FINITENESS OF CLASS NUMBERS FOR ALGEBRAIC GROUPS

BRIAN CONRAD

1. Introduction

Let G be a smooth affine group scheme over a global field F. We do *not* assume G to be connected. Let $\mathbf{A}_F$ denote the locally compact adele ring of F, and let S denote a fixed non-empty set of places of F which is assumed to contain the archimedean places in the characteristic 0 case.

Let $\mathbf{A}_F^S$ be the factor ring of adeles with vanishing component along S and let $F_S = \prod_{v \in S} F_v$, so $\mathbf{A}_F = F_S \times \mathbf{A}_F^S$. Recall (as we review in §2) that for any finite type affine scheme X over a topological ring R, the set $X(R)$ inherits a natural topology which is functorial in X and its formation is compatible with fiber products (in the categories of R-schemes and topological spaces respectively). In particular, $X(R)$ is locally compact when R is, and $X(R)$ is a topological group when X is an R-group scheme.

We are interested in the locally compact topological group $G(\mathbf{A}_F)$. Since F is a discrete subring of $\mathbf{A}_F$, it is clear that the subgroup $G(F)$ inside of $G(\mathbf{A}_F)$ is discrete (and closed). Let K be a compact open subgroup in $G(\mathbf{A}_F^S)$. Consider the double coset space

$$\Sigma_{G,K,S} = G(F)\backslash G(\mathbf{A}_F)/G(F_S)K = G(F)\backslash G(\mathbf{A}_F^S)/K.$$

Clearly for any two compact open subgroups K and K', $K \cap K'$ is again compact open and hence of finite index in each of K and K'. It follows that the finiteness property of $\Sigma_{G,K,S}$ is independent of the choice of K, and so is an intrinsic property of G.

Definition 1.1. The group G *has finite class numbers* (with respect to the fixed non-empty finite set of places S) if $\Sigma_{G,K,S}$ is finite for one (equivalently, every) compact open subgroup $K \subseteq G(\mathbf{A}_F^S)$.

In the case $G = \mathbf{G}_m$ and suitable K's, the sets $\Sigma_{G,K,S}$ coincide with certain classical generalized ideal class groups. For this reason, the size of the set $\Sigma_{G,K,S}$ (when finite) is called the (K,S) *class number* of G. Obviously G has finite class numbers (with respect to a fixed choice of S) if and only if for every compact open subgroup K in $G(\mathbf{A}_{F,f})$ (with $\mathbf{A}_{F,f}$ the factor ring of "finite adeles", corresponding to all non-archimedean places of F) the double coset space $G(F)\backslash G(\mathbf{A}_F)/G(F_S)K$ is finite; the distinction here is that K is a compact open in $G(\mathbf{A}_{F,f})$ rather than $G(\mathbf{A}_F^S)$. In practice one often sees this latter point of view because it uses compact open subgroups of the group $G(\mathbf{A}_{F,f})$ that has nothing to do with the choice of S, but for our purposes it will be more convenient to use compact open subgroups of $G(\mathbf{A}_F^S)$ rather than compact open subgroups of $G(\mathbf{A}_{F,f})$.

When $G(F)\backslash G(\mathbf{A}_F)$ is compact then obviously G has finite class numbers (for any S). This includes the case $G = \mathbf{G}_a$, but for this case one can be more precise: the classical strong approximation theorem says that $G(F) = F$ is dense in $G(\mathbf{A}_F^S) = \mathbf{A}_F^S$, so $\Sigma_{G,K,S}$ is a singleton set in this case for any K and hence $\mathbf{G}_a$ has trivial finite class numbers (for any S and any K). The classical theorem on finiteness of generalized ideal class groups (or compactness of $F^\times\backslash(\mathbf{A}_F^\times)^{\|\cdot\|_F=1}$) implies that $\mathbf{G}_m$ has finite class numbers (for any S).

It is a general theorem of Borel [1, Thm. 5.1] that any G as above has finite class numbers for any S *when F has characteristic* 0. In particular, one does not need to assume connectivity or reductivity. The strategy of Borel's proof is to first reduce the general case to the connected case by means of Lang's trick [2, 16.5(i)] and a compactness argument, and then to use structure theorems for connected affine algebraic groups to reduce the connected case to the unipotent and reductive cases separately. He then uses a similar technique to reduce the unipotent case to that of $\mathbf{G}_a$, which we treated above, and for the reductive case he uses some serious technology, essentially results in the theory of "Siegel domains" as developed in earlier work of Borel and Harish-Chandra. In particular, he does *not* make use of reduction to the semisimple case.

The situation in positive characteristic is a bit technical for two reasons: the unipotent radical of $G_{\overline{F}}$ may fail to be "defined" over the ground field F (as a subgroup of the F-group G), and non-trivial unipotent F-groups may fail to have non-trivial global points. As an example of the latter, if $F = k(t)$ for a perfect field k of characteristic $p > 0$ then the subgroup $U \subseteq \mathbf{G}_a \times \mathbf{G}_a$ defined by $y^p = x + tx^p$ is unipotent (it becomes isomorphic to $\mathbf{G}_a$ over $F^{1/p}$) and it is a pleasant exercise to check that $U(F) = \{(0,0)\}$ when $p > 2$ (whereas if $p = 2$ then $U(F)$ contains the non-trivial element $(1/t, 0)$). Despite the subtle nature of $U(F)$ in positive characteristic, we shall see that for all unipotent groups U over a global field one has that $U(F)\backslash U(\mathbf{A}_F)$ is compact, and thus U has finite class numbers for any S (of the usual sort).

The purpose of these notes is to address the following generalization of Borel's theorem:

Theorem 1.2. *Let G be a smooth affine group scheme over a global field F. If the unipotent radical $\mathscr{R}_u(G_{\overline{F}})$ is defined and split over F as a unipotent subgroup of G, then G has finite class numbers with respect to any S.*

Remark 1.3. The hypothesis on $\mathscr{R}_u(G_{\overline{F}})$ is trivially satisfied if F has characteristic 0 or if G^0 is reductive.

We begin our discussion with some preliminary "general nonsense" in §2 concerning adelic points of separated F-schemes of finite type (using the rather different viewpoints of Weil and Grothendieck), and then in §3 we explain why it suffices to prove Theorem 1.2 for the identity component G^0. We then show in §4 that $U(F)\backslash U(\mathbf{A}_F)$ is compact for unipotent F-groups U and this is used to prove that for connected G it suffices to prove Theorem 1.2 for the reductive $G^0/\mathscr{R}_u(G)$. Next, in §5 we give a direct proof of Theorem 1.2 for tori. Building on this, in §6 we follow some suggestions of G. Prasad to reduce Theorem 1.2 for a connected reductive G to the case of semisimple groups.

The semisimple case over global fields can be proved by using hard finiteness theorems on volumes of quotients of semisimple adelic groups by certain discrete subgroups; we refer the reader to [3, 3.9] for details (strictly speaking, one has to read carefully to verify that the arguments in [3, §3] work for any semisimple group over a global field, and do not require the additional restrictions that are universally imposed throughout the rest of [3]).

In these notes, we understand the phrase "algebraic group" to mean smooth affine group scheme of finite type (over whatever base is being considered). We do *not* make any connectivity assumptions on fibers. It is recommended that the reader merely skim §2 on a first reading.

2. General nonsense on adelic points

In the pre-Grothendieck era, Weil defined a process of "adelization" of algebraic groups over global fields. In modern language, the underlying set is the set of points of the group with values in the adele ring (an algebra over the ground field on which our schemes live). But sometimes it is useful to know that this slick definition is equivalent to a more concrete one of the sort that Weil and his contemporaries used. In this section, we prove in the most general setting that Weil's "adelization" process in [9, Ch. 1] recovers the same thing that post-Grothendieck people use.

Remark 2.1. A rather breezy discussion of this material may also be found in [7, Ch. I, §3].

Let F be a global field and let S be a finite non-empty set of places of F, always understood to contain the infinite places when F has characteristic 0. Let $\mathbf{A}_{F,S} \subseteq \mathbf{A}_F$ denote the open subring of adeles which are integral at all places away from S (not to be confused with the factor ring $\mathbf{A}_F^S$ that is not a subring of $\mathbf{A}_F$!), so $\mathbf{A}_F$ is the direct limit of the open subrings $\mathbf{A}_{F,S}$ over increasing S.

Let X denote a separated finite type F-scheme. We would like to endow the set $X(\mathbf{A}_F)$ with a natural structure of Hausdorff locally compact topological space in a manner which is functorial in $\mathbf{A}_F$ and compatible with formation of fiber products (for topological spaces and F-schemes). The "easy" case is affine X. Here the basic idea is that if the coordinate ring $\Gamma(X, \mathscr{O}_X)$ is F-isomorphic to $F[t_1, \dots, t_n]/(f_1, \dots, f_m)$, then as a set $X(\mathbf{A}_F)$ is identified with the closed subset of $\mathbf{A}_F^n$ where the functions $f_j : \mathbf{A}_F^n \to \mathbf{A}_F$ vanish. This set thereby inherits a locally compact subspace topology. To see that this is independent of the choice of presentation of $\Gamma(X, \mathscr{O}_X)$, it is more elegant to uniquely characterize this construction by means of functorial properties. The starting point is:

Theorem 2.2. *Let R be a topological ring. There is a unique way to topologize $X(R)$ for affine finite type R-schemes X in a manner which is functorial in X, compatible with formation of fiber products, carries closed immersions to topological embeddings, and for $X = \operatorname{Spec} R[t]$ gives $X(R) = R$ its usual topology.*

If R is Hausdorff, then $X(R)$ is Hausdorff and closed immersions $X \hookrightarrow X'$ induce closed embeddings $X(R) \to X'(R)$. If an addition R is locally compact, then $X(R)$ is locally compact.

The Hausdorff property is necessary to require if we want closed immersions to go over to closed embeddings. Indeed, taking the origin of the affine line shows that such a topological property would force the identity point in R to be closed, and this forces R to be Hausdorff since (viewing R as an additive group) a topological group whose identity point is closed must be Hausdorff (diagonal of a group object is a base change on the identity section in any category admitting products).

Proof. To see uniqueness, we pick a closed immersion $i : X \hookrightarrow \operatorname{Spec} R[t_1, \dots, t_n]$. Forming the induced map on R-points and using compatibility with products (view affine n-space as product of n copies of the affine line) as well as the assumption on closed immersions, the induced set map $X(R) \hookrightarrow R^n$ is a topological embedding into R^n endowed with its usual topology. This proves the uniqueness, and that $X(R)$ has to be Hausdorff compact when R is Hausdorff. Likewise, we see that $X(R)$ is closed in R^n in the Hausdorff case, so when R is also locally compact then so is $X(R)$.

There remains the issue of existence, and for this we pick an R-algebra isomorphism

$$\Gamma(X, \mathscr{O}_X) \simeq R[t_1, \dots, t_n]/I \tag{1}$$

for an ideal I, and identify $X(R)$ with the subset of R^n on which the elements of I (viewed as functions $R^n \to R$) all vanish. The main issue is to check that this construction is intrinsic to X and enjoys all of the desired properties. For independence of the choice of (1), the slickest procedure is to observe that if $f : X \to X'$ is a map of affine finite type R-schemes (e.g., the identity) and we choose respective closed immersions i and i' of X and X' into affine n-space and affine n'-space over R, then f lifts (in many ways) to a morphism between the affine spaces carrying $i(X(R))$ over into $i'(X'(R))$ on R-points of the affine spaces. Hence, the continuity of polynomial maps $R^n \to R^{n'}$ provides what we need for well-definedness of this construction. The same argument establishes that this construction is functorial in X.

If $i : X \hookrightarrow X'$ is a closed immersion and we pick a closed immersion of X' into an affine space and use this to make the topologies on both $X(R)$ and $X'(R)$ (this being legal in view of the well-definedness we have already checked), then it is clear that $i : X(R) \to X'(R)$ is an embedding of topological spaces (even closed when R is Hausdorff). By forming products of closed immersions into affine spaces, we see that $(X \times_R X')(R) \to X(R) \times X'(R)$ is a topological isomorphism, ultimately because of the corresponding result on affine spaces, which is obvious.

Finally, to see that $(X \times_Y Z)(R) \to X(R) \times_{Y(R)} Z(R)$ is a topological isomorphism (for given R-scheme maps $X \to Y$ and $Z \to Y$), consider the isomorphism

$$X \times_Y Z \simeq (X \times_R Z) \times_{Y \times_R Y} Y$$

and its topoological counterpart. Since we have already checked compatibility with absolute products (over the final object in the category), the separatedness of Y over R reduces us to the case in which one of the structure maps of the fiber product is a topological embedding. But we have already seen that closed immersions are carried into topological embeddings, so we're done. ■

Example 2.3. If $R \to R'$ is a continuous map of topological rings (e.g., the inclusion of F inside of $\mathbf{A}_F$ or of $\mathscr{O}_{F,S}$ into $\mathbf{A}_{F,S}$, with the subring having the discrete topology in both cases), then for any affine finite type R-scheme X with base change X' to R', the natural map $X(R) \to X(R') = X'(R')$ is continuous, and when $R \to R'$ is a topological embedding then so is $X(R) \to X(R')$. Moreover, if R' is closed in R then $X(R) \to X(R')$ is a closed embedding. These claims are immediate from the construction of the topologies by means of closed immersions of X into an affine space over R (and the base change on this to give a closed immersion of X' into an affine space over R'). The same argument shows that if R is discrete in R', then $X(R)$ is discrete in $X(R')$.

Example 2.4. Since F is discrete in $\mathbf{A}_F$, so F^n is discrete in $\mathbf{A}_F^n$, it follows that for any *affine* finite type F-scheme X, $X(F) \to X(\mathbf{A}_F)$ is a topological embedding onto a discrete subset. Similarly, if X_S is affine of finite type over $\mathscr{O}_{F,S}$, then $X(\mathscr{O}_{F,S})$ is a discrete subset of $X(\mathbf{A}_{F,S})$.

The construction in the proof of Theorem 2.2 used quite crucially that X is affine. In the non-affine case, we immediately run into a problem: if U is open affine in X, then $U(R) \to X(R)$ need not be an open embedding; it may even fail to be a topological embedding. For example, if X is the affine line over F and U is the complement of the origin, then $U(R) \hookrightarrow X(R)$ is the map $R^\times \to R$ where R has its usual topology but $R^\times$ has a structure of topological group coming from the affine model $U = \mathbf{G}_m \simeq \operatorname{Spec} R[x,y]/(xy-1)$ inside of the plane (i.e., $r, r' \in R^\times$ are close when r is near r' in R and r^{-1} is near ${r'}^{-1}$ in R). But as we know well from the example of adele rings, the unit group of a topological ring need *not* be a topological group with respect to the induced topology from the ring. But since the topology on $R^\times = \mathbf{G}_m(R)$ is a topological group structure, we see that in such examples the inclusion $R^\times \to R$ cannot be a topological embedding.

What is going wrong more generally is that if $X = \operatorname{Spec} A$ and $U = \operatorname{Spec} A_f$ with $A = R[t_1, \dots, t_n]/I$, then on the zero locus $X(R)$ of I in R^n, the subset $(\operatorname{Spec} A_f)(R) \subseteq X(R)$ is where $f : X(R) \to R$ is *unit*-valued – the preimage of the subset $R^\times$ – and this preimage might *not* be open! Indeed, such openness in general amounts to the claim that the set of non-units in R is a closed set, and this can easily fail: one may have a sequence of non-units whose limit is a unit. For example, in the adele ring of $\mathbf{Q}$ the sequence of adeles which are 1 in more and more slots but p in the pth slot for large primes p is a sequence of non-units which converges to the unit 1.

Of course, if R is a topological (Hausdorff) field, such as a local field, then the non-units consist of a single closed point $\{0\}$. More generally, since failure of openness of units is the only obstacle to basic open affines inducing open embeddings on spaces of R-points, globalization of the topology can be done when the units are open. The following result is a simple exercise (note that any map from a local scheme S to a scheme X must factor through an open affine in X):

Theorem 2.5. *If $R^\times$ is open in R, then for any open immersion $U \hookrightarrow X$ of affine finite type R-schemes, the induced map $U(R) \hookrightarrow X(R)$ is an open topological embedding. In such cases, when R is local then there is a unique way to topologize $X(R)$ for arbitrary locally finite type R-schemes X subject to the requirements of functoriality, carrying closed (resp. open) immersions of schemes into embeddings (resp. open embeddings) of topological spaces, and compatibility with fiber products.*

This agrees with the earlier construction for affines, and when R is Hausdorff then $X(R)$ is Hausdorff when X is separated over R. If is locally compact and Hausdorff, then $X(R)$ is locally compact.

Remark 2.6. The above theorem is most important in the case when R is a Hausdorff topological field, or a complete discrete valuation ring. For example, if X is a locally finite type scheme over a local field k (such as $\mathbf{C}$ or $\mathbf{Q}_p$), then this makes $X(k)$ into a locally compact topological space in a reasonable manner. The same goes for $X(\mathscr{O})$ with a complete discrete valuation ring $\mathscr{O}$ and a locally finite type $\mathscr{O}$-scheme X.

Remark 2.7. Even for affine X, if Z is closed in X and U is its open complement, then usually the disjoint subsets $Z(R)$ and $U(R)$ in $X(R)$ do *not* fill up $X(R)$. The point is that "Zariski open" corresponds to a "unit" condition, while "Zariski closed" corresponds to a nilpotence condition. Thus, the fact that elements of R may be both non-nilpotent and non-units presents an obstruction to $X(R)$ generally being the union of $U(R)$ and $Z(R)$. More geometrically, if we consider maps $\operatorname{Spec} R \to X$, when $\operatorname{Spec} R$ is "big" then its image might hit both Z and U. Of course, for R a local artin ring (so $\operatorname{Spec} R$ is one point), this problem does not arise. This is why the construction $X(R)$ is particularly straightforward when R is a field.

In view of the above discussion, it is a somewhat remarkable fact that in the special case when $R = \mathbf{A}_F$ is the adele ring of a global field, one can *still* topologize $X(R)$ for arbitrary finite type R-schemes X (e.g., X an abelian variety). It will no longer be the case that immersions of schemes are carried into topological embeddings, so the construction is a bit precarious (it will be functorial and compatible with fiber products, and will recover the earlier construction in the affine case and will carry closed immersions into closed immersions). Nevertheless, since [9, Ch. 1] makes such a construction in an *ad hoc* manner, we would like to present a more Grothendieck-style development of Weil's construction.

Remark 2.8. Except for issues of local compactness, the diligent reader will readily check that everything that follows works with the adele ring of a global field replaced by the "adele ring" associated to any connected Dedekind scheme (e.g., an algebraic curve over $\mathbf{C}$). Of course, without finiteness for residue fields at closed points one does not get local compactness, so the value of such topologies is questionable.

The key to everything is the theory of limits of schemes from §8–§11 in EGA IV. In concrete terms at the affine level, if X is a finite type F-algebra (for a global field F), then by chasing denominators in a finite presentation of the coordinate ring of X we can find a large finite set S of places (containing the archimedean places in the number field case) and a finite-type algebra over $\mathscr{O}_{F,S}$ whose generic fiber is the coordinate ring of X. Geometrically, we can find an affine finite type $\mathscr{O}_{F,S}$-scheme X_S whose generic fiber is X. Grothendieck's global technique of "denominator chasing" (which is what his theory of limits comes down to, up to the harder task of moving *properties* of morphisms and schemes through such limits) shows the same holds in a strong sense for finite type F-schemes which need not be affine. Keeping in mind that $F = \varinjlim \mathscr{O}_{F,S}$, the limit taken over increasing S, the following result is essential (if you assume all rings in the theorem are noetherian, then "finite presentation" means "finite type"):

Theorem 2.9. *Let $\{A_i\}$ be a directed system of rings, with direct limit A. Let X be a finitely presented A-scheme. Then there exists some i_0 and a finitely presented A_{i_0}-scheme X_{i_0} whose base change to A is isomorphic to X.*

Moreover, if X_{i_0} and Y_{i_0} are two finitely presented A_{i_0}-schemes for some i_0, and we define the base changes X_i, Y_i over A_i for all $i \geq i_0$, and likewise X and Y over A, then the natural map of sets

$$\varinjlim \mathrm{Hom}_{A_i}(X_i, Y_i) \to \mathrm{Hom}_A(X, Y)$$

is bijective. A map $f_{i_0} : X_{i_0} \to Y_{i_0}$ acquires property $\mathbf{P}$ upon base change to some A_i if and only if the induced map $f : X \to Y$ over A has property $\mathbf{P}$, where $\mathbf{P}$ is any of the following properties: closed immersion, separated, proper, smooth, affine, open immersion, finite, fibers geometrically connected of pure dimension d.

The "smearing out" X_{i_0} of a finitely presented A-scheme X is essentially unique up to essentially unique isomorphism in the following sense: for finitely presented A_{i_0}-schemes X_{i_0} and X'_{i_0} with A-fiber X, there exists some $i \geq i_0$ and an isomorphism $h_i : X_i \simeq X'_i$ compatible with the common identification with X upon base change to A, and this h_i is essentially unique in the sense that if h_i and H_i are two such isomorphisms then for some $i' \geq i$ the induced isomorphisms $h_{i'}$ and $H_{i'}$ are equal.

Proof. Though the basic outcome is not a "surprise", the moving of *properties* through direct limits does require some genuine technique. The whole task is somewhat lengthy, and involves systematically developing the theory is just the right order. See §8–§11 in EGA IV for complete details. The full list of properties $\mathbf{P}$ which descend through limits is much longer, but we only need the ones indicated above (but in practice one wants much more). ∎

We would like to apply the above theorem to a finite type F-scheme X. To this end, we pick a big finite set of places S as usual and a finite type $\mathscr{O}_{F,S}$-scheme X_S with generic fiber X. For any S' containing S, we define $X_{S'}$ over $\mathscr{O}_{F,S'}$ by base change. Note that if we are given a morphism of $\mathscr{O}_{F,S'}$-schemes Spec $\mathbf{A}_{F,S'} \to X_{S'}$ for some S', then for any finite set S'' containing S' we get an induced map of $\mathscr{O}_{F,S''}$-schemes Spec $\mathbf{A}_{F,S''} \to X_{S''}$ by base change, and likewise by passing to generic fibers we get an F-scheme map Spec $\mathbf{A}_F \to X$. Putting this together, we get a natural map of sets

$$\varinjlim X_{S'}(\mathbf{A}_{F,S'}) = \varinjlim X_S(\mathbf{A}_{F,S'}) \to X_S(\mathbf{A}_F) = X(\mathbf{A}_F) \tag{2}$$

(which is readily checked to equal the limit of base change maps). In this limit we are working only with S' containing S, and increasing S a bit has no impact. Theorem 2.9 makes precise the sense in which this direct limit is intrinsic to X. For example, we leave it as a pleasant exercise (using Theorem 2.9) to endow the left side of (2) with a structure of (set-valued) *functor* in X.

Actually, we can do better. I claim that the left side of (2) is naturally a topological space in a manner which respects functoriality in X, and that (2) is *bijective*. This will provide us with a functorial topological structure on $X(\mathbf{A}_F)$, and the other desired properties (compatibility with fiber products, behavior for affine

X and for closed immersions) will drop out. It is the bijectivity of (2) that is really the key point, since the left side of (2) is *by definition* what Weil regarded as the "adelization" of (what we call) a finite type F-scheme X. In a nutshell, the right side of (2) is the Grothendieck viewpoint and the left side of (2) is the Weil viewpoint. It is by means of this bijection that we can trasport a topological structure to the right side of (2) for non-affine X.

There is one case in which bijectivity of (2) is obvious: affine X. Indeed, in this case the bijectivity claim comes down to the assertion that if $F[t_1, \dots, t_n]/(f_1, \dots, f_m) \to \mathbf{A}_F$ is a map of F-algebras, then for some large finite set of places S, the t_j's lands in some $\mathbf{A}_{F,S}$ and the f_j's have coefficients in $\mathscr{O}_{F,S}$. This is denominator-chasing in its most basic form. In order to establish bijectivity in a more global situation (i.e., not assuming X to be affine), one needs to use Grothendieck's "global denominator chasing".

The point is that since $\mathbf{A}_F = \varinjlim \mathbf{A}_{F,S'}$ and X_S is locally of finite type over the noetherian ring $\mathscr{O}_{F,S}$, one can appeal to Grothendieck's *functorial characterization* of locally finitely presented morphisms, EGA IV, 8.14.2: a map of schemes $Y \to T$ is locally of finite presentation if and only if for any directed system $\{A_i\}$ of rings over T with direct limit A, the natural map $\varinjlim Y(A_i) \to Y(A)$ is bijective. Of course, one should not impose noetherian assumptions (at least on the A_i's, but also not on Y and T while one is at it), since adele rings are not noetherian. Applying this to $X_S \to \operatorname{Spec} \mathscr{O}_{F,S}$ and the directed system of $\mathbf{A}_{F,S'}$'s for S' containing S, we get the bijectivity of (2). The only case we'll really need is affine X, for which we have just seen the denominator-chasing by hand without needing to appeal to Grothendieck's globalization (which in any case comes down to formalism building upon the evident affine case).

Now let us establish the desired topological properties of (2). First we need some notation. For an $\mathscr{O}_{F,S}$-scheme X_S and a place v of F not in S (i.e., v is a maximal ideal of $\mathscr{O}_{F,S}$), we will write $X_{S,v}$ to denote the base change to the completion $\mathscr{O}_v$ at v. For any v, we write X_v to denote the base change to the fraction field F_v of $\mathscr{O}_v$.

Theorem 2.10. *Let X_S be a finite type $\mathscr{O}_{F,S}$-scheme. Using the projections from $\mathbf{A}_{F,S}$ to F_v for $v \in S$ and to $\mathscr{O}_v$ for $v \notin S$, the natural map of sets*

$$X_S(\mathbf{A}_{F,S}) \to \prod_{v \in S} X_v(F_v) \times \prod_{v \notin S} X_{S,v}(\mathscr{O}_v) \tag{3}$$

is a bijection. When X is affine and we give both sides their natural topologies, using product topology on the right side, this is a homeomorphism.

In general, if we use the above bijection to define a topology on $X_S(\mathbf{A}_{F,S})$, then for any finite sets of places $S' \subseteq S''$ containing S, the natural map $X_{S'}(\mathbf{A}_{F,S'}) \to X_{S''}(\mathbf{A}_{F,S''})$ is an open continuous map of topological spaces which is injective when X_S is separated over $\mathscr{O}_{F,S}$.

In the statement of this theorem, we are using Remark 2.6 to give the $X_v(F_v)$'s and $X_{S,v}(\mathscr{O}_v)$'s their natural topologies.

Proof. The bijectivity aspect amounts to the claim that a morphism of $\mathscr{O}_{F,S}$-schemes $\operatorname{Spec} \mathbf{A}_{F,S} \to X_S$ is both uniquely determined by its restriction to the open subschemes $\operatorname{Spec} F_v$ ($v \in S$) and $\operatorname{Spec} \mathscr{O}_v$ ($v \notin S$), and may be constructed from such arbitrary given data. The fact that the quasi-compact $\operatorname{Spec} \mathbf{A}_{F,S}$ is *not* the disjoint union of these infinitely many open affine subschemes is the reason we have to give the matter some thought.

For injectivity of (3), consider $f, g \in X_S(\mathbf{A}_{F,S})$ which induce the same F_v-points for $v \in S$ and the same $\mathscr{O}_v$-points for all $v \notin S$. To prove $f = g$, consider the product map

$$(f, g) : \operatorname{Spec} \mathbf{A}_{F,S} \to X_S \times_{\mathscr{O}_{F,S}} X_S.$$

Since the diagonal of X_S is an immersion, we can pick an open subscheme U in $X_S \times X_S$ around the diagonal through which $\Delta_{X_S/\mathscr{O}_{F,S}}$ factors as a closed immersion (if we assumed X_S to be separated, then we could even take $U = X_S \times X_S$). We will first prove that (f, g) lands in U, and that pullback along this factorized map through U carries the ideal sheaf of the diagonal closed subscheme to 0 on $\operatorname{Spec} \mathbf{A}_{F,S}$. This will provide a factorization through the diagonal, so $f = g$ as desired.

In order to factor (f, g) through the open U, pick a coherent ideal sheaf $\mathscr{I}$ on $X_S \times X_S$ whose zero locus is the complement of U. The desired factorization of (f, g) through the open subscheme U is a set-theoretic property, and amounts to the claim that the pullback of the *coherent* $\mathscr{I}$ to an ideal sheaf on $\mathrm{Spec}\,\mathbf{A}_{F,S}$ is a quasi-coherent ideal sheaf with no zeros: it is the unit ideal sheaf. By functoriality, the pullback of this to each $\mathrm{Spec}\,F_v$ ($v \in S$) and $\mathrm{Spec}\,\mathscr{O}_v$ ($v \notin S$) *is* the unit ideal. Also, because $\mathscr{I}$ arises by pullback from a coherent sheaf, it is locally finitely generated. Under the dictionary between quasi-coherent ideal sheaves and ideals on arbitrary affine schemes, the ideal I associated to $\mathscr{I}$ is *finitely generated*. This follows from:

Lemma 2.11. *Let A be a commutative ring and M an A-module. If the associated quasi-coherent sheaf $\widetilde{M}$ is locally finitely generated as an $\mathscr{O}_X$-module, with $X = \mathrm{Spec}\,A$, then M is a finite A-module.*

Proof. Let $\{\mathrm{Spec}\,A_i\}$ be a finite open affine covering over which the restriction of $\widetilde{M}$ admits finitely many global generators. Thus, by exactness of global sections on quasi-coherent sheaves over affines, $M_i = A_i \otimes_A M$ is a finite A_i-module. For $A' = \prod A_i$, we conclude that $A' \otimes_A M$ is a finite A'-module. But A' is faithfully flat over A, and a module which becomes finite after a faithfully flat extension of scalars had to be finite to begin with (hint: express module generators upstairs as finite sums of elementary tensors and consider the submodule downstairs generated by module elements coming from these elementary tensors). ■

We are reduced to the task of proving that a finitely generated ideal I in $\mathbf{A}_{F,S}$ is the unit ideal if it induces the unit ideal in each F_v ($v \in S$) and $\mathscr{O}_v$ ($v \notin S$). The finiteness property of I is crucial; it is easy to construct ideals in $\mathbf{A}_{F,S}$ which are not finitely generated but which generate the unit ideal in each standard factor ring (consider the ideal generated by elements which are uniformizers in all but finitely many places).

Let $a_1, \dots, a_n$ be generators of I. By hypothesis, for each $v \in S$ one of the $a_{i,v} \in F_v$ is nonzero, and for each $v \notin S$, one of the $a_{i,v} \in \mathscr{O}_v$ is a unit. Since there are just *finitely many* of these a_i's, it is easy to make an $\mathbf{A}_{F,S}$-linear combination which is equal to 1. Thus, $I = (1)$. This proves that (f, g) factors through U, and now we run through a similar game with pullback of the coherent ideal sheaf of the diagonal closed subscheme in U: the algebraic problem is to show that if I is an ideal in $\mathbf{A}_{F,S}$ which projects to 0 in F_v for $v \in S$ and projects to 0 in $\mathscr{O}_v$ for each $v \notin S$, then $I = 0$. But this is trivial, since $\mathbf{A}_{F,S}$ is the product of such F_v's and $\mathscr{O}_v$'s. This completes the proof that (3) is injective. Note that so far we have only required X_S to be *locally* of finite type over $\mathscr{O}_{F,S}$ (though without having assume X to be finite type over F, we could not have ever made suitable integral models X_S in the first place; quasi-compactness conditions in Theorem 2.9 are quite essential).

Now we prove that (3) is surjective. Assume we are given maps $x_v : \mathrm{Spec}\,F_v \to X_v$ for $v \in S$ and $x_v : \mathrm{Spec}\,\mathscr{O}_v \to X_{S,v}$ for all $v \notin S$ (all maps over $\mathscr{O}_{F,S}$). We claim that there exists $x \in X_S(\mathbf{A}_{F,S})$ inducing the given local data. Let $\{U_i\}$ be a *finite* open affine covering of X_S (here we use that X_S is quasi-compact). Since each x_v has source a local ring, its images lands in some U_i (chase the closed point). Pick one for each v, and let V_i be the set of v's for which we have selected U_i as an open affine through which x_v factors. We have a natural finite product decomposition $\mathbf{A}_{F,S} = \prod_i \mathbf{A}_{F,V_i}$, where $\mathbf{A}_{F,V_i}$ is the subproduct of the product ring $\mathbf{A}_{F,S}$ corresponding to local factors for places $v \in V_i$. Since the Spec functor carries *finite* products into disjoint unions, we may focus on each $\mathbf{A}_{F,V_i}$ separately. Since U_i is affine, we are reduced to the claim that if $\phi_\alpha : \mathrm{Spec}\,R_\alpha \to \mathrm{Spec}\,B$ are maps of affine schemes over some affine base $\mathrm{Spec}\,C$, then there exists a map of C-schemes $\phi : \mathrm{Spec}(\prod R_\alpha) \to \mathrm{Spec}\,B$ inducing each ϕ_α. By restating in terms of ring maps, this is obvious.

With bijectivity of (3) settled, we wish to use this bijection to endow $X_S(\mathbf{A}_{F,S})$ with a topological structure. We must check that for affine X_S, this recovers the topology constructed earlier. By picking a finite presentation of the coordinate ring of X_S as an $\mathscr{O}_{F,S}$-algebra, and recalling how the topology on points of affine schemes (of finite type) was defined by means of embeddings into affine spaces, the whole issue comes down to the claim that the product topology on $\mathbf{A}_{F,S}^n$ agrees with the product topology on

$$\prod_{v \in S} F_v^n \times \prod_{v \notin S} \mathscr{O}_v^n.$$

This is trivial.

Finally, we have to check that if $S' \subseteq S''$ is an inclusion of finite sets of places containing S, then the map $X_{S'}(\mathbf{A}_{F,S'}) \to X_{S''}(\mathbf{A}_{F,S''})$ is an open embedding of topological spaces. Via (3), this map is (topologically!) the product of three maps: the identity maps on $\prod_{v \in S'} X_v(F_v)$ and on $\prod_{v \notin S''} X_{S,v}(\mathscr{O}_v)$, and the base change map

$$\prod_{v \in S''-S'} X_{S,v}(\mathscr{O}_v) \to \prod_{v \in S''-S'} X_v(F_v).$$

Thus, the whole problem comes down to show that for $v \notin S$, the natural map $X_{S,v}(\mathscr{O}_v) \to X_v(F_v)$ is continuous and open, and even injective when X_S is separated. The injectivity for separated X_S follows from the valuative criterion for separatedness, so we just have to check continuity and openness.

In general, for a finite type scheme X over a complete discrete valuation ring $\mathscr{O}$ with fraction field K given its natural topology, we claim that $X(\mathscr{O}) \to X_K(K)$ is a continuous open map. If U is an open subscheme of X, then by Theorem 2.5, $U(\mathscr{O})$ is open in $X(\mathscr{O})$. Since $X(\mathscr{O})$ is the union of the $U_i(\mathscr{O})$'s for $\{U_i\}$ an open covering of X, our problem is of local nature on X. Hence, we may assume X is affine. By picking a closed immersion of X into an affine space over $\mathscr{O}$, the fact that $\mathscr{O}^n$ is open in K^n then provides what we need. ■

Using Theorem 2.10 to topologize $X_S(\mathbf{A}_{F,S})$ for finite type $\mathscr{O}_{F,S}$-schemes X_S, it is immediate from the construction that this topology is functorial in X_S, has a countable base of opens, carries fiber product into fiber products, and carries closed immersions into closed embeddings (use Theorem 2.5 and the fact that an arbitrary product of closed embeddings is a closed embedding). For open immersions $U_S \hookrightarrow X_S$ it is not true that $U_S(\mathbf{A}_{F,S}) \to X_S(\mathbf{A}_{F,S})$ is an open embedding, though it is a topological embedding. Indeed, an arbitrary product of open embeddings is a topological embedding but usually does not have open image. This is the reason that the construction of the topology on $X_S(\mathbf{A}_{F,S})$ in the non-affine case had to be done globally via the product decomposition in (3), without trying to glue topologies coming from open affines in X_S.

Corollary 2.12. *Let X_S be a finite type $\mathscr{O}_{F,S}$-scheme. The topological space $X_S(\mathbf{A}_{F,S})$ is locally compact, and is Hausdorff when X_S is separated.*

Proof. Since our topology construction does commute with products and carries closed immersions to closed embeddings, it is clear that when X_S is separated then $X_S(\mathbf{A}_{F,S})$ is Hausdorff. As for local compactness, we want the infinite product space $X_S(\mathbf{A}_{F,S})$ to be locally compact. Since the factor spaces $X_v(F_v)$ are locally compact for $v \in S$, we just have to check that $X_{S,v}(\mathscr{O}_v)$ is compact for $v \notin S$. More generally, for any compact complete discrete valuation ring R and any finite type R-scheme X, we claim $X(R)$ is compact. Indeed, Theorem 2.5 shows that a finite open affine covering $\{U_i\}$ of X makes $\{U_i(R)\}$ a finite open covering of $X(R)$, so the problem comes down to the affine case, which in turn is reduced to the trivial case of affine space (R^n is compact since R is compact). ■

Let X be a finite type F-scheme. We use Theorems 2.9 and 2.10 with the bijection (2) to give $X(\mathbf{A}_F)$ a *functorial* topological structure. To make sense of this, we need to briefly recall how one topologizes direct limits. If $\{T_\alpha\}$ is a directed system of topological spaces, with direct limit set T, we declare $U \subseteq T$ to be open if and only if the preimage of U in each T_α is open. This is readily checked to be a direct limit in the topological category. In general such abstract topologies are hard to handle (e.g., it is not generally true that they commute with formation of finite products). However, the case when transition maps are open holds no surprises: if $T_\alpha \to T_{\alpha'}$ is an open continuous map for all $\alpha' \geq \alpha$, then T is the rising union of the images U_α of the T_α's, and by giving each U_α the quotient topology from T_α it is clear that the topology on T is characterized by declaring the U_α's to be open.

The final part of Theorem 2.10 shows that the case of open transition maps is exactly what we need. Since the behavior of quotient topologies with respect to fiber products (or even absolute products) is quite tricky, the topology on $X(\mathbf{A}_F)$ is probably quite bad unless we impose a hypothesis on X to ensure injectivity of the transition maps in the limit of $X_{S'}(\mathbf{A}_{F,S'})$'s. As we see from the final part of Theorem 2.10, as well as Theorem 2.9, assuming X to be separated over F eliminates all of these problems. It is a pleasant exercise to check that when X is F-separated, then (2) expresses $X(\mathbf{A}_F)$ as a direct limit of locally compact Hausdorff spaces with transition maps that are open embeddings. In this way, we see that $X(\mathbf{A}_F)$ is locally compact and Hausdorff (with a countable base of opens) when X is F-separated, and moreover that this topology is compatible with fiber products. This construction does *not* convert open immersions into topological embeddings in general, but closed immersions do go over to closed embeddings of topological spaces (exercise; use openness of transition maps).

To summarize: for a finite type separated F-scheme X, the set $X(\mathbf{A}_F)$ acquires a functorial structure of locally compact Hausdorff topological space with a countable base of opens, and this topology is compatible with fiber products and converts closed immersions of such F-schemes into closed embeddings of topological spaces. Moreover, if X is the generic fiber of a separated finite type $\mathscr{O}_{F,S}$-scheme X_S, then $X_S(\mathbf{A}_{F,S})$ is naturally an open subset of $X(\mathbf{A}_F)$. As a special case, when X is a group scheme of finite type over F (automatically separated), the set $X(\mathbf{A}_F)$ is naturally a locally compact Hausdorff topological group.

Example 2.13. As an illustration of how confusing adelic points are to pre-Grothendieck people, take a look at the discussion of adelic points in [8]. Not only is their discussion incredibly muddled and hard to read (they have to go through serious contortions in the non-affine case due to their lack of the functorial criterion for locally finitely presented maps, as we used above), but they seem to have a fundamental misunderstanding because at one point they say that if $\{U_i\}$ is an open affine cover of X then $\{U_i(\mathbf{A}_F)\}$ covers $X(\mathbf{A}_F)$ set-theoretically. This is completely wrong, since a morphism $\operatorname{Spec} \mathbf{A}_F \to X$ need not have image entirely contained in any of the U_i's. One may object and say that they're defining a "different" notion of adelic points, but this cannot be. Indeed, in the affine case we agree, and the preceding remark makes clear that the formation of the set $\cup U_i(\mathbf{A}_F)$ inside of $\prod_v X(F_v)$ is not at all independent of $\{U_i\}$, and in particular is not intrinsic to X. Thus, one cannot take $\cup U_i(\mathbf{A}_F)$ as a definition of "adelic points". They seem to be trying to describe Weil's construction, but the lack of the right language leads them into nonsense. Of course, in the end [8] needs only the affine case (for algebraic groups), so such a mistake is irrelevant for the purposes of [8].

Though Example 2.3 shows that $X(F)$ is a discrete closed set in $X(\mathbf{A}_F)$ for finite type *affine* F-schemes X (as F is discrete and closed in $\mathbf{A}_F$), globalizing to the non-affine case usually destroys such properties. For example, consider the sequence (x_n) in $\mathbf{P}^1(\mathbf{A}_\mathbf{Q})$ where $x_n = [a_n, b_n]$ with $a_n, b_n \in \mathbf{Z} - \{0,1\}$ having a_n converging to zero in $\mathbf{Q}_p$ for half the primes, and converging to 1 (though local unit values) in the other non-archimedean factors, and vice-versa for b_n. Using the bijection in Theorem 2.10 we see

$$x_n \in \mathbf{P}^1(\mathbf{Q}) \subseteq \mathbf{P}^1(\mathbf{A}_\mathbf{Q}^\infty) = \mathbf{P}^1(\mathbf{R}) \times \mathbf{P}^1(\widehat{\mathbf{Z}}) = \mathbf{P}^1(\mathbf{R}) \times \prod_p \mathbf{P}^1(\mathbf{Z}_p) = \mathbf{P}^1(\mathbf{R}) \times \prod_p \mathbf{P}^1(\mathbf{Q}_p)$$

for all n, with the infinite product correctly describing the topology. This is a sequence of rational points which converges to $[0,1]$ in half of the non-archimedean factors, and to $[1,0]$ in the others (and we may pass to a subsequence to get convergence at the compact ∞ factor $\mathbf{P}^1(\mathbf{R})$). Thus, the limit is the point $[e, e']$ where $e, e' \in \mathbf{A}_\mathbf{Q}$ are mutually orthogonal non-trivial idempotents away from ∞, and no $\mathbf{A}_\mathbf{Q}^\times$-scaling can bring us to a representative with homogeneous coordinates in $\mathbf{Q}$. Thus, the limit is not a $\mathbf{Q}$-rational point. This shows that the set of rational points inside of the adelic points need not be closed in general (for non-affine X).

However, the *induced* topology on the set of rational points is always discrete:

Theorem 2.14. *Let X be a separated finite type F-scheme. The natural map $X(F) \to X(\mathbf{A}_F)$ is injective and the induced topology on $X(F)$ is the discrete topology.*

Beware that the failure of $X(F)$ to be closed in $X(\mathbf{A}_F)$ renders the discreteness of the induced topology somewhat useless. The situation is like that of $\{1/2^n\}$ inside $\mathbf{R}^\times$.

Proof. The injectivity is obvious (e.g., since $\mathbf{A}_F$ is faithfully flat over F). Since $X(\mathbf{A}_F)$ is a Hausdorff space with a countable base of opens, the discreteness of the induced topology on $X(F)$ amounts to the fact that if (x_n) is a sequence in $X(F)$ converging in $X(\mathbf{A}_F)$ to some $x \in X(F)$, then $x_n = x$ for large n. Pick an open affine U in X with $x \in U(F)$. Choose a place v_0. The projection $X(\mathbf{A}_F) \to X(F_{v_0})$ is clearly continuous, so $x_n \to x$ in $X(F_{v_0})$. But $U(F_{v_0})$ is open in $X(F_{v_0})$, so for large n we have $x_n \in U(F_{v_0}) \cap X(F)$ inside of $X(F_{v_0})$. This overlap is obviously equal to $U(F)$, so by dropping some initial terms $x_n \in U(F)$ for all n. Thus, the convergence $x_n \to x$ in $X(\mathbf{A}_F)$ takes place inside of the subset $U(\mathbf{A}_F)$. But $U(\mathbf{A}_F) \to X(\mathbf{A}_F)$ is a topological embedding, so $x_n \to x$ inside the topological space $U(\mathbf{A}_F)$. But U is affine, so $U(F)$ is discrete and closed in $U(\mathbf{A}_F)$. Thus, $x_n = x$ for large n. ∎

There is one final aspect of this nonsense which seems worth making explicit, while we're in a foundational mood. This applies, for example, to abelian varieties.

Theorem 2.15. *Assume X is an proper F-scheme. The locally compact Hausdorff topological space $X(\mathbf{A}_F)$ is compact. In particular, if X_S is a finite type $\mathscr{O}_{F,S}$-scheme with generic fiber X, then $X(\mathbf{A}_F) = X_{S'}(\mathbf{A}_{F,S'})$ for sufficiently large finite S' containing S.*

The subset $X(F)$ is closed in $X(\mathbf{A}_F)$ if and only if it is finite.

Proof. By slightly increasing S if necessary, Theorem 2.9 shows that we can assume X_S is proper over $\mathscr{O}_{F,S}$. For any $v \notin S$, the valuative criterion for properness tells us that the map $X_{S,v}(\mathscr{O}_v) \to X_v(F_v)$ is bijective. But near the end of the proof of Theorem 2.10 we saw that this map is also an open embedding, so it is a homeomorphism. It then follows that for any finite S' containing S, the natural map $X_S(\mathbf{A}_{F,S}) \to X_{S'}(\mathbf{A}_{F,S'})$ is a homeomorphism. Thus, passing to the direct limit on these homeomorphisms shows that $X_S(\mathbf{A}_{F,S}) \to X(\mathbf{A}_F)$ is a homeomorphism. We are faced with proving just that $X_S(\mathbf{A}_{F,S})$ is compact. Being a product of $X_v(F_v)$ for $v \notin S$ and $X_{S,v}(\mathscr{O}_v) \simeq X_v(F_v)$ for $v \in S$, we just have to prove that if X is a proper scheme over a locally compact field K, then the topological space $X(K)$ is compact.

First assume K is non-archimedean. If $\mathscr{O}$ denotes the compact valuation ring of K and there is a closed immersion $X \hookrightarrow X'_K$, where X'_K is the generic fiber of a proper $\mathscr{O}$-scheme X', then topologically $X'_K(K) = X'(\mathscr{O})$, and $X(K)$ is closed in here. Since $X'(\mathscr{O})$ is compact (being covered by $U_i(\mathscr{O})$ for finitely many open affines U_i, for which $U_i(\mathscr{O})$ is closed in some compact $\mathscr{O}^{n_i}$), the compactness of $X(K)$ would follow. Thus, the problem comes down to finding a closed immersion of X into the generic fiber of a proper $\mathscr{O}$-scheme X'. If X is projective, then we can take X' to be a projective space over $\mathscr{O}$. This handles abelian varieties, for example.

In general, it seems necessary to use a hard theorem: the Nagata compactification theorem ensures that any separated finite type scheme over a noetherian base Y admits an open immersion into a proper Y-scheme. Thus, there is an open immersion of X into a proper $\mathscr{O}$-scheme X'. By replacing X' with the scheme-theoretic image of this quasi-compact open immersion, we may assume X is a schematically dense open in X'. This is preserved by flat base change to the generic fiber, so X is a schemetically dense open in the K-proper X'_K. But by hypothesis X is K-proper, so by separatedness of X'_K we see that the open immersion $X \hookrightarrow X'_K$ is a closed immersion. This is also schematically dense, so the kernel ideal sheaf must vanish. Thus, $X \simeq X'_K$, as desired.

In the archimedean case, if $K = \mathbf{R}$ then $X(\mathbf{R})$ is a closed subset of $X_{\mathbf{C}}(\mathbf{C})$, so we may reduce to the case $X = \mathbf{C}$. In this case, Chow's Lemma provides us with a projective $\mathbf{C}$-scheme X' and a surjection of schemes $X' \to X$. Since $\mathbf{C}$ is algebraically closed, the continuous map $X'(\mathbf{C}) \to X(\mathbf{C})$ must also be surjective (Nullstellensatz). Thus, it suffices to prove compactness of X', so we may assume X is projective. But a closed immersion $X \hookrightarrow \mathbf{P}^n_{\mathbf{C}}$ induces a closed embedding of $X(\mathbf{C})$ into $\mathbf{P}^n(\mathbf{C})$, and one checks from the gluing construction of projective space as a scheme that the topological space $\mathbf{P}^n(\mathbf{C})$ is what topologists call $\mathbf{CP}^n$, and this is seen to be compact in the usual manner (scale to represent points using homogenous coordinates bounded by 1 in absolute value).

With $X(\mathbf{A}_F)$ now known to be compact Hausdorff, if $X(F)$ is finite then it is certainly a closed subset, and if $X(F)$ is closed then it is a closed subset (hence compact) with the *discrete* topology, so it has to be finite. ■

The nature of $X(F)$ inside of $X(\mathbf{A}_F)$ for F-proper X is a subject of study by various people (such as Manin, with cubic surfaces).

From now on we will only require the case of affine X, so the real purpose of the above is to provide us with a hands-on description of the topology on the adelic points (Weil's approach) which is going to be useful in some later proofs. The equivalence of the Weil and Grothendieck descriptions in the affine case can be verified with much less theory than was invoked above, but it is pleasaint to see that the theory works without affineness restrictions.

3. Connected components

Now let G be an algebraic group over a global field F. Recall that this means G is a smooth affine group scheme over F, not assumed to be connected. Let G^0 denote the connected component of the identity. Recall from EGA IV$_2$, 4.5.13, that G^0 is geometrically connected over F. It follows that G^0 is functorially a normal subgroup of G (as this amounts to a statement concerning conjugation which may be checked in the universal case, and by geometric reducedness and the Nullstellensatz it suffices to show that $G^0(\overline{F})$ is normal in $G(\overline{F})$, which in turn holds because $(G^0)_{\overline{F}}$ is the connected component of $G_{\overline{F}}$). Thus, $G^0(A)$ is a normal subgroup of $G(A)$ for any F-algebra A. We first want to show that G has finite class numbers for a fixed choice of S if G^0 does.

Since G^0 is a closed subscheme of G, it follows from the theory in §2 that $G(\mathbf{A}_F)$ is a locally compact and Hausdorff topological group containing $G^0(\mathbf{A}_F)$ as a closed (normal) subgroup. In particular, the quotient space $G(\mathbf{A}_F)/G^0(\mathbf{A}_F)$ is locally compact Hausdorff, and is even a topological group. Moreover, Theorem 2.9 ensures that we can find a smooth affine group scheme G_S over Spec $\mathscr{O}_{F,S}$ for a sufficiently large finite set of places S and another such group scheme G_S^0 with a closed and open immersion $G_S^0 \to G_S$ whose generic fiber is $G^0 \to G$ and whose closed fibers G_x^0 over closed points $x \in \operatorname{Spec} \mathscr{O}_{F,S}$ are (geometrically) connected smooth algebraic groups over the finite field $\kappa(x)$. This latter connectivity property for fibers over finite fields is crucial in the proof of:

Theorem 3.1. *The quotient $G(\mathbf{A}_F)/G^0(\mathbf{A}_F)$ is compact.*

Before we give the proof, to get a feel for this theorem (which is [1, 1.9]) consider the special case when G is a constant group $\mathbf{Z}/n\mathbf{Z}$. In this case G^0 is trivial, and $G(\mathbf{A}_F)$ is the set of n-tuples of mutually orthogonal idempotents in $\mathbf{A}_F$ with sum adding up to 1. In other words, if V_F denotes the set of places of F (index set for the "factors" of $\mathbf{A}_F$), then $G(\mathbf{A}_F)$ is the set $\operatorname{Hom}_{\mathrm{Set}}(V_F, \mathbf{Z}/n\mathbf{Z})$, or $\prod_{V_F} \mathbf{Z}/n\mathbf{Z}$ (product with index set V_F). The topology is equal to the product topology (check!), so compactness follows from Tychonoff's theorem. This space is big, but compact.

Proof. Pick a large finite set of places S for which we have $G_S^0 \to G_S$, as considered above the theorem. For S' containing S, $G_{S'}^0(\mathbf{A}_{F,S'})$ is a closed subgroup of $G_{S'}(\mathbf{A}_{F,S'})$. This is the inclusion

$$\prod_{v\in S'} G^0(F_v) \times \prod_{v\notin S'} G_{S,v}^0(\mathscr{O}_v) \to \prod_{v\in S'} G(F_v) \times \prod_{v\notin S'} G_{S,v}(\mathscr{O}_v)$$

using product topologies. By Theorem 2.5, the inclusions in each factor are open and closed embeddings. Clearly $G(F_v)/G^0(F_v)$ injects into $(G/G^0)(F_v)$, which is a finite set, and likewise for $v \notin S$ we see that $G_S(\mathscr{O}_v)/G_S^0(\mathscr{O}_v)$ injects into the finite set $G(F_v)/G^0(F_v)$ (since $G_S^0(\mathscr{O}_v) = G^0(F_v) \cap G_S(\mathscr{O}_v)$ inside of $G(F_v)$, due to the fact that G_S^0 is closed and open in G_S, while Spec $\mathscr{O}_v$ is connected). Thus, the topological quotient $G_S(\mathbf{A}_{F,S})/G_S^0(\mathbf{A}_{F,S})$ is *topologically* a product of finite discrete sets, so it is compact Hausdorff.

For finite S' containing S, since $\mathbf{A}_{F,S'} = \mathscr{O}_{F,S'} \otimes_{\mathscr{O}_{F,S}} \mathbf{A}_{F,S}$ and G_S^0 is closed and open in G_S we see that

$$G_S^0(\mathbf{A}_{F,S}) = G_S(\mathbf{A}_{F,S}) \cap G_{S'}^0(\mathbf{A}_{F,S'}) = G_S(\mathbf{A}_{F,S}) \cap G_S^0(\mathbf{A}_{F,S})$$

inside of $G_{S'}(\mathbf{A}_{F,S'}) = G_S(\mathbf{A}_{F,S'})$. Thus, the continuous map of compact Hausdorff groups

$$G_S(\mathbf{A}_{F,S})/G_S^0(\mathbf{A}_{F,S}) \to G_{S'}(\mathbf{A}_{F,S'})/G_{S'}^0(\mathbf{A}_{F,S'})$$

is injective and hence a closed embedding. However, $G_S(\mathbf{A}_{F,S})$ is open in $G_{S'}(\mathbf{A}_{F,S'})$, so this is also an open embedding (thus with finite index). The same holds with the pair (S, S') replaced by (S', S'') for any pair of finite sets of places containing S with $S' \subseteq S''$.

Recalling that $G(\mathbf{A}_F)$ is the rising union of open subgroups $G_{S'}(\mathbf{A}_{F,S'})$, and similarly for G^0 with the $G_{S'}^0$'s, it is easy to deduce from the preceding paragraph that $G(\mathbf{A}_F)/G^0(\mathbf{A}_F)$ is the rising union of open subgroups $G_{S'}(\mathbf{A}_{F,S'})/G_{S'}^0(\mathbf{A}_{F,S'})$. with their compact quotient topologies. What we will prove is that this rising chain stops. By (3), it is enough to show that for large S', $G_{S'}(\mathbf{A}_{F,S'})G^0(F_v)$ contains $G(F_v)$ for every $v \notin S'$. That is, we claim $G_S(\mathscr{O}_v)G^0(F_v) = G(F_v)$ for all but finitely many v (outside of S). Here is the key: if we consider the short exact sequence

$$1 \to G^0 \to G \to G/G^0 \to 1 \tag{4}$$

of F-group schemes, the right map is smooth and faithfully flat (i.e., surjective), with G/G^0 a finite (étale) F-group scheme, so by Theorem 2.9, if we enlarge S a little we may find a finite (étale) $\mathscr{O}_{F,S}$-group scheme E_S with generic fiber G/G^0 and a smooth faithfully flat group scheme map $\pi_S : G_S \to E_S$ with generic fiber $G \to G/G^0$. The kernel $H_S = \ker(\pi_S)$ is a smooth finite type $\mathscr{O}_{F,S}$-group scheme, and its generic fiber is identified with G^0. Thus, again by Theorem 2.9, by increasing S a little we may find an isomorphism $H_S \simeq G^0_S$ compatible with the closed immersion into G_S. The upshot is that (by increasing S) we can "smear out" (4) to a diagram

$$1 \to G^0_S \to G_S \to E_S \to 1$$

of finite type $\mathscr{O}_{F,S}$-group schemes, with the left map exactly as above and $G_S \to E_S$ a smooth surjection having E_S finite (even finite étale) over $\mathscr{O}_{F,S}$.

Now it suffices to prove the following general claim. Suppose R is a complete (or just henselian) discrete valuation ring with fraction field K and *finite* residue field k, and G is a smooth separated finite type group schemes over R. Suppose there is a smooth surjection $G \to E$ onto a finite R-group scheme E for which the (necessarily smooth) kernel G^0 has (necessarily geometrically) connected fibers. Then we claim that $G(K) = G^0(K)G(R)$. Since $E(K) = E(R)$ by finiteness of E and $G(K) \to E(K)$ has kernel $G^0(K)$, for $g \in G(K)$ with image $\overline{g} \in E(K) = E(R)$ we just have to show that $\overline{g}$ is in the image of $G(R)$. Consider the cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & G \arrow[d] \\ \operatorname{Spec} R \arrow[r, "\overline{g}"'] & E \end{tikzcd}$$

We want $X(R)$ to be non-empty.

From the hypotheses, $G \to E$ is a smooth surjection which is a principal homogenous space for the E-group scheme $E \times_R G^0$, so applying pullback along $\overline{g}$ shows that X admits a structure of fiberwise non-empty principal homogenous space for the R-group G^0. The closed fiber X_0 is a non-empty principal homogenous space for a smooth *connected* group scheme G^0_0 over a *finite* field k. But Lang's trick shows that such homogeneous spaces are always trivial, which is to say that $X(k)$ must be non-empty. Since X is R-smooth and R is henselian local, a rational point on the closed fiber lifts to an R-point. ■

Let us now show that Theorem 1.2 holds for G (with a fixed set of places S) if it holds for G^0 (and the same set of places S). The inclusion

$$G(F)/G^0(F) \hookrightarrow (G/G^0)(F)$$

implies that $G(F)/G^0(F)$ is finite (since G/G^0 is F-finite).

Let K be a compact open subgroup in $G(\mathbf{A}_F^S)$, so $K^0 = K \cap G^0(\mathbf{A}_F^S)$ is a compact open subgroup of $G^0(\mathbf{A}_F^S)$ (since $G^0(\mathbf{A}_F^S)$ is a closed subgroup of $G(\mathbf{A}_F^S)$). By the hypothesis that G^0 has finite class numbers with respect to S, there exists a finite set $\{\gamma_j^0\}$ in $G^0(\mathbf{A}_F^S)$ such that

$$G^0(\mathbf{A}_F^S) = \coprod G^0(F)\gamma_j^0 K^0.$$

By Theorem 3.1, $G(\mathbf{A}_F^S)/G^0(\mathbf{A}_F^S)$ is compact, so there exists a finite subset $\{g_i\}$ in $G(\mathbf{A}_F^S)$ such that

$$G(\mathbf{A}_F^S) = \coprod G^0(\mathbf{A}_F^S) g_i K = \coprod G^0(F)\gamma_j^0 K^0 g_i K.$$

Since $G^0(F) \subseteq G(F)$ and each compact open subset $K^0 g_i K$ in $G(\mathbf{A}_F^S)$ is a finite union of right cosets $g_{i,\alpha}K$, we obtain finiteness of $\Sigma_{G,K,S}$.

From now on we shall restrict our attention to connected algebraic F-groups G such that $\mathscr{R}_u(G_{\overline{F}})$ is defined over F as a subgroup of G, with $\mathscr{R}_u(G)$ denoting this descended unipotent F-subgroup of G.

4. Unipotent groups

Our aim in this section is two-fold: we will prove $U(F)\backslash U(\mathbf{A}_F)$ is compact for any unipotent F-group U, and this will be used to prove that if $G/\mathscr{R}_u(G)$ has finite class numbers (with respect to S) and $\mathscr{R}_u(G)$ is F-split then G has finite class numbers (with respect to S).

Let us first recall a general result in the theory of topological groups that will often be used in what follows.

Theorem 4.1. *Let G be a locally compact Hausdorff topological group, and X a second-countable locally compact Hausdorff topological space endowed with a continuous G-action, say on the right. Let $x \in X$ be a point and let $G_x \subseteq G$ be its stabilizer for the G-action. If the orbit $x \cdot G$ is locally closed in X then the natural map $G_x\backslash G \to X$ induced by $g \mapsto xg$ is a homeomorphism.*

Proof. See [5, Ch. IX, §5] for a proof in a more general setting. The role of second-countability is so that the Baire category theorem may be applied. ■

Definition 4.2. Let H be an algebraic F-group. For each $\chi \in \mathrm{X}_F(H) = \mathrm{Hom}_F(H, \mathbf{G}_m)$, define

$$|\chi| : H(\mathbf{A}_F) \to \mathbf{R}_{>0}$$

to be the continuous composite of $\chi : H(\mathbf{A}_F) \to \mathbf{G}_m(\mathbf{A}_F) = \mathbf{A}_F^\times$ and the idelic norm $\|\cdot\|_F : \mathbf{A}_F^\times \to \mathbf{R}_{>0}$.

The closed subgroup $H(\mathbf{A}_F)^0 \subseteq H(\mathbf{A}_F)$ is defined to be

$$H(\mathbf{A}_F)^0 = \bigcap_{\chi \in \mathrm{X}_F(H)} \ker |\chi|.$$

Example 4.3. If H is a (connected) semisimple F-group, a unipotent F-group, an anisotropic F-torus, or more generally $\mathrm{X}_F(H) = \{1\}$, then obviously $H(\mathbf{A}_F)^0 = H(\mathbf{A}_F)$.

Our initial interest in Definition 4.2 is due to:

Lemma 4.4. *Let $H \hookrightarrow H'$ be a closed algebraic F-subgroup of an algebraic F-group. The natural continuous map*

$$H(F)\backslash H(\mathbf{A}_F)^0 \to H'(F)\backslash H'(\mathbf{A}_F)^0$$

is a closed embedding. In particular, $H(F)\backslash H(\mathbf{A}_F)^0 \to H'(F)\backslash H'(\mathbf{A}_F)$ is a closed embedding.

Proof. Since the target is a locally compact Hausdorff space admitting a continuous right action by $H'(\mathbf{A}_F)^0$ and hence by $H(\mathbf{A}_F)^0$, and $H(\mathbf{A}_F)^0$ is a second-countable locally compact Hausdorff group, it follows from Theorem 4.1 that for any $x \in H'(F)\backslash H'(\mathbf{A}_F)^0$ with stabilizer subgroup S_x in $H(\mathbf{A}_F)^0$, the natural orbit map

$$S_x\backslash H(\mathbf{A}_F)^0 \to H'(F)\backslash H'(\mathbf{A}_F)^0$$

is a homeomorphism onto the orbit of x if the orbit is closed. Taking x to be the coset of the identity gives $S_x = H'(F) \cap H(\mathbf{A}_F)^0 = H(F)$, and so the problem is to prove that this orbit is closed. That is, we want $H'(F)H(\mathbf{A}_F)^0$ to be closed in $H'(\mathbf{A}_F)^0$. A short elegant proof of this is given in [7, Ch. IV, §1.1]. ■

Lemma 4.5. *Let*

$$1 \to H' \to H \to H'' \to 1$$

be a short exact sequence of connected algebraic F-groups such that H' admits a filtration by closed algebraic subgroups with successive quotients of the form $\mathbf{G}_a$ or $\mathrm{Res}_{F'/F}(\mathbf{G}_m)$ for various finite separable extensions F'/F.

For every place v the map $H(F_v) \to H''(F_v)$ is a topological fibration, and the map $H(\mathbf{A}_F) \to H''(\mathbf{A}_F)$ is a topological fibration. Also,

$$H(F)\backslash H(\mathbf{A}_F) \to H''(F)\backslash H''(\mathbf{A}_F)$$

is an open locally split map and if H' is central in H then this map is a fibration whose fibers are identified with $H'(F)\backslash H'(\mathbf{A}_F)$ via $H(\mathbf{A}_F)$-translation.

Proof. By Hilbert's theorem 90 and the hypothesis on H', for any field L/F the map $H(L) \to H''(L)$ is surjective. Thus, $H(F_v) \to H''(F_v)$ is a surjection of topological groups and (by the implicit function theorem for F_v-manifolds) it locally admits sections because $H \to H''$ is smooth, so this map of F_v-points is a fibration.

To handle the adelic points, we may "smear out" the given exact sequence to an exact sequence

$$1 \to \mathscr{H}' \to \mathscr{H} \to \mathscr{H}'' \to 1$$

of affine group schemes of finite type over Spec $\mathscr{O}_{F,S}$ for a sufficiently large set of places S, and we can arrange moreover that $\mathscr{H}'$ is $\mathscr{O}_{F,S}$-smooth with geometrically connected fibers (we can also arrange the same for $\mathscr{H}$ and $\mathscr{H}''$, but we do not need such properties). Of course, by "short exact sequence" we mean that $\mathscr{H} \to \mathscr{H}''$ is faithfully flat and quasi-compact with functorial kernel $\mathscr{H}'$. In particular, the *fpqc* morphism $\mathscr{H} \to \mathscr{H}''$ is a smooth morphism and expresses $\mathscr{H}$ as an $\mathscr{H}'$-torsor over $\mathscr{H}''$ for the étale topology. We claim that the map of topological groups $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$ (for $v \notin S$) is surjective for all $v \notin S$. For any $h'' \in \mathscr{H}''(\mathscr{O}_v)$, we have to prove that for the pullback $X_{h''}$ of the smooth $\mathscr{O}_v$-morphism $\mathscr{H}_{/\mathscr{O}_v} \to \mathscr{H}''_{/\mathscr{O}_v}$ along the map $h : \mathrm{Spec}\,\mathscr{O}_v \to X_{h''}$, the set $X_{h''}(\mathscr{O}_v)$ is non-empty. The $\mathscr{O}_v$-scheme $X_{h''}$ is an $\mathscr{H}'_{/\mathscr{O}_v}$-torsor for the étale topology due to the torsor property for $\mathscr{H} \to \mathscr{H}''$. In particular, $X_{h''}$ is $\mathscr{O}_v$-smooth, and so to construct an $\mathscr{O}_v$-point in $X_{h''}$ it suffices (by Hensel's lemma) to construct a rational point in the mod-v fiber of $X_{h''}$. This special fiber is a torsor (in the étale topology) for a connected smooth group over a finite field, and hence by Lang's theorem it is a trivial torsor. Thus, $X_{h''}$ must have a rational point in its special fiber at v, as required. This completes the proof that the map of $\mathscr{O}_v$-point groups $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$ is surjective.

Since this map of $\mathscr{O}_v$-points is induced by the fibration map $H(F_v) \to H''(F_v)$ via restriction to open subgroups, we can construct local cross-sections for the map of topological groups $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$, so this map is a fibration for all but finitely many v. Since $\mathscr{H}''(\mathscr{O}_v)$ is compact and totally disconnected for all $v \notin S$, there exists a global cross-section to $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$. It then follows immediately that the map of topological groups $H(\mathbf{A}_F) \to H''(\mathbf{A}_F)$ admits local cross-sections (for the adelic topology) and so is a fibration.

Finally, consider the map

$$\pi : H(F)\backslash H(\mathbf{A}_F) \to H''(F)\backslash H''(\mathbf{A}_F).$$

Since $H''(F)$ is discrete in $H''(\mathbf{A}_F)$, so

$$H''(\mathbf{A}_F) \to H''(F)\backslash H''(\mathbf{A}_F)$$

admits local cross-sections, we get local cross-sections for π by using local cross-sections for $H(\mathbf{A}_F) \to H''(\mathbf{A}_F)$. Since $H(F) \to H''(F)$ and $H(\mathbf{A}_F) \to H''(\mathbf{A}_F)$ are surjective, the right action by $H'(\mathbf{A}_F)$ is transitive on fibers of π and all fibers are homeomorphic. Thus, all fibers of π are homeomorphic to the fiber $\pi^{-1}(1)$.

The right action of $H'(\mathbf{A}_F)$ on fibers is continuous, and the stabilizer in $H'(\mathbf{A}_F)$ for the identity coset in $H(F)\backslash H(\mathbf{A}_F)$ is $H'(\mathbf{A}_F) \cap H(F) = H'(F)$. Since $\pi^{-1}(1)$ is closed, it follows from Theorem 4.1 that the natural map $H'(F)\backslash H'(\mathbf{A}_F) \to \pi^{-1}(1)$ is a homeomorphism. If H' is central, then $H'(F)\backslash H'(\mathbf{A}_F)$ naturally acts on the right. Since $H'(F)$ is discrete in $H'(\mathbf{A}_F)$ and $H(F)$ is discrete in $H(\mathbf{A}_F)$, we can use the local cross-sections and the $H'(F)\backslash H'(\mathbf{A}_F)$-action to verify that π is a topological fibration. ∎

As a brief digression, we note that the restrictive hypothesis on the kernel H' in Lemma 4.5 can be removed if we weaken the conclusion:

Theorem 4.6. *If*

$$1 \to H' \to H \to H'' \to 1$$

is a short exact sequence of smooth affine F-group schemes such that H' is connected, then the map $H(F_v) \to H''(F_v)$ is an open map (with local cross-sections over its image) for every place v, and $H(\mathbf{A}_F) \to H''(\mathbf{A}_F)$ is an open map with local cross-sections over its image. In particular, the natural maps of topological groups $H(F_v)/H'(F_v) \to H''(F_v)$ (for all v) and $H(\mathbf{A}_F)/H'(\mathbf{A}_F) \to H''(\mathbf{A}_F)$ are open embeddings.

Proof. Since the map $H \to H''$ is a smooth morphism of F-schemes, due to the F-smoothness of the kernel H' (and the fact that $H \to H''$ is a faithfully flat and quasi-compact map), Zariski-locally on H it factors as an étale map to an affine space over H''. The same conclusions hold after extension of scalars to F_v for any v, so the map $H(F_v) \to H''(F_v)$ is open for all v because étale maps of F_v-schemes induce open maps on F_v-points (by the structure theorem for étale morphisms). Moreover, since an étale map between F_v-analytic spaces is local homeomorphism (by the structure theorem for quasi-finite maps of F_v-analytic spaces), there are local cross-sections.

To handle adelic points, we can use the same "smearing out" argument (resting on Lang's trick) as in the proof of Lemma 4.5, because that part of the argument did not use the restrictive assumptions on H' in Lemma 4.5 that are not being assumed here. ■

Theorem 4.7. *If U is a unipotent F-group, then $U(F)\backslash U(\mathbf{A}_F)$ is compact.*

Our proof uses Weil restriction of scalars with respect to a finite inseparable extension of fields. A nice development of Weil restriction in the context of schemes is given in [4, §7.6], and we leave it to the reader to check that in the special case of Galois covers (of fields, or more generally connected schemes) it recovers the explicit Galois descent construction as used in [9] and many other works on algebraic groups.

Proof. The case $U = \mathbf{G}_a$ is classical. If U admits a filtration by connected algebraic F-subgroups with successive quotients isomorphic to $\mathbf{G}_a$, then the derived series for U provides successive central extensions to which we may inductively apply Lemma 4.5 to get the result. This takes care of characteristic 0, so we may now assume F has positive characteristic p.

There exists $n \geq 1$ such that for the finite extension $F' = F^{1/p^n}$ over F the group $U_{F'}$ admits a filtration as just considered (over F'), and so $U(F')\backslash U(\mathbf{A}_{F'})$ is compact. The Weil restriction of scalars $U' = \mathrm{Res}_{F'/F}(U_{F'})$ is an F-group scheme of finite type that is affine (by construction) and smooth (by the functorial criterion). By the functorial property of the Weil restriction, clearly

$$U'(F)\backslash U'(\mathbf{A}_F) = U(F')\backslash U(\mathbf{A}_{F'}),$$

so this is compact. The natural F-group map $U \to U'$ is a closed immersion (as it is a functorial monomorphism between smooth affine groups), so by Lemma 4.4 we conclude that $U(F)\backslash U(\mathbf{A}_F)$ is compact. ■

Corollary 4.8. *Let G be a connected algebraic F-group such that the unipotent radical $\mathscr{R}_u(G_{\overline{F}})$ is defined over F and is F-split. If the reductive F-group $G/\mathscr{R}_u(G)$ has finite class numbers with respect to a given finite set of places S, then so does G.*

Proof. Let $G' = G/\mathscr{R}_u(G)$. By Lemma 4.5 and Theorem 4.7, the natural map

$$G(F)\backslash G(\mathbf{A}_F) \to G'(F)\backslash G'(\mathbf{A}_F)$$

is an open and locally split with compact fibers.

Let K be a compact open subgroup in $G(\mathbf{A}_F^S)$, so by Lemma 4.5 the image K' of K in $G'(\mathbf{A}_F^S)$ is a compact open subgroup. Since $G(F_S) \to G'(F_S)$ is surjective by Lemma 4.5, it follows that the induced map of discrete spaces

$$\Sigma_{G,K,S} = G(F)\backslash G(\mathbf{A}_F)/G(F_S)K \to G'(F)\backslash G'(\mathbf{A}_F)/G'(F_S)K' = \Sigma_{G',K',S}$$

has fibers that are compact and hence finite. By hypothesis $\Sigma_{G',K',S}$ is finite, and thus so is $\Sigma_{G,K,S}$. ■

The proof of Theorem 1.2 is now reduced to the case of reductive F-groups.

5. Tori

Lemma 5.1. *For any F-torus T, $T(F)\backslash T(\mathbf{A}_F)^0$ is compact.*

Proof. By Lemma 4.4, it suffices to prove this result for an F-torus containing T as a closed subtorus. For a finite separable extension F'/F that splits T, the canonical map $T \to \mathrm{Res}_{F'/F}(T_{F'})$ is a closed immersion of F-tori, and so we are reduced to the case $T = \mathrm{Res}_{F'/F}(\mathbf{G}_m)$ for a finite separable extension F'/F.

The character group $\mathrm{X}_F(\mathrm{Res}_{F'/F}(\mathbf{G}_m))$ is infinite cyclic and a non-trivial element is given by the norm map $\mathrm{N}_{F'/F}$. Since $\mathbf{R}_{>0}$ is uniquely divisible, for any algebraic F-group H the subgroup $H(\mathbf{A}_F)^0 \subseteq H(\mathbf{A}_F)$ may be computed as the intersection of $\ker|\chi|$'s for elements $\chi \in \mathrm{X}_F(H)$ generating a finite-index sublattice. Thus,

$$(\mathrm{Res}_{F'/F}(\mathbf{G}_m))(\mathbf{A}_F)^0 = \ker(\|\cdot\|_F \circ \mathrm{N}_{F'/F}) = \ker(\|\cdot\|_{F'}),$$

so for $T = \mathrm{Res}_{F'/F}(\mathbf{G}_m)$ we have

$$T(F)\backslash T(\mathbf{A}_F) = {F'}^{\times}\backslash(\mathbf{A}_{F'}^{\times})^{\|\cdot\|_{F'}=1},$$

and this is compact by the classical theory of global fields. ■

Theorem 5.2. *If T is an algebraic torus over a global field F, then T has finite class numbers with respect to any choice of the (non-empty) finite set of places S.*

Proof. Let T' be the maximal split subtorus of T, so $T'' = T/T'$ is F-anisotropic. By Lemma 4.5,

$$1 \to T'(F)\backslash T'(\mathbf{A}_F) \to T(F)\backslash T(\mathbf{A}_F) \to T''(F)\backslash T''(\mathbf{A}_F) \to 1$$

is an exact sequence of topological groups (with second step a fibration). Likewise

$$1 \to T'(\mathbf{A}_F) \to T(\mathbf{A}_F) \to T''(\mathbf{A}_F) \to 1$$

is an exact sequence of topological groups (with second step a fibration), and the same holds for $\mathbf{A}_F^S$-points.

Let K be a compact open subgroup in $T(\mathbf{A}_F^S)$, with K'' its compact open image in $T''(\mathbf{A}_F^S)$ and $K' = K \cap T'(\mathbf{A}_F^S)$ a compact open subgroup in $T'(\mathbf{A}_F^S)$. The sequence of groups

$$T'(F)\backslash T'(\mathbf{A}_F)/T'(F_S)K' \to T(F)\backslash T(\mathbf{A}_F)/T(F_S)K \to T''(F)\backslash T''(\mathbf{A}_F)/T''(F_S)K'' \to 1$$

is a right exact sequence of abstract groups, and so to prove finiteness of the middle term $\Sigma_{T,K,S}$ it suffices to prove finiteness of the outer terms $\Sigma_{T',K',S}$ and $\Sigma_{T'',K'',S}$. Hence, we are reduced to separately treating the split case and the anisotropic case.

If T is split, so $T \simeq \mathbf{G}_m^n$ for some n, then we immediately reduce to the case $n = 1$, which is the classical finiteness for class numbers. If T is anisotropic, then $T(F)\backslash T(\mathbf{A}_F)$ is compact, by Lemma 5.1, so T trivially has finite class numbers. ■

6. The reductive case

Let G be a connected reductive F-group over a global field F. We want to show that if the semisimple group $\widetilde{G} = G/Z_{\mathrm{red}}^0$ has finite class numbers (using a fixed non-empty finite set of places S as usual), where Z_{red} denotes the "reduced center" of G, then so does G (recall from [2, 18.2(*ii*)] that for any connected reductive group H over a field k, the scheme-theoretic center Z has underlying reduced scheme that is a k-smooth diagonalizable subgroup; it is called the *reduced center*, and its identity component is a torus). This will complete the reduction of Theorem 1.2 to the case of semisimple groups.

We begin with some finiteness lemmas over local fields k. For an arbitrary k-torus T, we define

$$T(k)^0 = \bigcap_{\chi \in \mathrm{X}_k(T)} \ker|\chi|.$$

For example, $T(k)^0 = T(k)$ if T is k-anisotropic.

Lemma 6.1. *The subgroup $T(k)^0$ is compact for any k-torus T.*

Proof. Since $T(k)^0$ can be calculated using just a set of χ's that generate a finite-index subgroup of $\mathrm{X}_k(T)$, for any closed immersion of k-tori $T_1 \to T_2$ the closed embedding map $T_1(k) \to T_2(k)$ carries $T_1(k)^0$ into $T_2(k)^0$. Applying this to the closed immersion of k-tori

$$T \hookrightarrow \mathrm{Res}_{k'/k}(T_{k'})$$

for a finite separable extension k'/k that splits T reduces us to the special case $T = \mathrm{Res}_{k'/k}(\mathbf{G}_m)$. In this case $\mathrm{X}_k(T)$ is infinite cyclic and $\mathrm{N}_{k'/k}$ is a nontrivial element, so

$$T(k)^0 = \ker(\mathrm{N}_{k'/k} : T(k) \to \mathbf{G}_m(k)) = \ker(\mathrm{N}_{k'/k} : {k'}^{\times} \to k^{\times}).$$

This kernel is the compact $\mathscr{O}_{k'}^{\times}$ in the non-archimedean case and it is easily checked to be compact in the archimedean case. ■

Lemma 6.2. *Let G be a connected reductive group over a local field k. Let $T = G/G^{\mathrm{der}}$ and $\widetilde{G} = G/Z^0_{\mathrm{red}}$. The natural maps*

$$G(k) \to T(k),\ \ G(k) \to \widetilde{G}(k)$$

have open image with finite index.

This lemma is non-trivial in characteristic $p > 0$, since for tori T over a local field of characteristic p the subgroup $p \cdot T(k) \subseteq T(k)$ is often (always?) not open and often (always?) can fail to have finite index (consider $T = \mathbf{G}_m$).

Proof. Since $G \to T$ and $G \to \widetilde{G}$ are smooth, the implicit function theorem for k-analytic manifolds ensures that the maps on k-points have open images.

For the central torus $Z^0_{\mathrm{red}} \subseteq G$, the map $Z^0_{\mathrm{red}} \to T$ is an isogeny of tori. Thus, $Z^0_{\mathrm{red}}(k) \to T(k)$ has image containing $N \cdot T(k)$ for some nonzero integer N. Hence, to show that the image of $G(k) \to T(k)$ has finite index it suffices to prove that $T(k)/N \cdot T(k)$ is compact (as then the open subgroup $G(k)/N \cdot T(k)$ has finite index); note that $N \cdot T(k)$ is a closed subgroup of $T(k)$ because $N : T(k) \to T(k)$ is proper (this is clear if $T = \mathrm{Res}_{k'/k}(\mathbf{G}_m)$ for a finite separable extension k'/k and in general we use the closed immersion $T \hookrightarrow \mathrm{Res}_{k'/k}(T_{k'})$ for finite separable k'/k splitting T).

To prove that the locally compact Hausdorff group $T(k)/N \cdot T(k)$ is compact, let $T' \subseteq T$ be the maximal k-split subtorus and $T'' = T/T'$, so T'' is anisotropic. By Hilbert 90 the sequence of groups

$$1 \to T'(k) \to T(k) \to T''(k) \to 1$$

is exact, and $T(k) \to T''(k)$ is open because $T \to T''$ is smooth. Thus,

$$T'(k)/N \cdot T'(k) \to T(k)/N \cdot T(k) \to T''(k)/N \cdot T''(k) \to 1$$

is a right exact sequence with the second map open. Since all terms are locally compact and Hausdorff, if the outer terms are compact then the middle term is as well. Hence, we are reduced to the cases of split tori and anisotropic tori. The split case is trivial, and the anisotropic case follows from Lemma 6.1.

Let us now turn to the problem of showing that the open image of $G(k) \to \widetilde{G}(k)$ has finite index. Since we have an exact sequence of groups

$$G(k) \to \widetilde{G}(k) \to \mathrm{H}^1(k, Z^0_{\mathrm{red}}),$$

it suffices to proving that $\mathrm{H}^1(k, T)$ is finite for any k-torus T. This is trivial in characteristic 0, or more generally when the finite Galois splitting field k'/k for T has degree not divisibly by $\mathrm{char}(k)$. To handle the general case, one needs to do some real work. Let $G_k = \mathrm{Gal}(k_s/k)$ for a choice of separable closure k_s/k. By abstract nonsense, for any discrete G_k-module M whose underlying abelian group is a finite free $\mathbf{Z}$-module, there is a natural isomorphism

$$\mathrm{H}^n(k, \mathrm{Hom}_{\mathbf{Z}}(M, \cdot)) \simeq \mathrm{Ext}^n_{G_k}(M, \cdot)$$

of δ-functors on the category of discrete G_k-modules (where $\mathrm{Ext}^{\bullet}_{G_k}$ is the Ext-bifunctor on this category). Applying this isomorphism to $k_s^{\times}$ with M equal to the geometric character group $\mathrm{X}(T) = \mathrm{Hom}_{k_s}(T_{k_s}, \mathbf{G}_m)$, we get $\mathrm{H}^1(k, T) \simeq \mathrm{Ext}^1_{G_k}(\mathrm{X}(T), \mathbf{G}_m)$. Thus, it suffices to prove in general that $\mathrm{Ext}^1_{G_k}(M, \mathbf{G}_m)$ is finite for any M as above.

By local class field theory, for any $r \in \mathbf{Z}$ and any discrete G_k-module M there is a natural pairing

$$\mathrm{Ext}^r_{G_k}(M, \mathbf{G}_m) \times \mathrm{H}^{2-r}(k, M) \to \mathrm{H}^2(k, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z},$$

and so there is a natural map

$$\alpha^r_k(M) : \mathrm{Ext}^r_{G_k}(M, \mathbf{G}_m) \to \mathrm{Hom}_{\mathbf{Z}}(\mathrm{H}^{2-r}(k, M), \mathbf{Q}/\mathbf{Z})$$

that is clearly a map of δ-functors (recall that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group). Using the full force of local class field theory, it follows from [6, Ch. I, Thm. 1.8(a)] that $\alpha^r_k(M)$ is an isomorphism whenever M is a finitely generated $\mathbf{Z}$-module and $r \geq 2$, and also that $\alpha^1_k(M)$ is an isomorphism when M is finite free as a

1 $\mathbf{Z}$-module. [1] Thus, for M finite and free over $\mathbf{Z}$ we have that $\mathrm{Ext}^1_{G_k}(M, \mathbf{G}_m)$ is the $\mathbf{Q}/\mathbf{Z}$-dual of $\mathrm{H}^1(k, M)$, and so the desired finiteness follows from the obvious finiteness of $\mathrm{H}^1(k, M)$ (use inflation-restriction with respect to a finite Galois extension k'/k that splits the discrete G_k-module M). ∎

The essential result we need is:

Theorem 6.3 (Prasad)**.** *Let G be a connected reductive F-group. The natural map*

$$\mathrm{Ad}^0 : G(F)\backslash G(\mathbf{A}_F)^0 \to G^{\mathrm{ad}}(F)\backslash G^{\mathrm{ad}}(\mathbf{A}_F)$$

is proper.

Let us initially grant Theorem 6.3 and see how it is used to prove that G has finite class numbers (with respect to a fixed choice of S) if the semisimple $\widetilde{G}$ does. We may factorize Ad^0 as follows:

$$G(F)\backslash G(\mathbf{A}_F)^0 \stackrel{\mathrm{pr}^0}{\to} \widetilde{G}(F)\backslash \widetilde{G}(\mathbf{A}_F) \to G^{\mathrm{ad}}(F)\backslash G^{\mathrm{ad}}(\mathbf{A}_F),$$

and since these spaces are Hausdorff with the composite a proper map, the first step pr^0 must be proper.

Choose a compact open subgroup K' in $\widetilde{G}(\mathbf{A}_F^S)$. Since $\widetilde{G}$ has finite class numbers with respect to S (by hypothesis, or rather because Borel and Prasad proved it for all connected semisimple groups over global fields and all choices of the auxiliary non-empty finite set of places S, containing the archimedean places in the number field case), there exist finitely many $y_i \in \widetilde{G}(\mathbf{A}_F^S)$ such that

$$\widetilde{G}(\mathbf{A}_F) = \bigcup_i \widetilde{G}(F) y_i \widetilde{G}(F_S) K'; \tag{5}$$

note that K' and $\widetilde{G}(F_S)$ commute since $\mathbf{A}_F = F_S \times \mathbf{A}_F^S$. Since pr^0 is proper, each preimage

$$(\mathrm{pr}^0)^{-1}(\widetilde{G}(F)\backslash \widetilde{G}(F) y_i K') \subseteq G(F)\backslash G(\mathbf{A}_F)^0 \tag{6}$$

is compact in $G(F)\backslash G(\mathbf{A}_F)^0$.

Pick a compact open K in $G(\mathbf{A}_F^S)$, so $G(F_S)K = K \cdot G(F_S)$ is an open subgroup of $G(\mathbf{A}_F)$. Since the union in (5) is finite and the image of $G(F_S)$ in $\widetilde{G}(F_S)$ has finite index (by Lemma 6.2), it follows from the compactness of (6) that $G(\mathbf{A}_F)^0$ is contained in finitely many double cosets

$$G(F) z_j G(F_S) K.$$

Thus, the union of these double cosets contains the subgroup $G(\mathbf{A}_F)^0 \cdot G(F_S)$ in $G(\mathbf{A}_F)$. By Lemma 6.4 below, there is a finite set $\{x_r\}$ of representatives for $G(\mathbf{A}_F)^0 \cdot G(F_S)\backslash G(\mathbf{A}_F)$, so by running through the above argument using $\cap_r x_r^{-1} K x_r$ in the role of K gives the desired finiteness for $\Sigma_{G,K,S}$.

Lemma 6.4. *The subgroup $G(\mathbf{A}_F)^0 \cdot G(F_S)$ in $G(\mathbf{A}_F)$ has finite index.*

Proof. Let $T = G/G^{\mathrm{der}}$, so $\mathrm{X}_F(T) = \mathrm{X}_F(G)$. Hence, we have a natural injection of groups

$$G(\mathbf{A}_F)/G(\mathbf{A}_F)^0 \to T(\mathbf{A}_F)/T(\mathbf{A}_F)^0.$$

Consider the commutative diagram of groups

$$\begin{tikzcd} G(\mathbf{A}_F)/G(\mathbf{A}_F)^0 \arrow[r] & T(\mathbf{A}_F)/T(\mathbf{A}_F)^0 \\ G(F_S) \arrow[u] \arrow[r] & T(F_S) \arrow[u] \end{tikzcd}$$

We need to prove that the map along the left has image with finite index, so by injectivity of the top row it is enough to prove the maps along the bottom and right sides have images with finite index.

Since $T(\mathbf{A}_F)^0$ contains any compact open subgroup of $T(\mathbf{A}_F^S)$ (as it contains any compact subgroup at all) and it also contains $T(F)$, the desired finite-index property for the map along the right side follows from the finiteness of class numbers for tori. By Lemma 6.2, $G(F_S) \to T(F_S)$ has image with finite index. ∎

[1] The diligent reader should check that the diagram [6, Ch. I, (1.9.1)] commutes, at least in the cases needed for the proof; this ultimately reduces to the δ-functorial characterization of corestriction.

It remains to prove Theorem 6.3. We shall first use a standard argument to circumvent the possibility that the reduced center of the connected reductive F-group G may have non-vanishing degree-1 Galois cohomology over various extensions of F. Choose a closed immersion $i : G \hookrightarrow \mathrm{GL}_n$ over F for some n, and let $R \subseteq \mathrm{M}_n(F)$ be the F-subalgebra generated by the "reduced center" Z of G. Since Z is diagonalizable, as we noted earlier, the finite commutative F-algebra R is semisimple. Hence, the F-group $C = R^\times$ is a torus in GL_n that centralizes G and contains Z, so $G' = C \cdot G$ is a connected reductive F-subgroup of GL_n whose reduced center is C. Define $\underline{G} = G'/C$, so $\underline{G}$ is semisimple and the natural map $G^{\mathrm{ad}} \to \underline{G}^{\mathrm{ad}}$ is an isomorphism, and moreover the isogeny $\underline{G} \to \underline{G}^{\mathrm{ad}}$ has infinitesimal kernel (that is, it is purely inseparable) by the construction of $\underline{G}$.

By Lemma 4.4, the natural map

$$G(F)\backslash G(\mathbf{A}_F)^0 \to G'(F)\backslash G'(\mathbf{A}_F)^0$$

is a closed embedding. Thus, to prove Theorem 6.3 it suffices to show that the natural map

$$G'(F)\backslash G'(\mathbf{A}_F)^0 \to \underline{G}^{\mathrm{ad}}(F)\backslash \underline{G}^{\mathrm{ad}}(\mathbf{A}_F)$$

is proper.

We shall show that each step in the factorization

$$G'(F)\backslash G'(\mathbf{A}_F)^0 \to \underline{G}(F)\backslash \underline{G}(\mathbf{A}_F) \to \underline{G}^{\mathrm{ad}}(F)\backslash \underline{G}^{\mathrm{ad}}(\mathbf{A}_F)$$

is proper. For the second step we recall that $\underline{G} \to \underline{G}^{\mathrm{ad}}$ is a purely inseparable isogeny between connected semisimple F-groups, and so we invoke:

Lemma 6.5. *Let $H \to H'$ be a purely inseparable isogeny between connected algebraic groups over a global field k, and assume $\mathrm{X}_k(H) = \{1\}$. The natural continuous map*

$$H(k)\backslash H(\mathbf{A}_k) \to H'(k)\backslash H'(\mathbf{A}_k)$$

is a closed embedding.

Proof. Since the map in question is a continuous injection between Hausdorff spaces, it is a closed embedding if and only if it is proper.

The case of characteristic 0 is trivial, so we may assume $\mathrm{char}(k) = p > 0$. The kernel of $H \to H'$ is a finite commutative local k-group, and so it is killed by some iterate of its relative Frobenius. Thus, for sufficiently large n we may factorize the n-fold relative Frobenius $H \to H^{(p^n)}$ through $H \to H'$. In general, if $f : X \to Y$ and $f' : Y \to Z$ are continuous maps between topological spaces with f' separated and $f' \circ f$ proper, then f is proper. Thus, it suffices to treat the case of the map $H \to H^{(p^n)}$.

Let $k' = k^{1/p^n}$, a global field of degree p^n over k. We have a natural isomorphism of topological groups $H^{(p^n)}(\mathbf{A}_k) = H(\mathbf{A}_{k'})$, and this carries $H^{(p^n)}(k)$ over to $H(k')$. The composite

$$H(\mathbf{A}_k) \to H^{(p^n)}(\mathbf{A}_k) = H(\mathbf{A}_{k'})$$

is induced by functoriality with respect to the map of k-algebras $\mathbf{A}_k \to \mathbf{A}_{k'}$, and likewise

$$H(k) \to H^{(p^n)}(k) = H(k')$$

is induced by functoriality with respect to the map of fields $k \to k'$. Hence, our problem is to prove that the natural map

$$H(k)\backslash H(\mathbf{A}_k) \to H(k')\backslash H(\mathbf{A}_{k'})$$

is a closed embedding.

Rather more generally, I claim that for any finite extension of global fields k'/k and any connected algebraic k-group H with $\mathrm{X}_k(H) = \{1\}$, the natural map

$$H(k)\backslash H(\mathbf{A}_k) \to H(k')\backslash H(\mathbf{A}_{k'})$$

is a closed embedding. Letting $H' = \mathrm{Res}_{k'/k}(H_{k'})$, this map is the map

$$H(k)\backslash H(\mathbf{A}_k) \to H'(k)\backslash H'(\mathbf{A}_k)$$

induced by the canonical closed immersion $H \to H'$. Since $H(\mathbf{A}_k) = H(\mathbf{A}_k)^0$ (as $\mathrm{X}_k(H) = \{1\}$), we may apply Lemma 4.4 to the map $H \to H'$ to conclude the proof. ■

To complete the proof of Theorem 6.3, it remains to prove that the natural map

$$\pi : G'(F)\backslash G'(\mathbf{A}_F)^0 \to \underline{G}(F)\backslash \underline{G}(\mathbf{A}_F)$$

is proper. The smooth quotient map map $G' \to \underline{G}$ has kernel C that is the unit group of a finite semisimple commutative F-algebra, so we can apply Lemma 4.5 to conclude that $G'(F) \to \underline{G}(F)$ is surjective, $G'(F_v) \to \underline{G}(F_v)$ is a fibration for every place v, $G'(\mathbf{A}_F) \to \underline{G}(\mathbf{A}_F)$ is a fibration carrying $G'(F)$ onto $\underline{G}(F)$, and

$$G'(F)\backslash G'(\mathbf{A}_F) \to \underline{G}(F)\backslash \underline{G}(\mathbf{A}_F)$$

is a fibration whose fibers are orbits for the continuous free right action of $C(F)\backslash C(\mathbf{A}_F)$ on $G'(F)\backslash G'(\mathbf{A}_F)$ (recall that C is central in G'). Lemma 4.5 also ensures that the map $C(F)\backslash C(\mathbf{A}_F) \to G'(F)\backslash G'(\mathbf{A}_F)$ is a closed embedding.

We now separately treat the cases of number fields and function fields, due to the different structure of the idelic norm and idelic topology in the two cases. The case $C = 1$ is trivial, so we can assume $C \neq 1$.

Suppose F is a number field. In this case, we will show that the local cross-sections to the central fibration $G'(\mathbf{A}_F) \to \underline{G}(\mathbf{A}_F)$ can be chosen to land inside of $G'(\mathbf{A}_F)^0$. This will provide local cross-sections to π, showing that π is a fibration whose fibers are orbits for the continuous free action of $C(F)\backslash C(\mathbf{A}_F)^0$, so by Theorem 4.1 the properness of π will follow from an application of Lemma 5.1 to the torus C. To build local cross-sections landing in $G'(\mathbf{A}_F)^0$, it suffices to construct a continuous map of topological spaces $c : G'(\mathbf{A}_F) \to C(\mathbf{A}_F)$ such that $c(g)^{-1}g \in G'(\mathbf{A}_F)^0$ for all g.

Let $T \subseteq C$ be the maximal k-split subtorus of C (clearly $T \neq 1$ if $C \neq 1$, due to the construction of C). Since $\mathrm{X}_F(G')$ is a finite-index lattice in $\mathrm{X}_F(T)$, we can choose a basis $\chi_1, \ldots, \chi_r$ of $\mathrm{X}_F(T)$ such that $\chi_1^{e_1}, \ldots, \chi_r^{e_r}$ is a basis of $\mathrm{X}_F(G')$. We use the χ_j's to define an isomorphism $T \simeq \mathbf{G}_m^r$, so upon choosing an archimedean place $v \in S$ we get a closed embedding

$$t : \mathbf{R}_{>0}^r \hookrightarrow (F_v^\times)^r = T(F_v) \hookrightarrow T(\mathbf{A}_F) \subseteq C(\mathbf{A}_F)$$

via the canonical inclusion $\mathbf{R}_{>0} \hookrightarrow F_v^\times$ for the archimedean place v. For any $g \in G'(\mathbf{A}_F)$, define

$$c(g) = t(\|\chi_1^{e_1}(g)\|_F^{1/e_1}, \cdots, \|\chi_r^{e_r}(g)\|_F^{1/e_r}).$$

It is clear that c has the desired property, due to unique divisibility of $\mathbf{R}_{>0}$ and how the χ_j's were chosen.

Next, suppose F is a global function field with constant field of size q, so the idelic norm on $\mathbf{A}_F^\times$ has image $Q = q^{\mathbf{Z}}$ in $\mathbf{R}_{>0}$. Once again, let $\{\chi_1, \ldots, \chi_r\}$ be a basis of $\mathrm{X}_F(T)$ such that $\{\chi_1^{e_1}, \ldots, \chi_r^{e_r}\}$ is a basis of $\mathrm{X}_F(G') = \mathrm{X}_F(G'/G'^{\mathrm{der}})$. The continuous homomorphism

$$|\chi| = (\|\chi_1^{e_1}\|_F, \ldots, \|\chi_r^{e_r}\|_F) : G'(\mathbf{A}_F) \to Q^{\oplus r} \subseteq \mathbf{R}_{>0}^{\oplus r}$$

has image equal to a subgroup $\Gamma \subseteq Q^{\oplus r}$ (even of finite index, though we do not use this fact), and the restriction of this map to $C(\mathbf{A}_F)$ has image that we denote $\Lambda \subseteq \Gamma$. Since the natural map $C \to G'/G'^{\mathrm{der}}$ is an isogeny of F-tori, there is a second isogeny in the other direction such that their composite is multiplication by some nonzero integer on C. Hence, Γ/Λ is killed by this nonzero integer and so Λ has finite index in Γ.

For each $\underline{g} \in \underline{G}(\mathbf{A}_F)$ the elements $g \in G'(\mathbf{A}_F)$ mapping to $\underline{G}(F) \cdot \underline{g}$ in $\underline{G}(F)\backslash \underline{G}(\mathbf{A}_F)$ all give rise to the same left coset $C(\mathbf{A}_F)G'(\mathbf{A}_F)^0 g$, and so by local constancy of the idelic norm we arrive at a natural decomposition of $\underline{G}(F)\backslash \underline{G}(\mathbf{A}_F)$ into finitely many open and closed subsets Y_j labelled by the elements $\overline{\gamma}_j \in \Gamma/\Lambda$. For each such j, let $g_j \in G'(\mathbf{A}_F)$ be an element whose image in Γ/Λ is $\overline{\gamma}_j$, so we get a finite disjoint union decomposition

$$G'(F)\backslash G'(\mathbf{A}_F) = \coprod G'(F)\backslash C(\mathbf{A}_F)G'(\mathbf{A}_F)^0 g_j$$

into the open and closed preimages of the Y_j's. Define

$$E = \coprod G'(\mathbf{A}_F)^0 g_j = \coprod g_j G'(\mathbf{A}_F)^0,$$

an open and closed set in $G'(\mathbf{A}_F)$ that is stable under left and right translations by the normal subgroup $G'(\mathbf{A}_F)^0$.

Since $G'(\mathbf{A}_F)$ has a topological base of compact open sets, on the open and closed subgroup

$$|\chi|^{-1}(\Lambda) = C(\mathbf{A}_F)G'(\mathbf{A}_F)^0 \subseteq G'(\mathbf{A}_F)$$

it is trivial to construct a (locally constant) continuous map $c : |\chi|^{-1}(\Lambda) \to C(\mathbf{A}_F)$ such that $c(g)^{-1}g \in G'(\mathbf{A}_F)^0$ for all $g \in |\chi|^{-1}(\Lambda)$. Thus, by using the continuous maps

$$g \mapsto c(gg_j^{-1})^{-1}g \in G'(\mathbf{A}_F)^0 g_j$$

on each $C(\mathbf{A}_F)G'(\mathbf{A}_F)^0 g_j$ and using local cross sections to $G'(\mathbf{A}_F) \to \underline{G}(\mathbf{A}_F)$ we see that the restriction of the $C(F)\backslash C(\mathbf{A}_F)$-equivariant fibration

$$G'(F)\backslash G'(\mathbf{A}_F) \to \underline{G}(F)\backslash \underline{G}(\mathbf{A}_F)$$

to the $C(F)\backslash C(\mathbf{A}_F)^0$-stable open and closed set $G'(F)\backslash E$ admits local cross-sections. The resulting map

$$\pi_E : G'(F)\backslash E \to \underline{G}(F)\backslash \underline{G}(\mathbf{A}_F)$$

must therefore be a $C(F)\backslash C(\mathbf{A}_F)^0$-equivariant fibration (the topology is easy to follow because E is open and closed in $G'(\mathbf{A}_F)$ and we are using quotients by discrete subgroups).

The continuous free action of $C(F)\backslash C(\mathbf{A}_F)^0$ on fibers of π_E is transitive. Thus, the fibration π_E has fibers homeomorphic to $C(F)\backslash C(\mathbf{A}_F)^0$. Since $C(F)\backslash C(\mathbf{A}_F)^0$ is compact, by Lemma 5.1, the map π_E is therefore a fibration with compact fibers and thus it is proper. The restriction π of π_E to the closed subset $G'(F)\backslash G'(\mathbf{A}_F)^0$ is therefore also proper, as desired.

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: `bdconrad@umich.edu`