrak{h}$ defined by $(x + iy; z) \mapsto (x + zy; z)$ carries the "twisted" embedding of $\mathbf{Z}^2 \times \mathfrak{h}$ into $\mathbf{C} \times \mathfrak{h}$ over to the "constant" embedding as $\mathbf{Z}[i] \times \mathfrak{h}$. This makes the topological picture obvious, and since the "action" of $\mathbf{Z} \times \Delta^*$ on $\mathbf{C}^\times \times \Delta^*$ is analytic, the descent of the analytic Δ^*-group structure through the covering map is straightfoward to check.

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We call the quotient constructed in the preceding lemma the *standard Tate family*

$$\underline{\mathrm{Tate}}_1 \to \Delta^*.$$

This is a genuine analytic morphism which "interpolates" the classical Tate construction $\mathbf{C}^\times/q^{\mathbf{Z}}$ for *varying* $q \in \Delta^*$. If we pull back along the finite covering map $\Delta^* \to \Delta^*$ defined by $q \mapsto q^M$, we get another such analytic Δ^*-group

$$\underline{\mathrm{Tate}}_M \to \Delta^*.$$

This has analytic fiber group $\mathbf{C}^\times/q^{M\mathbf{Z}}$ over $q \in \Delta^*$. It can also be directly constructed by using $\mathbf{Z} \times \Delta^* \hookrightarrow \mathbf{C}^\times \times \Delta^*$ defined by $(n, q) \mapsto (q^{nM}, q)$ and arguing as in the proof of the preceding lemma.

In any case, we see that for all positive integers m there are natural closed immersions of Δ^*-groups

$$\mu_m \times \Delta^* \hookrightarrow \underline{\mathrm{Tate}}_M$$

compatible with the finite covering maps $\underline{\mathrm{Tate}}_{M'} \to \underline{\mathrm{Tate}}_M$ for $M|M'$, and there are canonical short exact sequences over Δ^* given by

$$0 \to \mu_m \times \Delta^* \to \underline{\mathrm{Tate}}_M[m] \to (\mathbf{Z}/m) \times \Delta^* \to 0,$$

inducing the classical short exact sequence for m-torsion of $\mathbf{C}^\times/q^{M\mathbf{Z}}$ on fibers. More importantly, there is a canonical section

$$q : \Delta^* \to \underline{\mathrm{Tate}}_m^*[m]$$

defined by $q \mapsto (q \bmod q^{m\mathbf{Z}}, q)$, and this splits the exact sequence for m-torsion on $\underline{\mathrm{Tate}}_m$.

A more sophisticated formulation of how we described the structures classified on $Y_0(N)$ near $[\infty]$ is to say that if we restrict

$$(\underline{\mathrm{Tate}}_1, \mu_N \times \Delta^*) \to \Delta^*$$

over a small punctured disc Δ_r^* of radius $r \ll 1$ around the origin, then Δ_r^* may be identified with a punctured neighborhood of $[\infty] \in Y_0(N)$ in such a manner that the fiber $(\mathbf{C}^\times/q^{\mathbf{Z}}, \mu_N)$ of our Tate family over $q \in \Delta_r^*$ is a representative for the isomorphism class which is classified by the point q when viewed in $\Delta_r^* \hookrightarrow Y_0(N)$.

What is the analogous picture near the other cusps? No such picture is described by Gross-Zagier, but it is actually essential in order to make an accurate translation between cusps on arithmetic and analytic models of modular curves. Before we begin, we note that the description given in Gross-Zagier is incorrect. For a reduced form fraction m/n with $d = \gcd(n, N)$, they define $f_d = \gcd(d, N/d)$ and consider the element $m(n/d)^{-1} \bmod f_d$, upon noting that $n/d \in (\mathbf{Z}/f_d)^\times$. It is claimed that this construction is $\Gamma_0(N)$-invariant and leads to a (not very meaningful) description of the cusps on $X_0(N)$ as a bare set. However, for

$$\gamma = \begin{pmatrix} x & y \\ Nu & v \end{pmatrix} \in \Gamma_0(N)$$

consider the reduced form fraction

$$\gamma(m/n) = (xm + yn)/(Num + vn)$$

for which the equality $\gcd(Num + vn, N) = \gcd(vn, N) = d$ holds (so at least d and f_d are $\Gamma_0(N)$-invariant quantities).

Clearly

$$(Num + vn)/d \equiv v(n/d) \bmod f_d$$

and

$$xm + yn \equiv xm \bmod f_d,$$

so applying the same Gross-Zagier construction to $\gamma(m/n)$ yields $xv^{-1}m(n/d)^{-1} \bmod f_d$. Hence, if this construction were going to be well-defined, we'd need $xv^{-1} \equiv 1 \bmod f_d$. But in fact $xv - yNu = 1$, so $xv \equiv 1 \bmod f_d$, whence we encounter problems in general if we follow Gross-Zagier here.

Such problems would go away if we had worked with $(m(n/d))^{-1} \bmod f_d$, and one of our aims here is to explain via geometry where such (corrected) formulas "come from". The same method can be used to conceptually classify the cusps on other standard modular curves. We again emphasize that the big advantage of our geometric approach is that when we use the arithmetic theory, we can detect conceptually which arithmetic cusps correspond to which analytic cusps in terms of our explicit upper half-plane models. Without this sort of method, it is baffling to me how one would perceive (with any sense of rigor) the dictionary between analytic and arithmetic cusps.

We must give a word of warning at this point concerning what will happen at other cusps. In certain favorable cases, we will be able to write down a "universal family" over a punctured disc around a cusp. But such descriptions typically require specifying a generator of the cyclic subgroup, and this cannot always be done (essentially it only happens when the map $X_1(N) \to X_0(N)$ is somewhere unramified over the cusp in question). For squarefree N we will be in good shape at all cusps, but otherwise we will have to deal with some mild complications.

2. Relative Tate curve calculations

In order to streamline the subsequent discussion, it will be helpful to first give some examples of what we seek to understand, and at the end it will be shown that the examples we give here constitute "all" of the possible examples.

Definition 2.1. For any positive integer r, we let $\zeta_r = e^{2\pi i/r}$.

To get started, we choose $d|N$ and define $f_d = \gcd(d, N/d)$. Choose a unit $u \in (\mathbf{Z}/f_d)^\times$ and fix $u' \in \mathbf{Z}/N$ lifting u with $u' \bmod d$ a unit. The section q^{1/f_d} of $\underline{\mathrm{Tate}}_{N/df_d} \to \Delta^*$ is not globally well-defined, but the Δ^*-subgroup

$$\langle \zeta_N^{u'} q^{1/f_d} \rangle \hookrightarrow \mathrm{Tate}_{N/df_d} \tag{1}$$

is well-defined. Indeed, for any $q \in \Delta^*$ and any f_dth root q^{1/f_d} of q, since N/f_d is divisible by d we compute

$$(\zeta_N^{u'} q^{1/f_d})^{N/f_d} = \zeta_{f_d}^{u} q^{(d/f_d)(N/df_d)} \equiv \zeta_{f_d}^{u} \bmod q^{(N/df_d)\mathbf{Z}},$$

so this subgroup contains μ_{f_d}. Thus, the choice of f_dth root really doesn't matter: we get a well-defined Δ^*-sub*group*.

It may seem that this formula was pulled out of thin air, but we'll see in the next section how one can be naturally led to such families. It simplifies the exposition to explicate these examples before we launch into the geometric analysis near cusps on $X_0(N)$.

Of course, when $f_d = 1$ then the subgroup in (1) even has a global generator given by the section $\zeta_N^{u'} q$. In general this doesn't happen, but that's OK. Observe that the Δ^*-subgroup we've constructed is cyclic of order N on fibers. Indeed, to see that

$$\zeta_N^{u'} q^{1/f_d} \in \mathbf{C}^\times / q^{(N/df_d)\mathbf{Z}}$$

has exact order N, it suffices to raise to the (N/d)th power and to check that $\zeta_d^{u'}$ has exact order d. But recall the requirement that $u' \bmod d \in \mathbf{Z}/d$ should be a *unit*. That condition takes care of what we need. We conclude that the structure

$$(\underline{\mathrm{Tate}}_{N/df_d}, \langle \zeta_N^{u'} q^{1/f_d} \rangle) \to \Delta^*$$

is an analytic elliptic curve over Δ^* endowed with a relative cyclic subgroup of exact order N.

Let's show that, up to Δ^*-isomorphism, this data does not depend on the choice of lifting $u' \in \mathbf{Z}/N$ of $u \in (\mathbf{Z}/f_d)^\times$ (subject to the condition $u' \bmod d$ is a unit in $(\mathbf{Z}/d)^\times$). We first give a more workable description of the data with which we're working. Consider the elliptic curve $\underline{\mathrm{Tate}}_{N/d} \to \Delta^\times$ endowed with the section $\zeta_N^{u'} q$. For $\zeta \in \mu_{f_d}$, the map

$$\mathbf{C}^\times \times \Delta^* \to \mathbf{C}^\times \times \Delta^*$$

defined by $(t, q) \mapsto (t, \zeta q)$ covers the action $q \mapsto \zeta q$ on Δ^* and although it moves the section $\zeta_N^{u'} q$ to the section $\zeta_N^{u'} \zeta q$, the Δ^*-subgroup $\langle \zeta_N^{u'} q \rangle$ contains

$$(\zeta_N^{u'} q)^{N/d} = \zeta_d^{u'} q^{N/d} \equiv \zeta_d^{u'} \bmod q^{(N/d)\mathbf{Z}},$$

so since $u' \bmod d$ is a unit it follows that $\mu_d \times \Delta^*$ lies in this Δ^*-subgroup and hence our action of μ_{f_d} on $\mathbf{C}^\times \times \Delta^*$ covering its standard action on Δ^* allows us to descent everything throught the quotient map $\Delta^* \to \Delta^*$ defined by $q \mapsto q^{f_d}$. The descended quotient is exactly the structure

$$(\underline{\mathrm{Tate}}_{N/df_d}, \langle \zeta_N^{u'} q^{1/f_d} \rangle)$$

considered already.

What happens if we change the choice of $u' \in \mathbf{Z}/N$ lifting $u \in (\mathbf{Z}/f_d)^\times$? For any (N/d)th root of unity $\zeta = \zeta_N^{ds}$ we have

$$\zeta_N^{u'} q = \zeta_N^{u' - ds}(\zeta q),$$

and the action $(t, q) \mapsto (t, \zeta q)$ on $\mathbf{C}^\times \times \Delta^*$ preserves the "lattice"

$$\{(q^{N/d}, q) \,|\, q \in \Delta^*\}$$

defining the quotient $\underline{\mathrm{Tate}}_{N/d}$, so we see that the pullback of $\underline{\mathrm{Tate}}_{N/d} \to \Delta^*$ by the automorphism $q \mapsto \zeta q$ on the base Δ^* is isomorphic to $\underline{\mathrm{Tate}}_{N/d} \to \Delta^*$ in a manner which carries $\zeta_N^{u'} q$ back to $\zeta_N^{u'-ds} q$. Of course $u' - ds \in \mathbf{Z}/N$ is still the same as u' modulo d (in particular, it is a unit mod d), but as we run through all such ζ's we get all other choices of u' which lift the same unit modulo d. Moreover, this all clearly commutes with the μ_{f_d}-action whose quotient defines the actual structure over Δ^* in which we are most interested (using q^{1/f_d}). To summarize, up to reparameterization ($q \leftrightarrow \zeta q$) on the base disc our construction (up to isomorphism) depends only on our lift of u to a unit in $(\mathbf{Z}/d)^\times$.

But even the lifting to $(\mathbf{Z}/d)^\times$ doesn't matter (i.e., only the original choice of $u \in (\mathbf{Z}/f_d)^\times$ matters). Indeed, since it is just the *subgroup* rather than the specific generator in which we are interested, we can consider other generators $(\zeta_N^{u'} q)^r$ for $r \in (\mathbf{Z}/N)^\times$ with $r \equiv 1 \bmod N/d$ (so $q^r \equiv q \bmod q^{(N/d)\mathbf{Z}}$). The resulting generator $\zeta_N^{u'r} q \in \mathbf{C}^\times / q^{(N/d)\mathbf{Z}}$ has root of unity coefficient with exponent $u'r \in \mathbf{Z}/N$ whose image modulo d is $u'r \bmod d$ with $u' \in (\mathbf{Z}/d)^\times$ a unit and $r \bmod d$ running through all unit values modulo $f_d = \gcd(d, N/d)$ since the only constraint on r is $r \equiv 1 \bmod N/d$. We conclude that, up to reparameterization $q \leftrightarrow \zeta q$ on the base disc, the Tate family example we can construct depends (up to isomorphism) only on the choice of $d|N$ and $u \in (\mathbf{Z}/f_d)^\times$.

3. Local cusp calculations

Since we have already described the relative picture near $[\infty] \in X_0(N)$, in order to carry this over to the other cusps, let's now choose $m/n \in \mathbf{Q}$ a reduced-form fraction (we even allow for the possibility that $[m/n] = [\infty] \in X_0(N)$). This choice will be fixed for the rest of this write-up. Choose a matrix

$$\gamma = \begin{pmatrix} m & m' \\ n & n' \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

such that $\gamma(\infty) = m/n$, so we have the relationship

$$\gamma^{-1} \mathrm{Stab}_{m/n}(\Gamma_0(N)) \gamma = \mathrm{Stab}_\infty(\gamma^{-1} \Gamma_0(N) \gamma).$$

Since $\gamma^{-1}\Gamma_0(N)\gamma$ is a subgroup of $\mathrm{SL}_2(\mathbf{Z})$, its stabilizer at ∞ is a subgroup of

$$\mathrm{Stab}_\infty(\mathrm{SL}_2(\mathbf{Z})) = \{\pm 1\} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\mathbf{Z}}.$$

Thus, there is a unique natural number d', a priori depending on m/n and γ, such that $\mathrm{Stab}_\infty(\gamma^{-1}\Gamma_0(N)\gamma)$ is equal to one of the following distinct types of groups:

$$\begin{pmatrix} 1 & d' \\ 0 & 1 \end{pmatrix}^{\mathbf{Z}}, \quad \begin{pmatrix} -1 & d' \\ 0 & -1 \end{pmatrix}^{\mathbf{Z}}, \quad \{\pm 1\} \cdot \begin{pmatrix} 1 & d' \\ 0 & 1 \end{pmatrix}^{\mathbf{Z}}.$$

These all have the same image in $\mathrm{PSL}_2(\mathbf{Z})$. Since unipotence is a conjugation-invariant property of a matrix, and cyclicity is a conjugation-invariant property of a subgroup, we see that the above three types of cases are independent of the choice of γ and d' is independent of γ and is even unaffected by left multiplication on γ by $\Gamma_0(N)$. Thus, d' is intrinsic to $[m/n] \in X_0(N)$. In fact, since the analytic map $j : X_0(N) \to \mathbf{P}^1$ defined by the classical j-function is just the composite of the natural map $X_0(N) \to X_0(1)$ followed by the j-map $j : X_0(1) \to \mathbf{P}^1$ which is unramified at ∞, we see that d' is intrinsic because it is the ramification degree at $[m/n] \in X_0(N)$ for the map $j : X_0(N) \to \mathbf{P}^1$.

Note that right now we have not explained how to directly compute d' in terms of m, n, N. We begin by using "moduli" reasoning to prove the following lemma:

Lemma 3.1. *With notation as above, let $d = \gcd(N, n)$. Then $d' = (N/d)/f_d$ where $f_d = \gcd(d, N/d)$.*

From this lemma, we see that the simplest situation is when $f_d = 1$. When N is squarefree this always happens.

Proof. For z's "near" m/n in $\mathfrak{h}$, we have the associated data $(\mathbf{C}/[1,z],\langle 1/N\rangle)$, with two such pairs isomorphic if and only if the corresponding z's are in the same orbit for $\mathrm{Stab}_{m/n}(\Gamma_0(N))$. Using the coordinates given by γ^{-1}, we see that the formula $(\mathbf{C}/[1,\gamma(z)],\langle 1/N\rangle)$ equally well describes such data in terms of which "nearness to $[m/n]$" in $X_0(N)$ becomes "nearness to ∞" in some punctured disc

$$\{\mathrm{Im}(z)>M\}/(z\sim z+d').$$

More specifically, due to how d' was defined, and using the bijection between $\Gamma_0(N)\setminus\mathfrak{h}$ and isomorphism classes of pairs (E,C_N), to study the moduli data near $[m/n]\in X_0(N)$ is to study the data

$$(\mathbf{C}/[1,\gamma(z)],\langle 1/N\rangle)\overset{(nz+n')}{\simeq}(\mathbf{C}/[1,z],\langle (nz+n')/N\rangle)$$

for $\mathrm{Im}(z)\gg 0$, with $z\sim z+d'$.

Since $d=\gcd(n,N)$, we can write

$$n=dn_0,\quad N=dN_0,$$

where $\gcd(n_0,N_0)=1$. By definition of d', for z with large imaginary part and $r\in\mathbf{Z}$ there exists an isomorphism

$$(\mathbf{C}/[1,z+r],\langle (n(z+r)+n')/N\rangle)\simeq(\mathbf{C}/[1,z],\langle (nz+n')/N\rangle)$$

if and only if $r\in d'\mathbf{Z}$. We'll find another way to classify such r's, and this will lead to the determination of d' as advertised.

Since the lattices $[1,z+r]$ and $[1,z]$ coincide for $r\in\mathbf{Z}$ and, for $\mathrm{Im}(z)>1$, do not admit endomorphisms by $\mathbf{Z}[\zeta_3]$ or $\mathbf{Z}[\zeta_4]$, it follows that the only *iso*morphisms $\mathbf{C}/[1,z+r]\simeq\mathbf{C}/[1,z]$ are ±1. Since subgroups are invariant under inversion, we conclude

$$r\in d'\mathbf{Z}\Leftrightarrow\left\langle\frac{n(z+r)+n'}{N}\right\rangle=\left\langle\frac{nz+n'}{N}\right\rangle \bmod [1,z]\Leftrightarrow\frac{nr}{N}\in\left\langle\frac{nz+n'}{N}\right\rangle \bmod [1,z].$$

This latter condition says exactly $nr\equiv a(nz+n') \bmod [N,Nz]$ for some $a\in\mathbf{Z}$. That is, we have

$$nr\equiv au' \bmod N,\quad an\equiv 0 \bmod N$$

for some $a\in\mathbf{Z}$.

Since $d=\gcd(n,N)$ and $n=dn_0$ with $\gcd(n_0,N/d)=1$, the condition $an\equiv 0 \bmod N$ says

$$a\equiv 0 \bmod N/d,$$

so the condition on r becomes exactly $nr\in\langle n'\cdot N/d\rangle \bmod N$. That is, $nr=(N/d)r'$ with $r'\in\langle n'\rangle \bmod d$, yet the condition

$$\begin{pmatrix} m & m'\\ n & n'\end{pmatrix}\in\mathrm{SL}_2(\mathbf{Z}) \tag{2}$$

forces $\gcd(n',n)=1$, so $\gcd(n',d)=1$. This implies that n' generates $\mathbf{Z}/d$, so $r\in d'\mathbf{Z}$ if and only if

$$nr\equiv 0 \bmod N/d.$$

Since $n=dn_0$ with $\gcd(n_0,N/d)=1$, we obtain that $r\in d'\mathbf{Z}$ if and only if $r\equiv 0 \bmod (N/d)/f_d$, where $f_d=\gcd(d,N/d)$. This yields the desired formula for d'. ■

With the computation of d' settled, we now turn to consider the data

$$(\mathbf{C}/[1,z],\langle (nz+n')/N\rangle)=(\mathbf{C}/[1,z],\langle n_0z/(N/d)+n'/N\rangle).$$

In order to encode the equivalence relation $z\sim z+d'$ with $d'=(N/d)/f_d$, it will turn out to be more convenient to first mod out by $z\sim z+N/d$ and then to deal with the extra factor of f_d. We apply $e^{2\pi i(\cdot)}$ to our situation, so $q'=e^{2\pi iz/(N/d)}$ is an f_dth root of a parameter q on $X_0(N)$ near $[m/n]$ (for $|q'|\ll 1$), in terms of which the data for points on $X_0(N)$ near $[m/n]$ looks like

$$(\mathbf{C}^\times/q'^{(N/d)\mathbf{Z}},\langle\zeta_N^{n'}q'^{n_0}\rangle),$$

where q' is (locally) an f_dth root of a parameter in a small punctured disc Δ_r^* around $[m/n]$. Recall also that n' satisfies $n' \equiv m^{-1} \bmod n$ due to (2).

Since $n_0 = n/\gcd(n,N)$ is relatively prime to $N/\gcd(n,N) = N/d$, we can find $a \in \mathbf{Z}$ with

$$a \equiv n_0^{-1} \bmod N/d$$

and $a \in (\mathbf{Z}/N)^\times$. Thus, we can replace the indicated subgroup generator $\zeta_N^{n'} q'^{n_0}$ with its ath power, and this is multiplicatively congruent to $\zeta_N^{n'a} q'$ modulo $q'^{(N/d)\mathbf{Z}}$. We have now normalized our desciption to involve a subgroup generator which is a root of unity times q'. Thus, we have shown that we can identify a small punctured neighborhood of $[m/n] \in X_0(N)$ with a punctured disc Δ_r^* such that the restriction of the family

$$(\underline{\mathrm{Tate}}_{N/df_d}, \langle \zeta_N^{n'a} q^{1/f_d} \rangle) \to \Delta^*$$

over Δ_r^* describes (via its fibers) exactly the structure classified by points in $X_0(N)$ near $[m/n]$.

Our earlier study of relative Tate curve families made explicit that this latter construction in fact only depends (up to reparameterization of the base disc, reselection of the "local generator" of the subgroup, and isomorphism) on $n'a \bmod f_d$. But $a \bmod N/d = n_0^{-1} \bmod N/d = (n/d)^{-1} \bmod N/d$, as we have already seen, and $n' \equiv m^{-1} \bmod d$ since $n'm \equiv 1 \bmod n$. Thus, modulo f_d we have

$$n'a \equiv m^{-1}(n/d)^{-1} \bmod f_d.$$

It follows that the structures classified by a neighborhood of $[m/n] \in X_0(N)$ arise from our Tate construction for the divisor $d = \gcd(n,N)$ of N and the unit $u = m^{-1}(n/d)^{-1} \in (\mathbf{Z}/f_d)^\times$ (with $f_d = \gcd(d, N/d)$). We have already noted that both $d = \gcd(n,N)$ and $m^{-1}(n/d)^{-1} \in (\mathbf{Z}/f_d)^\times$ depend only on the $\Gamma_0(N)$-orbit of $m/n \in \mathbf{P}^1(\mathbf{Q})$ (where $\infty = 1/0$, with $d = N$ and $u = 1$), and since the points of $Y_0(N)$ are in bijection with isomorphism classes of pairs (E, C_N), distinct cusps on $X_0(N)$ must give rise to distinct pairs (d,u). Our method of analysis on $Y_0(N)$ shows that the evident set-theoretic maps

$$\Delta_r^* \to Y_0(N)$$

are in fact analytic isomorphisms onto punctured neighborhoods of cusps.

This makes the description

$$\mathrm{cusps}(X_0(N)) = \coprod_{d|N} (\mathbf{Z}/f_d)^\times$$

both explicit and geometric, and "explains" the recipe $m/n \mapsto (m \cdot (n/d))^{-1} \bmod f_d$. For a cusp $x \in X_0(N)$ giving rise to parameters $d|N$ and $u \in (\mathbf{Z}/f_d)^\times$, and $u' \in \mathbf{Z}/N$ a fixed lift of u which is a unit modulo d, the structures parameterized by $Y_0(N)$ near x are given by the fibers of the structure

$$(\underline{\mathrm{Tate}}_{N/df_d}, \langle \zeta_N^{u'} q^{1/f_d} \rangle) \to \Delta^* \tag{3}$$

over $q \in \Delta^*$ with $0 < |q| \ll 1$.

4. The translation to the arithmetic case

We only make some brief motivating remarks here. Consider the situation at a cusp $x \in X_0(N)$ with associated paraemters $d|N$ and $u \in (\mathbf{Z}/f_d)^\times$. The "universal family" on $Y_0(N)$ near x is described by (3), which depends on a choice of primitive Nth root of unity ζ_N. Changing this choice amounts to changing the choice of u', and the only condition we need to have an isomorphic structure is that we not change u' modulo f_d.

It is essentially for this reason that the point corresponding to x on the arithmetic model $X_0(N)_{/\mathbf{Q}}$ has residue field $\mathbf{Q}(x)$ which is a priori a subfield of $\mathbf{Q}(\mu_N)$ but with $\mathrm{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q}(x)) = \mathrm{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q}(\mu_{f_d}))$, so in fact $\mathbf{Q}(x) = \mathbf{Q}(\mu_{f_d})$. For example, when N is squarefree then all cusps on $X_0(N)_{/\mathbf{Q}}$ are $\mathbf{Q}$-rational points. Completely justifying our suggestive argument (and extending it to the situation over $\mathbf{Z}$ for arbitrary N) requires using the arithmetic theory of the Tate curve, and in particular the "integral" version of this theory over $\mathbf{Z}[\![q]\!]$ where q is *not* inverted (which is analogous to carrying out the non-trivial task of doing our preceding analytic work without removing the origin from discs).