L-FUNCTIONS FOR CM ABELIAN VARIETIES

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The purpose of these notes is to determine the L-function of an abelian variety A over a number field L_0 when A obtains CM over a finite extension L/L_0. This L-function can be realized as a product of L-functions of Grössencharaktere of a minimal such L, at least under the assumption that the image of a CM field in the geometric endomorphism ring of A is stable under the action of $G = \mathrm{Gal}(L/L_0)$. This includes as an important special case those A that are absolutely simple. The main references are Milne [4] and Lang [3], though we will be following [4] more closely, as Milne's treatment is somewhat simpler (at the expense, perhaps, of greater abstraction).

To establish some notation: if L is a field, we denote by $\overline{L}$ a separable closure of L and by G_L the absolute Galois group $\mathrm{Gal}(\overline{L}/L)$ of L. If L is a global, resp. non-archimedean local, field and v is a, resp. the, non-archimedean valuation of L, then we denote by π_v a choice of uniformizer at v. In this situation, we denote by $\mathbf{N}v$ the size of the residue field $k(v)$ at v.

1. The CM case

In this section, we use the Main Theorem of complex multiplication to determine the L-function of a CM abelian variety.

1.1. CM characters.

Let (A, ι) be a CM abelian variety of type (F, Φ) defined over a number field L. Define

$$\rho : G_L \longrightarrow \mathbf{A}_{F,\mathrm{f}}^{\times}$$

to be the (continuous) homomorphism given the action on the total Tate module $V_{\mathrm{f}}(A)$. Denote by

$$N_\Phi : \mathbf{A}_K^{\times} \longrightarrow \mathbf{A}_F^{\times}$$

the reflex norm and set $\eta = (N_\Phi \circ N_{L/K})^{-1} : \mathbf{A}_L^{\times} \to \mathbf{A}_F^{\times}$. In this context, the Main Theorem of complex multiplication is:

Theorem 1. *There is a unique homomorphism $\alpha : \mathbf{A}_L^{\times} \to F^{\times}$, continuous for the discrete topology on $F^{\times}$, such that, if we define $\psi = \alpha\eta$, then the diagram*

$$(*)\qquad \begin{array}{ccc} \mathbf{A}_L^{\times} & \xrightarrow{\psi} & \mathbf{A}_F^{\times} \\ {\scriptstyle \mathrm{rec}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ G_L^{\mathrm{ab}} & \xrightarrow[\rho]{} & \mathbf{A}_{F,\mathrm{f}}^{\times} \end{array}$$

commutes.

Proposition 2. *The homomorphism $\psi : \mathbf{A}_L^{\times} \longrightarrow \mathbf{A}_F^{\times}$ in Theorem 1 is trivial on $L^{\times} \subseteq \mathbf{A}_L^{\times}$.*

Proof. If $x \in L^\times$, then $\psi(x)_{\text{fin}} = 1$ by the commutativity of $(*)$. Note that $\eta(x) \in F^\times$, since $N_\Phi|_{K^\times}$ carries $K^\times$ into $F^\times$. Thus, $\psi(x) \in F^\times$ and hence $\psi(x) = 1$. $\square$

Consider the topological product

$$\mathbf{A}_F^\times = \mathbf{A}_{F,\mathrm{f}}^\times \times F_\infty^\times \subseteq \prod_p (F \otimes \mathbf{Q}_p)^\times \times F_\infty^\times,$$

where $F_\infty = F \otimes \mathbf{R} \cong \prod_{\phi \in \Phi} \mathbf{C}_\phi$. Accordingly, we can write each $s \in \mathbf{A}_F^\times$ as $(s_{\mathrm{f}}, s_\infty)$, where $s_{\mathrm{f}} = (s_p)_p$ and $s_\infty = (s_\phi)_\phi$. For any rational prime p, we define

$$\psi_p : \mathbf{A}_L^\times \longrightarrow (F \otimes \mathbf{Q}_p)^\times : x \mapsto \psi(x)_p.$$

This map factors as

$$\mathbf{A}_L^\times \xrightarrow{\text{rec}} G_L^{\text{ab}} \xrightarrow{\rho} (F \otimes \mathbf{Q}_p)^\times$$

and, by $(*)$, ψ_p thereby gives the action of G_L on the p-adic Tate module $T_p(A)$.

If τ is any embedding of F into $\mathbf{C}$, then we define the Grössencharakter $\psi_\tau : \mathbf{A}_L^\times \to \mathbf{C}^\times$ by the composition

$$\mathbf{A}_L^\times \xrightarrow{\psi_\infty} F_\infty^\times = (F \otimes \mathbf{R})^\times \xrightarrow{\tau \otimes 1} \mathbf{C}^\times.$$

Exactly one of τ or $\overline{\tau}$ is equal to an element $\phi \in \Phi$, so we can give an equivalent definition of ψ_τ by the formula $\psi_\tau(x) = \tau(\psi(x)_\phi)$.

Theorem 3. *Let v be a prime of L. Then α, resp. ψ_τ for all $\tau : F \hookrightarrow \mathbf{C}$, is unramified at v if and only if A has good reduction at v. In case this holds, $\iota\big(\alpha(\pi_v)\big)$ reduces at v to the Frobenius endomorphism Fr_v of the v-fiber $A_{k(v)}$.*

Proof. Choose any rational prime ℓ such that $v \nmid \ell$. By definition, α is unramified at v if and only if ψ_ℓ, resp. ψ_τ for all τ, is unramified at v, since $\eta(L_v^\times)_\ell = 1$, resp. $\eta(L_v^\times)_\tau = 1$ for all τ. That this is equivalent to A being unramified at v follows from the fact that ψ_ℓ gives the action of G_L on the ℓ-adic Tate module of A. The statement that $\iota\big(\alpha(\pi_v)\big)$ reduces to Fr_v at v can be seen from $(*)$. $\square$

1.2. ***L*-functions.** We pause here to recall some generalities regarding L-functions of Grössencharaktere and abelian varieties. Let M be a number field and let $\chi : \mathbf{A}_M^\times \to \mathbf{C}^\times$ be a Grössencharakter of M. For all places v of M, we define the local L-function

$$L_v(\chi, T) = \begin{cases} (1 - \chi(\pi_v)T)^{-1} & \chi \text{ unramified at } v, \\ 1 & \chi \text{ ramified at } v. \end{cases}$$

The global L-function of χ is given by $L(\chi, s) = \prod_v L_v(\chi, \mathbf{N}v^{-s})$. This Euler product converges absolutely and uniformly in some right half-plane, has analytic continuation to all of $\mathbf{C}$ (provided χ isn't a power of the adelic norm), and satisfies a functional equation of standard type.

Similarly, if A is any abelian variety defined over M, for all places v of M we define the local L-function

$$L_v(A/M, T) = \det(1 - \mathrm{Fr}_v\, T \mid T_\ell(A)^{I_v})^{-1},$$

where ℓ is any rational prime such that $v \nmid \ell$. As before we define the global L-function of A over M by $L(A/M, s) = \prod_v L_v(A/M, \mathbf{N}v^{-s})$. By the Riemann Hypothesis for abelian varieties over finite fields, this product converges in some right half-plane, but the analytic continuation and functional equation of this L-function is only known in a few special cases (some of which are proved in these notes).

We are now in a position to identify the L-function of a CM abelian variety as the product of the L-functions of its associated Grössencharaktere. In order to compute the local factors at the bad primes, however, we will need the following result of Serre-Tate:

Theorem 4. *If (A, ι) is a CM abelian variety defined over L, then for any place v of L at which A has bad reduction and for any rational prime ℓ not divisible by v, $T_\ell(A)^{I_v} = 0$.*

Proof. See Remark (2) on p. 504 of Serre-Tate [7] or Brian's notes [2, Thm. 6.5]. $\square$

Theorem 5. *If (A, ι) is a CM abelian variety defined over a number field L, then*

$$L(A/L, s) = \prod_{\tau : F \hookrightarrow \mathbf{C}} L(\psi_\tau, s),$$

where ψ_τ is the Grössencharakter defined above Theorem 3.

Proof. We check that the local L-functions agree at all primes v of L. By Theorems 3 and 4, we only need to check at places v where A has good reduction, or equivalently where ψ_τ is unramified for all τ. Since $V_\ell(A)$ is a free $F \otimes \mathbf{Q}_\ell$-module of rank 1 on which Fr_v acts as $\alpha(\pi_v) \in F$, the characteristic polynomial of the action of Fr_v on $V_\ell(A)$ is the same as the characteristic polynomial of the action of $\alpha(\pi_v)$ on $F \otimes \mathbf{C} \cong \prod_\tau \mathbf{C}_\tau$, which is precisely

$$\prod_\tau \bigl(X - \tau(\alpha(\pi_v))\bigr) = \prod_\tau \bigl(X - \psi_\tau(\pi_v)\bigr).$$

From this it is clear that $L_v(A/L, T) = \prod_\tau L_v(\psi_\tau, T)$. $\square$

Remark. No ψ_τ is a power of the adelic norm $\|\cdot\|_L$. Indeed, otherwise, by Thm. 3 the Riemann Hypothesis for abelian varieties over finite fields would show that $\psi_\tau = \|\cdot\|_L^{\frac{1}{2}}$. If v is any of the infinitely many degree 1 primes of L, then this would imply that $\psi_\tau(\pi_v)^2 = p$, where p is the rational prime over which v lies. But $\psi_\tau(\pi_v) \in \tau(F^\times)$, so F contains a square root of infinitely many rational primes, a contradiction.

2. The potential CM case

This section will be devoted to generalizing Theorem 5 to situations where A does not necessarily have complex multiplication defined over a given field of definition for A.

2.1. **Restriction of scalars.** We will need to discuss some generalities concerning the behavior of the restriction of scalars functor on abelian varieties.

Let L_0 be an arbitrary field and suppose L/L_0 be a finite separable field extension. Fix an L_0-embedding $L \hookrightarrow \overline{L}_0$, so we get an associated open subgroup $G_L \subseteq G_{L_0}$. If A is an abelian variety defined over L, then the restriction of scalars functor $\mathrm{Res}_{L/L_0} A : \mathrm{Sch}/L_0 \to \mathrm{Ab}$ defined by $X \mapsto A(X \otimes_{L_0} L)$ is represented by an L_0-scheme, which we still denote by $\mathrm{Res}_{L/L_0} A$ or (for ease of notation) by A_*. For the existence of the restriction of scalars without the affineness restriction imposed in Nick's talk [6, Thm. 2.1], see the Néron Models book [1, §7.6 Thm. 4]. As in Nick's talk [6, Thm. 2.5(1)], we have a canonical isomorphism

$$P : A_{*,\overline{L}_0} \xrightarrow{\sim} \prod_{\sigma : L \hookrightarrow \overline{L}_0} A^{\sigma}_{\overline{L}_0}$$

induced by the isomorphism $\overline{L}_0 \otimes_{L_0} L \cong \prod_\sigma (\overline{L}_0)_\sigma$ (both products are taken over embeddings $\sigma : L \hookrightarrow \overline{L}_0$ over L_0). This shows in particular that A_* is an abelian variety.

On $\overline{L}_0$-points, we have an isomorphism

$$A_*(\overline{L}_0) \xrightarrow{\sim} \mathbf{Z}[G_{L_0}] \otimes_{\mathbf{Z}[G_L]} A(\overline{L}_0)$$

of G_{L_0}-modules defined by $(x_\sigma) \mapsto \sum_\sigma \widetilde{\sigma} \otimes \widetilde{\sigma}^{-1} x_\sigma$, where $\widetilde{\sigma}$ is any lift of σ to G_{L_0}. The inverse maps sends $\sigma \otimes x$ to $(0, \ldots, \sigma x, \ldots, 0)$ (all entries 0 except in the $\sigma|_L$ spot). This shows in particular that

$$V_\ell(A_*) \cong \mathbf{Q}_\ell[G_{L_0}] \otimes_{\mathbf{Q}_\ell[G_L]} V_\ell(A)$$

for all rational primes ℓ, i.e., $V_\ell(A_*) \cong \mathrm{Ind}_L^{L_0} V_\ell(A)$ as representations of G_{L_0}.

Proposition 6. *With notation as above,*

$$L(A_*/L_0, s) = L(A/L, s).$$

Proof. Using the description via induced modules, one shows that

$$\prod_{w|v} L_w(A/L, \mathbf{N}w^{-s}) = L_v(A/L_0, \mathbf{N}v^{-s})$$

for all primes v of L_0 by a tedious calculation. See Neukirch [5, Thm. VII.10.4(iv)] for the details. □

2.2. ***L*-functions, bis.** Now assume that A is an abelian variety defined over a number field L_0 such that $(A_{\overline{L}_0}, \iota)$ has CM of type (F, Φ). We assume that $\iota(F) \subseteq \mathrm{End}^0_{\overline{L}_0}(A)$ is stable under the action of G_{L_0}. (This assumption holds in particular when A is absolutely simple, as in this case $\iota(F) = \mathrm{End}^0_{\overline{L}_0}(A)$.) Let $L \subseteq \overline{L}_0$ be the smallest extension of L_0 such that $\iota(F) \subseteq \mathrm{End}^0_L(A)$, so that L/L_0 is a finite Galois extension. Set $G = \mathrm{Gal}(L/L_0)$. This Galois group acts on $\mathrm{End}^0_L(A)$ and hence on F via the embedding ι. We denote the fixed field of this action on F by F_0, so $\iota(F_0) = \iota(F) \cap \mathrm{End}^0_{L_0}(A)$. Note that by definition of L and Artin's lemma, we have that $[F : F_0] = [L : L_0]$ and the action of G on F gives an identification $G \cong \mathrm{Gal}(F/F_0)$. The set $\Sigma = \{\tau : F \hookrightarrow \mathbf{C}\}$ has a natural action of G on the right.

Proposition 7. *In the situation above, there is an L_0-isogeny*

$$A^n \sim \mathrm{Res}_{L/L_0}(A_L),$$

where $n = [L : L_0] = [F : F_0]$.

Proof. We are free to replace A by an isogenous abelian variety, so we may assume without loss of generality that $\iota(\mathcal{O}_{L_0}) \subseteq \mathrm{End}^0_{L_0}(A)$. Let $\mathcal{O}$ be the order of L satisfying $\iota(\mathcal{O}) = \iota(\mathcal{O}_L) \cap \mathrm{End}^0_L(A)$, which is a projective $\mathcal{O}_{L_0}$-module of generic rank n. As the canonical morphism $A \to \mathrm{Res}_{L/L_0}(A_L)$ is $\mathcal{O}_{L_0}$-linear (this can be checked on $\overline{L}_0$-points, e.g.), it extends to an $\mathcal{O}$-linear morphism

$$\beta : \mathcal{O} \otimes_{\mathcal{O}_{L_0}} A \longrightarrow \mathrm{Res}_{L/L_0}(A_L).$$

It suffices to prove that this is an isogeny, so we show that the tangent map at the identity

$$T_0\beta : F \otimes_{F_0} T_0A \longrightarrow T_0(A_L) = L \otimes_{L_0} T_0A$$

is an isomorphism. By our identification $G \cong \mathrm{Gal}(F/F_0)$ and the Normal Basis Theorem, F, resp. L, is a free module of rank 1 over $F_0[G]$, resp. $L_0[G]$. One sees from the definitions that the map $T_0\beta$ is G-equivariant. Therefore, it is enough to show that the associated map $(T_0\beta)^G$ between G-invariant subspaces is an isomorphism, which is clear, since $(T_0\beta)^G$ is the identity map on T_0A. ☐

Lemma 8. *For all $\tau \in \Sigma$ and $g \in G$, we have $L(\psi_\tau, s) = L(\psi_{\tau g}, s)$.*

Proof. We first check that $g\eta|_{L^\times} = \eta g|_{L^\times}$ for all $g \in G$. The map $\eta^{-1} = N_\Phi \circ N_{L/K}$, viewed as a map from $L^\times$ to $F^\times$, is given by $\eta^{-1}(x) = \det_F(x \mid T_0A)$, and if $x : T_0A \to T_0A$ has matrix M with respect to some F-basis $\{t_i\}$ of T_0A, then gx has matrix gM with respect to the F-basis $\{gt_i\}$. Thus, $\eta(gx)^{-1} = \det(gM) = g(\det M) = g(\eta(x)^{-1})$.

We now claim that α also commutes with the action of G, i.e., $g\alpha = \alpha g$ for all $g \in G$. This is clear from the fact that both $g\alpha$ and αg are each equal the unique continuous character $\mathbf{A}_L^\times \to F^\times$ which is equal to $g\eta = \eta g$ on $L^\times$ and sends π_w to the unique endomorphism of A reducing to Fr_{gw} at gw for primes w of L at which A has good reduction.

Let v be a prime of L_0. The above shows that for two primes w and w' of L lying over the same prime v of L_0, ψ_τ is unramified at w if and only if it is unramified at w'. (This can also be seen from the fact that A is defined over L_0 and hence has good reduction at w if and only if it has good reduction at w'.) If these equivalent conditions hold, then, for all $g \in G$, we have

$$\psi_{\tau g}(\pi_w) = \tau g\big(\alpha(\pi_w)\big) = \tau\big(\alpha(\pi_{gw})\big),$$

so

$$\prod_{w|v} L_w(\psi_\tau, T) = \prod_{w|v} L_w(\psi_{\tau g}, T)$$

since g permutes the primes over v. ☐

Theorem 9. *In the above situation,*

$$L(A/L_0, s) = \prod_{\tau \in \Sigma/G} L(\psi_\tau, s).$$

Proof. Let $n = |G| = [L : L_0] = [F : F_0]$ and choose a prime v of L_0. For ease of notation, set $A_* = \mathrm{Res}_{L/L_0}(A_L)$. We compute:

$$\begin{aligned}
L_v(A/L_0, \mathbf{N}v^{-s})^n &= L_v(A_*/L_0, \mathbf{N}v^{-s}) && \text{by Prop. 7}\\
&= \prod_{w|v} L_w(A_L/L, \mathbf{N}w^{-s}) && \text{by Prop. 6}\\
&= \prod_{w|v}\prod_{\tau\in\Sigma} L_w(\psi_\tau, \mathbf{N}w^{-s}) && \text{by Thm. 5}\\
&= \prod_{w|v}\prod_{\tau\in\Sigma/G} L_w(\psi_\tau, \mathbf{N}w^{-s})^n && \text{by Lemma 8}
\end{aligned}$$

It follows that

$$L_v(A/L_0, \mathbf{N}v^{-s}) = \prod_{w|v} \prod_{\tau \in \Sigma/G} L_w(\psi_\tau, \mathbf{N}w^{-s}),$$

since both sides are inverses of polynomials in $\mathbf{N}v^{-s}$ with constant term 1. $\square$

References

1. S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Springer-Verlag, 1990.
2. B. Conrad, *Main Theorem of complex multiplication*, VIGRE CM seminar, 2005.
3. S. Lang, *Complex multiplication*, Springer-Verlag, 1983.
4. J. Milne, *On the arithmetic of abelian varieties*, Invent. Math. **17** (1972), 177–190.
5. J. Neukirch, *Algebraic number theory*, Springer-Verlag, 1999.
6. N. Ramsey, *Weil restriction and adelic points*, VIGRE CM seminar, 2005.
7. J-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517.