MATH 172: MOTIVATION FOR FOURIER SERIES: SEPARATION OF VARIABLES

ANDRÁS VASY

Separation of variables is a method to solve certain PDEs which have a 'warped product' structure. First, on $\mathbb{R}^n$, a linear PDE of order m is of the form

$$\sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha u = f(x),$$

where a_α, f are given functions on $\mathbb{R}^n$, and where we write

$$\partial^\alpha = \partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n},$$

and

$$|\alpha| = \alpha_1 + \ldots + \alpha_n.$$

Typical examples include

(i) Laplace's equation on domains Ω in $\mathbb{R}^n$:

$$\Delta u = f,\ \Delta u = \partial_1^2 u + \ldots + \partial_n^2 u,$$

where u is the electrostatic potential generated by a charge distribution f, or the steady state temperature generated by a heat source f,

(ii) the wave equation on $\mathbb{R}_t \times \Omega$, Ω a domain in $\mathbb{R}^n$,

$$\partial_t^2 u - c^2\Delta_x u = f,$$

where c is the speed of waves, u is the displacement of a membrane from equilibrium, or a component of the electromagnetic field,

(iii) and the heat equation on $(0,\infty)_t \times \Omega$

$$\partial_t u - k\Delta_x u = f$$

where k is (essentially) the heat conductivity, and u is the temperature generated by a heat source distribution f.

In all these cases one needs to impose some boundary conditions if one is working in proper subdomains, and in the latter two some initial conditions. For instance, for the heat equation one may impose the Dirichlet boundary condition (DBC)

$$u|_{\partial\Omega} = 0,\ \text{resp. } u|_{\mathbb{R}\times\partial\Omega} = 0,\ \text{resp. } u|_{(0,\infty)\times\partial\Omega} = 0,$$

representing an electrically grounded or thermally insulating boundary for Laplace's equation, a fixed membrane edge for the wave equation, and a body whose surface is kept at 0 temperature for the heat equation. In addition, one would need to specify the initial temperature

$$u|_{\{0\}\times\Omega} = \phi$$

for the heat equation, ϕ a given function on Ω, and the initial position and velocity for the wave equation

$$u|_{\{0\}\times\Omega} = \phi,\ (\partial_t u)|_{\{0\}\times\Omega} = \psi,$$

with ϕ, ψ given functions on Ω. Another common boundary condition is the Neumann boundary condition,

$$\frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \text{ resp. } \frac{\partial u}{\partial n}|_{\mathbb{R}\times\partial\Omega} = 0, \text{ resp. } \frac{\partial u}{\partial n}|_{(0,\infty)\times\partial\Omega} = 0,$$

representing a thermally insulated boundary for Laplace's equation, a free membrane edge for the wave equation, and an body whose surface is insulated for the heat equation. In all these conditions, the 0 on the right hand side can be replaced by a specific function, e.g. for Laplace's equation, $u|_{\partial\Omega} = h$ means that the boundary is kept at a fixed potential h.

The general idea of separation of variables is the following: suppose we have a linear PDE $Lu = 0$ on a space $M_x \times N_y$. We look for solutions $u(x, y) = X(x)Y(y)$. In general, there are no non-trivial solutions (the identically 0 function being trivial), but in special cases we might be able to find some. We cannot expect even then that all solutions of the PDE are of this form. However, if we have a family

$$u_n(x, y) = X_n(x)Y_n(y),\ n \in \mathcal{I},$$

of separated solutions, where $\mathcal{I}$ is some index set (e.g. the positive integers), then, this being a linear PDE,

$$u(x, y) = \sum_{n\in\mathcal{I}} c_n u_n(x, y) = \sum_{n\in\mathcal{I}} c_n X_n(x)Y_n(y)$$

solves the PDE as well for any constants $c_n \in \mathbb{C}$, $n \in \mathcal{I}$, provided the sum converges in some reasonable sense, and we may be able to choose the constants so that this in fact gives an arbitrary solution of the PDE.

We emphasize that our endeavor, in general, is very unreasonable. Thus, we may make assumptions as we find it fit – we need to justify our results *after* we derive them.

As an example, consider the wave equation

$$u_{tt} - c^2\Delta_x u = 0$$

on $M_x \times \mathbb{R}_t$, where M is the space – for instance, M is $\mathbb{R}^n$, or a cube $[a, b]^n$ or a ball $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \le 1\}$. A separated solution is one of the form $u(x, t) = X(x)T(t)$. Substituting into the PDE yields

$$X(x)T''(t) - c^2T(t)(\Delta_x X)(x) = 0.$$

Rearranging, and assuming T and X do not vanish,

$$\frac{T''(t)}{c^2T(t)} = \frac{\Delta_x X(x)}{X(x)}.$$

Now, the left hand side is a function independent of x, the right hand side is a function independent of t, so they are both equal to a constant, $-\lambda$, namely pick your favorite value of x_0 and t_0, and then for any x and t,

$$\mathrm{RHS}(x) = \mathrm{LHS}(t_0) = \mathrm{RHS}(x_0) = \mathrm{LHS}(t),$$

so the constant in question is $\mathrm{LHS}(t_0)$. Thus, we get two ODEs:

$$T''(t) = -\lambda c^2T(t),$$
$$(\Delta_x X)(x) = -\lambda X(x).$$

Now typically one has additional conditions. For instance, one has boundary conditions at ∂M:

$$u|_{\partial M\times\mathbb{R}} = 0 \text{ (DBC) or}$$
$$\frac{\partial u}{\partial n}|_{\partial M\times\mathbb{R}} = 0 \text{ (NBC)}.$$

Then $X(x)T(t)$ has to satisfy these conditions for all $x \in \partial M$ and all $t \in \mathbb{R}$. Taking some t for which $T(t)$ does not vanish, we deduce that the analogous boundary condition is satisfied, namely

$$X|_{\partial M} = 0 \text{ (DBC) or}$$
$$\frac{\partial X}{\partial n}|_{\partial M} = 0 \text{ (NBC)}.$$

We also have initial conditions, such as

$$u(x, 0) = \phi(x),\ u_t(x, 0) = \psi(x),$$

but as these are not homogeneous, we do not impose these at this point, and hope that we will have sufficient flexibility from the c_n to match these.

We start by solving the ODE for T, which is easy:

$$T(t) = A\cos(\sqrt{\lambda}ct) + B\sin(\sqrt{\lambda}ct),\ \lambda \neq 0,$$

and

$$T(t) = A + Bt$$

if $\lambda = 0$. (We could have used complex exponentials instead. If λ is not positive, the trigonometric functions should be thought of as given by the corresponding complex exponentials.)

Now, in general the spatial equation,

$$-\Delta X = \lambda X,$$
$$X|_{\partial M} = 0 \text{ (DBC) or} \tag{1}$$
$$\frac{\partial X}{\partial n}|_{\partial M} = 0 \text{ (NBC)}$$

is impossible to solve explicitly. However, we point out that it is an *eigenvalue equation* for Δ: the statement is that X is an eigenfunction of $-\Delta$ with eigenvalue λ, in the strong sense that it also satisfies the boundary condition. If we let $X_n(x)$, $n \in \mathbb{N}$, be the eigenfunctions of $-\Delta$ with this boundary condition, with corresponding eigenvalue λ_n, then the conclusion is that

$$u(x, t) = \sum_{n\in\mathbb{N}} A_n \cos(\sqrt{\lambda_n}ct)X_n(x) + B_n \sin(\sqrt{\lambda_n}ct)X_n(x)$$

is the general separated solution. Note that matching the initial conditions requires

$$\phi(x) = \sum_{n\in\mathbb{N}} A_n X_n(x),\ \psi(x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}cB_n X_n(x),$$

i.e. writing the given functions ϕ and ψ as infinite linear combination of the eigenfunctions X_n – of course, in addition to finding some A_n and B_n which *should* work, we need to actually prove that these series indeed converges to the desired limit. Due to the decomposition of u into eigenfunctions of the spatial operator, X, these methods for solving the PDE are also called *spectral methods*.

In a simple situation, such as when $M = [0, \ell]$, we can find the eigenfunctions of $-\Delta$ explicitly. Namely, for Dirichlet boundary condition, the equation is

$$-\frac{d^2X}{dx^2} = \lambda X,\ X(0) = 0 = X(\ell),$$

the solution of the ODE (without the boundary conditions) is

$$\begin{aligned} X(x) &= C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x),\ \lambda \neq 0, \\ X(x) &= C + Dx,\ \lambda = 0. \end{aligned} \tag{2}$$

Evaluating at $x = 0$ and enforcing $X(0) = 0$ yields $C = 0$ in either case. Evaluating at ℓ yields $D = 0$ if $\lambda = 0$, so $\lambda = 0$ is of no interest (we are only interested in non-trivial solutions). If $\lambda \neq 0$, then evaluation at $x = \ell$ yields that either $D = 0$, which again would give a trivial solution, or $\sin(\sqrt{\lambda}\ell) = 0$, which thus we assume is the case. But the zeros of sine are at $n\pi$, $n \in \mathbb{Z}$, so

$$\sqrt{\lambda}\ell = n\pi,$$

hence

$$\lambda = \left(\frac{n\pi}{\ell}\right)^2.$$

Note that n and $-n$ give essentially the same X_n (up to an overall minus sign, which is irrelevant, as we are allowing an arbitrary coefficent D), while $n = 0$ yields the trivial solution, so we conclude that

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right),\ \lambda_n = \left(\frac{n\pi}{\ell}\right)^2,\ n > 0,\ n \in \mathbb{Z}.$$

Returning to the full problem, we deduce that

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n}ct)\sin\left(\frac{n\pi x}{\ell}\right) + B_n \sin(\sqrt{\lambda_n}ct)\sin\left(\frac{n\pi x}{\ell}\right).$$

It remains to determine the coefficients A_n and B_n, which is a subject of the next lecture.

Next, we consider the Neumann boundary conditions.

$$-\frac{d^2X}{dx^2} = \lambda X,\ X'(0) = 0 = X'(\ell),$$

Then (2) still holds. Substituting in $X'(0) = 0$ yields $D = 0$ whether $\lambda = 0$ or not. If $\lambda = 0$, the constant C satisfies the boundary condition at ℓ, so 1 is an eigenfunction with eigenvalue 0. If $\lambda \neq 0$, we obtain the requirement (as we want a non-trivial solution)

$$\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) = 0,$$

hence

$$\sqrt{\lambda}\ell = n\pi,\ n \in \mathbb{Z},$$

as above. Since $\lambda \neq 0$, we in fact have $n \neq 0$ in this case. Again, n and $-n$ give essentially the same eigenfunctions (in fact, exactly!), so our eigenfunctions are

$$\begin{aligned} &X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right),\ \lambda_n = \left(\frac{n\pi}{\ell}\right)^2,\ n > 0,\ n \in \mathbb{Z} \\ &X_0(x) = 1,\ \lambda_0 = 0. \end{aligned}$$

Note that $n = 0$ can be considered simply a special case of the general formula in this particular situation, and thus we may write our answer as

$$X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right),\ \lambda_n = \left(\frac{n\pi}{\ell}\right)^2,\ n \geq 0,\ n \in \mathbb{Z},$$

hence the general separated solution as

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n}ct)\cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin(\sqrt{\lambda_n}ct)\cos\left(\frac{n\pi x}{\ell}\right).$$

Next, consider the heat equation on a space M, i.e.

$$u_t - k\Delta_x u = 0$$

with either Dirichlet or Neumann boundary conditions. For separated solutions $u(x,t) = X(x)T(t)$ we obtain

$$X(x)T'(t) - kT(t)\Delta_x X = 0,$$

so

$$\frac{T'}{kT} = \frac{\Delta_x X}{X}.$$

Again, both sides must be equal to a constant, $-\lambda$. The ODE for T,

$$T' = -\lambda k T,$$

is then easy to solve,

$$T(t) = Ae^{-\lambda k t}.$$

The PDE for X still has boundary conditions, and is

$$\begin{aligned} &-\Delta X = \lambda X, \\ &X|_{\partial M} = 0 \text{ (DBC) or} \\ &\frac{\partial X}{\partial n}|_{\partial M} = 0 \text{ (NBC)}, \end{aligned} \tag{3}$$

the same as (1). The solution is thus also the same. For instance, for Dirichlet boundary conditions on $[0,\ell]$,

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right),\ \lambda_n = \left(\frac{n\pi}{\ell}\right)^2,\ n > 0,\ n \in \mathbb{Z},$$

hence the general separated solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n t} \sin\left(\frac{n\pi x}{\ell}\right).$$

Similarly, for Neumann boundary conditions we obtain

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-k\lambda_n t} \cos\left(\frac{n\pi x}{\ell}\right).$$

Similar techniques also work for some other problems with less product-type behavior. Thus, consider Laplace's equation on the disk of radius R,

$$D = \mathbb{B}^2_R = \{x \in \mathbb{R}^2 :\ |x| < R\},$$

namely

$$\begin{aligned} &\Delta u = 0,\ x \in D, \\ &u|_{\partial D} = h, \end{aligned}$$

with h a given function on ∂D. We again ignore the inhomogeneous conditions, so we are left with the PDE. Now, we need to think of the disk as a product space. This is not the case in Cartesian coordinates. However, in polar coordinates, we can identify D with

$$[0,R)_r \times \mathbb{S}^1_\theta,$$

where $\mathbb{S}^1$ is the unit circle. The Laplacian in polar coordinates is

$$\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2.$$

Note that polar coordinates are singular at $r = 0$ (the whole circle at $r = 0$ is squashed into a single point), and Δ has coefficients which are not smooth at $r = 0$. We again consider separated solutions,

$$u(r, \theta) = R(r)\Theta(\theta).$$

Substituting into the PDE yields

$$R''\Theta + r^{-1}R'\Theta + r^{-2}\Theta''R = 0.$$

Dividing by ΘR and multiplying by r^2:

$$-\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta},$$

and as before both sides must be equal to a constant, $-\lambda$.

Thus, the Θ ODE is

$$\Theta'' = -\lambda\Theta,$$

where Θ is a function on the circle $\mathbb{S}^1$. A function on $\mathbb{S}^1$ can be thought of as a function on $[0, 2\pi]$ such that the values at 0 and 2π agree (since they represent the same point on $\mathbb{S}^1$). However, a better way of thinking of it as a 2π-periodic function on $\mathbb{R}$: the points $\theta + 2n\pi$, $n \in \mathbb{Z}$, correspond to the same point in the circle. Explicitly, if the circle is considered the unit circle in $\mathbb{R}^2$, as usual, the map $\mathbb{R} \to \mathbb{S}^1$ is

$$\mathbb{R} \ni \theta \mapsto (\cos\theta, \sin\theta) \in \mathbb{S}^1,$$

making the 2π-periodicity clear. Thus, we want solutions of the Θ ODE, considered as an ODE on the real line, which are 2π-periodic. Now, without the periodicity condition, the solution of the ODE is, as before (when this was the ODE for T),

$$\Theta(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta),\ \lambda \neq 0,$$

and

$$\Theta(\theta) = A + B\theta$$

if $\lambda = 0$. Adding the 2π periodicity condition we need that $\sqrt{\lambda} = n \in \mathbb{Z}$ if $\lambda \neq 0$, and $B = 0$ if $\lambda = 0$. If $\lambda \neq 0$, then this gives $\lambda = n^2$. As positive and negative values of n give rise to the same eigenfunctions, we can restrict to $n > 0$ (note $n \neq 0$ if $\lambda \neq 0$). Thus, in summary, the solutions are

$$\begin{aligned}\Theta_n(\theta) &= A_n\cos(n\theta) + B_n\sin(n\theta),\ n > 0,\ n \in \mathbb{Z},\\ \Theta_0(\theta) &= A_0.\end{aligned}$$

It remains to deal with the R ODE which is

$$r^2R'' + rR' - \lambda R = 0.$$

Note that the coefficient of the highest derivative, R^2, vanishes at $r = 0$, so this ODE is degenerate, or singular, there. However, note that each derivative comes with a factor of r, thus R' (one derivative falling on R) has a factor of r in the coefficient, R'' (two derivatives falling on R) has a factor of r^2. A slightly better way to rewrite it is

$$r(rR')' - \lambda R = 0,$$

then explicitly you can see that each time you differentiate, you multiply by r. Such an ODE is called a regular singular ODE (in general, one could multiply by a smooth function times r each times one differentiates), and can (usually) be solved in a power series (in general with non-integer powers) around $r = 0$, with possible logarithmic terms. Our ODE is a particularly simple regular singular ODE. This can be seen by changing variables $t = \log r$, writing $T(r) = \log r$, so $\frac{dT}{dr} = r^{-1}$, for then for a function f on $\mathbb{R}$,

$$r\frac{d(f \circ T)}{dr} = r\left(\frac{df}{dt} \circ T\right)\frac{dT}{dr} = \frac{df}{dt} \circ T,$$

or informally,

$$r\frac{df}{dr} = r\frac{df}{dt}\frac{dt}{dr} = \frac{df}{dt},$$

so the ODE for $F(t) = R(e^t)$ becomes

$$\frac{d^2}{dt^2}F - \lambda F = 0,$$

so

$$\begin{aligned} &F(t) = Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t},\ \lambda \neq 0,\\ &F(t) = A + Bt,\ \lambda = 0. \end{aligned}$$

Using $t = \log r$, we get for the original function R,

$$\begin{aligned} &R(r) = Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}},\ \lambda \neq 0,\\ &R(r) = A + B\log r,\ \lambda = 0. \end{aligned}$$

Recalling that $\sqrt{\lambda}$ is a non-negative integer from the Θ ODE, we conclude that

$$\begin{aligned} &R(r) = Ar^n + Br^{-n},\ n > 0,\ n \in \mathbb{Z},\\ &R(r) = A + B\log r,\ n = 0. \end{aligned}$$

Now, there is nothing special about the origin: it is just a point in the interior of the disk, and we know by elliptic regularity that the solution of Laplace's equation should be $\mathcal{C}^\infty$ in the disk, so in particular it should be bounded, hence we throw out the exponents that would yield unbounded terms to get

$$\begin{aligned} &R_n(r) = r^n,\ n > 0,\ n \in \mathbb{Z},\\ &R_0(r) = 1,\ n = 0, \end{aligned}$$

and the $n = 0$ case could be simply included in the $n > 0$ one by letting $n \geq 0$ there. Note that the 'badly behaved' solutions r^{-n} and $\log r$ arose because we used badly behaved 'coordinates' on D: recall that these themselves were singular at the origin.

Combining this with our results for Θ, we obtain the general separated solution

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta)).$$

Again, the question is whether we can determine the coefficients to match the boundary conditions. Explicitly, the boundary condition is

$$h(\theta) = A_0 + \sum_{n=1}^{\infty} R^n(A_n\cos(n\theta) + B_n\sin(n\theta)),$$

so we need to write the given function h as an infinite linear combinations of the sines and cosines, and discuss convergence of the result. This is the topic of the next lectures.