MATH 172: CONVERGENCE OF THE FOURIER SERIES

ANDRÁS VASY

We now discuss convergence of the Fourier series on compact intervals I. 'Convergence' depends on the notion of convergence we use, such as

(i) L^2: $u_j \to u$ in L^2 if $\|u_j - u\|_{L^2} \to 0$ as $j \to \infty$.
(ii) uniform, or C^0: $u_j \to u$ uniformly if $\|u_j - u\|_{C^0} = \sup_{x\in I} |u_j(x) - u(x)| \to 0$.
(iii) uniform with all derivatives, or $\mathcal{C}^\infty$: $u_j \to u$ in $\mathcal{C}^\infty$ if for all non-negative integers k, $\sup_{x\in I} |\partial^k u_j(x) - \partial^k u(x)| \to 0$.
(iv) pointwise: $u_j \to u$ pointwise if for each $x \in I$, $u_j(x) \to u(x)$, i.e. for each $x \in I$, $|u_j(x) - u(x)| \to 0$.

Note that pointwise convergence is too weak for most purposes, so e.g. just because $u_j \to u$ pointwise, it does not follow that $\int_I u_j(x)\,dx \to \int u(x)\,dx$. This would follow, however, if one assumes uniform convergence, or indeed L^2 (or L^1) convergence, since

$$\Big|\int_I u_j(x)\,dx - \int_I u(x)\,dx\Big| = \Big|\int_I (u_j(x) - u(x))\,dx\Big| = |\langle u_j - u, 1\rangle| \leq \|u_j - u\|_{L^2}\|1\|_{L^2}.$$

Note also that uniform convergence implies L^2 convergence since

$$\begin{aligned}\|u_j - u\|_{L^2} &= \left(\int_I |u_j - u|^2\,dx\right)^{1/2} \\ &\leq \left(\int_I \|u_j - u\|_{C^0}^2\,dx\right)^{1/2} = \left(\int_I 1\,dx\right)^{1/2} \|u_j - u\|_{C^0},\end{aligned}$$

so if $u_j \to u$ uniformly, it also converges in L^2. Uniform convergence also implies pointwise convergence directly from the definition. On the other hand, convergence in $\mathcal{C}^\infty$ implies uniform convergence directly from the definition.

On the failure of convergence side: the uniform limit of a sequence of continuous functions is continuous, so in view of the continuity of the complex exponentials, sines and cosines, the various Fourier series cannot converge uniformly *unless the limit is continuous*. On the other hand, even if the limit is continuous, the convergence may not be uniform: understanding conditions, under which it is, is one of our first tasks.

There are two issues regarding convergence: whether the series in question converges at all (in whatever sense we are interested in), and second whether it converges to the desired limit, in this case the function whose Fourier series we are considering. The first part is easier to answer: we have already seen that even the generalized Fourier series converges in L^2 (but not necessarily to the function!).

Now consider uniform convergence. Recall that the typical way one shows convergence of a series is that to show that each term in absolute value is $\leq M_n$, where M_n is a non-negative constant, such that $\sum_n M_n$ converges. Similarly, one shows that a series $\sum_n u_n(x)$ converges uniformly by showing that there are non-negative constants M_n such that $\sup_{x\in I} |u_n(x)| \leq M_n$ and such that $\sum_n M_n$ converges.

The Weierstrass M-test is the statement that under these assumptions the series $\sum_n u_n(x)$ converges uniformly. In particular, if one shows that for $n \neq 0$ this boundedness holds with $M_n = C/|n|^s$ where $C > 0$ and $s > 1$, the Weierstrass M-test shows that the series converges uniformly. (Since a finite number of terms do not affect convergence, one can always ignore a finite number of terms if it is convenient.)

So suppose that ϕ is a function on $[-\ell, \ell]$ and its 2ℓ-periodic extension, denoted by ϕ_{ext}, is C^k, $k \geq 1$ integer. The full Fourier series is

$$\sum_{n=-\infty}^{\infty} C_n e^{in\pi x/\ell},\ C_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} \phi(x)\, e^{-in\pi x/\ell}\, dx. \tag{1}$$

This gives us the bound

$$|C_n| \leq \frac{1}{2\ell}\int_{-\ell}^{\ell} |\phi(x)\, e^{-in\pi x/\ell}|\, dx = \frac{1}{2\ell}\int_{-\ell}^{\ell} |\phi(x)|\, dx \leq \sup_{x\in[-\ell,\ell]} |\phi(x)|, \tag{2}$$

so the coefficients are bounded, but it is not clear if they have any decay, hence the uniform convergence of the Fourier series is unclear.

Now we integrate by parts, putting additional derivatives on ϕ. Thus, for $n \neq 0$,

$$C_n = \frac{1}{2\ell}\left(\frac{\ell}{in\pi}\right)\int_{-\ell}^{\ell} (\partial_x\phi)(x)\, e^{-in\pi x/\ell}\, dx,$$

since the boundary terms vanish in view of the fact that ϕ_{ext} is C^k. Note that the right hand side looks like the original expression for C_n, except that ϕ has been replaced by $\partial_x\phi$, and that the factor $\frac{\ell}{in\pi}$ appeared in front. Thus, repeating this argument k times, we deduce that

$$C_n = \frac{1}{2\ell}\left(\frac{\ell}{in\pi}\right)^k \int_{-\ell}^{\ell} (\partial_x^k\phi)(x)\, e^{-in\pi x/\ell}\, dx.$$

In particular, using the analogue of the estimate (2) to estimate the integral, we deduce that

$$|C_n| \leq \left(\frac{\ell}{|n|\pi}\right)^k \sup_{x\in[-\ell,\ell]} |(\partial_x^k\phi)(x)| = \frac{C}{|n|^k},\ C = \left(\frac{\ell}{\pi}\right)^k \sup_{x\in[-\ell,\ell]} |(\partial_x^k\phi)(x)|.$$

In particular, for $k \geq 2$, we deduce by the Weierstrass M-test that the Fourier series converges uniformly.

It is possible to improve this conclusion as follows. Note that

$$C_n = \left(\frac{\ell}{in\pi}\right)^k C_{k,n},\ n \neq 0,$$

where $C_{k,n}$ is the Fourier coefficient of $\partial_x^k\phi$. Thus, as long as $\partial_x^k\phi$ is continuous, or indeed L^2, we have from Bessel's inequality, with $X_n(x) = e^{in\pi x/\ell}$, that

$$2\ell \sum_{n\in\mathbb{Z}} |C_{k,n}|^2 = \sum_{n\in\mathbb{Z}} |C_{k,n}|^2 \|X_n\|^2 \leq \|\partial_x^k\phi\|_{L^2}^2.$$

Now, in order to apply the Weierstrass M-test we need that $\sum_{n\in\mathbb{Z}} |C_n|$ converges. We now use the Cauchy-Schwarz inequality for sequences, with inner product on the sequences given by

$$\langle \{a_n\}, \{b_n\}\rangle_{\ell^2} = \sum_{n\in\mathbb{Z}} a_n \overline{b_n}.$$

Below by $\left\{\left(\frac{\ell}{|n|\pi}\right)^k\right\}$ we mean the sequence whose $n=0$ entry is 0. We thus obtain using $k>1/2$ that

$$\begin{aligned}\sum_{n\in\mathbb{Z},\ n\neq 0}|C_n| &= \sum_{n\in\mathbb{Z},\ n\neq 0}\left(\frac{\ell}{|n|\pi}\right)^k|C_{k,n}| = \left\langle \{|C_{k,n}|\}, \left\{\left(\frac{\ell}{|n|\pi}\right)^k\right\}\right\rangle \\ &\leq \|\{|C_{k,n}|\}\|_{\ell^2}\left\|\left\{\left(\frac{\ell}{|n|\pi}\right)^k\right\}\right\|_{\ell^2} = \left(\sum_{n\in\mathbb{Z}}|C_{k,n}|^2\right)^{1/2}\left(\sum_{n\in\mathbb{Z},\ n\neq 0}\left(\frac{\ell}{|n|\pi}\right)^{2k}\right)^{1/2} \\ &\leq \frac{1}{\sqrt{2\ell}}\|\partial_x^k\phi\|_{L^2}\left(\sum_{n\in\mathbb{Z},\ n\neq 0}\left(\frac{\ell}{|n|\pi}\right)^{2k}\right)^{1/2},\end{aligned}$$

so it indeed converges. We thus deduce by the Weierstrass M-test that the Fourier series converges uniformly if merely $k\geq 1$.

Now, the term-by-term m-times differentiated Fourier series is

$$\sum_{n=-\infty}^{\infty}C_n\left(\frac{in\pi}{\ell}\right)^m e^{in\pi x/\ell},\ C_n=\frac{1}{2\ell}\int_{-\ell}^{\ell}\phi(x)\,e^{-in\pi x/\ell}\,dx,$$

so for $n\neq 0$ its coefficients satisfy the estimate

$$\begin{aligned}&\left|C_n\left(\frac{in\pi}{\ell}\right)^m\right| \leq \left(\frac{\ell}{|n|\pi}\right)^{k-m}\sup_{x\in[-\ell,\ell]}|(\partial_x^k\phi)(x)| = \frac{C}{|n|^{k-m}}, \\ &C=\left(\frac{\ell}{\pi}\right)^k\sup_{x\in[-\ell,\ell]}|(\partial_x^k\phi)(x)|.\end{aligned}$$

We thus deduce that the term by term m times differentiated Fourier series converges uniformly for $m\leq k-2$, and thus (by the standard analysis theorem) the uniform limit of the Fourier series is actually $k-2$-times differentiable, with derivative given by the term-by-term differentiated series. Modifying this argument as above, using Bessel's inequality, we in fact get that the Fourier series is actually $k-1$-times differentiable, with derivative given by the term-by-term differentiated series.

In the extreme case, when the 2ℓ-periodic extension of ϕ is $\mathcal{C}^\infty$, this tells us that in fact the Fourier series converges in $\mathcal{C}^\infty$. One way of summarizing our results thus far is that the map

$$T:\phi\mapsto\sum_{n=-\infty}^{\infty}C_n e^{in\pi x/\ell},\ C_n=\frac{1}{2\ell}\int_{-\ell}^{\ell}\phi(x)\,e^{-in\pi x/\ell}\,dx,$$

is continuous as a map

$$T:L^2([-\ell,\ell])\to L^2([-\ell,\ell]),$$

and also as

$$T:C^1_{\text{per}}([-\ell,\ell])\to C^0_{\text{per}}([-\ell,\ell]),$$

as well as

$$T:C^\infty_{\text{per}}([-\ell,\ell])\to C^\infty_{\text{per}}([-\ell,\ell]).$$

where the subscript per states that the 2ℓ-periodic extensions of these functions must have the stated regularity properties. Note the loss of derivatives as a map on C^1, or more generally C^k. We would like to say that this is the identity map in each case; by the density of $C^\infty_{\text{per}}([-\ell,\ell])$ in these spaces it would suffice to show this in the last case.

A more systematic way of achieving the same conclusion regarding convergence is the following. The functions $X_n(x)=e^{in\pi x/\ell}$ are eigenfunctions of $A=-i\frac{d}{dx}$ with periodic boundary conditions and with eigenvalue $\lambda_n=n\pi/\ell$. Now, in general, suppose that X_n are orthogonal eigenfunctions of a symmetric operator $A:D\to V$ on an inner product space V with eigenvalue λ_n (which are thus real), and suppose that $\phi,A\phi,\ldots,A^k\phi\in D$. Then for n such that $\lambda_n\neq 0$, the generalized Fourier coefficients

$$C_n=\frac{\langle\phi,X_n\rangle}{\|X_n\|^2}$$

satisfy

$$C_n=\lambda_n^{-1}\frac{\langle\phi,\lambda_nX_n\rangle}{\|X_n\|^2}=\lambda_n^{-1}\frac{\langle\phi,AX_n\rangle}{\|X_n\|^2}=\lambda_n^{-1}\frac{\langle A\phi,X_n\rangle}{\|X_n\|^2}.$$

Repeating k-times, we deduce that

$$C_n=\lambda_n^{-k}\frac{\langle A^k\phi,X_n\rangle}{\|X_n\|^2}.$$

Thus,

$$|C_n|\leq|\lambda_n|^{-k}\|A^k\phi\|_{L^2}\|X_n\|_{L^2}^{-1}.$$

In the special case $X_n(x)=e^{in\pi x/\ell}$, we get the estimate

$$|C_n|\leq\left(\frac{\ell}{|n|\pi}\right)^k\|\partial^k\phi\|_{L^2}\frac{1}{2\ell}=\frac{C}{|n|^k},\ C=\frac{1}{2\ell}\left(\frac{\ell}{\pi}\right)^k\|\partial^k\phi\|_{L^2},$$

which is almost the same estimate we had beforehand, with a slightly different constant, and the norm of the derivative we use is weaker, not the uniform (or sup) norm, but the L^2-norm. But now note that this argument works even for instance for the cosine and sine Fourier series, using that both sine and cosine are bounded by 1 in absolute value, i.e. we do not have to write down these cases separately.

We have thus shown that under appropriate assumptions, depending on the notion of convergence, the various kinds of Fourier series all converge. Note that we may need stronger assumptions than the kind of convergence we would like, for instance we needed to know something about derivatives of ϕ to conclude uniform convergence. However, we still need to discuss *what* the Fourier series converge to!

Note that when the Fourier sine series,

$$\sum_{n=1}^\infty B_n\sin\left(n\pi x/\ell\right),\ B_n=\frac{2}{\ell}\int_0^\ell\phi(x)\,\sin(n\pi x/\ell)\,dx,$$

converges, as it does say in L^2 when $\phi\in L^2$, or uniformly if ϕ is C^2 and satisfies the boundary conditions $\phi(0)=0=\phi(\ell)$, then it converges to an odd 2ℓ-periodic function since each term $\sin(n\pi x/\ell)$ is such. Similarly, the Fourier cosine series, when it converges, converges to an even 2ℓ-periodic function. Moreover, note that if a function Φ is odd on $[-\ell,\ell]$, its full Fourier series,

$$A_0' + \sum_{n=1}^{\infty} A_n' \cos(n\pi x/\ell) + B_n' \sin(n\pi x/\ell)\,,$$

$$A_0' = \frac{1}{2\ell}\int_{-\ell}^{\ell} \Phi(x)\,dx,\ \ A_n' = \frac{1}{\ell}\int_{-\ell}^{\ell} \Phi(x)\,\cos(n\pi x/\ell)\,dx,$$

$$B_n' = \frac{1}{\ell}\int_{-\ell}^{\ell} \Phi(x)\,\sin(n\pi x/\ell)\,dx,\ \ n \geq 1,$$

satisfies

$$A_0' = 0,\ A_n' = 0,\ B_n' = \frac{2}{\ell}\int_0^{\ell} \Phi(x)\,\sin(n\pi x/\ell)\,dx,$$

i.e. if Φ is the odd 2ℓ-periodic extension of ϕ then $B_n' = B_n$, B_n being the Fourier sine coefficient of ϕ on $[0,\ell]$. In view of an analogous argument for the Fourier cosine series, convergence issues for both the Fourier sine and cosine series on $[0,\ell]$ can be reduced to those for the full Fourier series on $[-\ell,\ell]$, so we only consider the latter. Moreover, the change of variables of $\theta = \frac{n\pi x}{\ell}$ preserves all the notions of convergence, so it suffices to consider the Fourier series on $[-\pi,\pi]$. (The general case works directly, by the same arguments, but we have to write less after this rescaling.)

One way of proving the convergence of the Fourier series is the following modification of Hörmander's proof of the Fourier inversion formula. We write $\mathcal{C}^\infty(\mathbb{S}^1) = \mathcal{C}^\infty_{\text{per}}([-\pi,\pi])$, and $s(\mathbb{Z})$ for sequences $\{c_n\}_{n\in\mathbb{Z}}$ such that $|n|^k|c_n|$ is bounded for all $k \geq 0$ (s stands for 'Schwartz sequences'), with notion of convergence that a sequence $\{\{c_{j,n}\}_{n\in\mathbb{Z}}\}_{j=1}^\infty$ converges to $\{c_n\}_{n\in\mathbb{Z}}$ in $s(\mathbb{Z})$ as $j \to \infty$ if for all $k \geq 0$ integer,

$$\sup_{n\in\mathbb{Z}} |n|^k |c_{j,n} - c_n| \to 0$$

as $j \to \infty$. Thus, the map

$$\mathcal{F} : \mathcal{C}^\infty(\mathbb{S}^1) \to s(\mathbb{Z}),\ (\mathcal{F}\phi)_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)\,e^{-in\theta}\,d\theta,$$

is continuous, and it satisfies

$$\mathcal{F}\frac{d\phi}{d\theta} = in\mathcal{F}\phi,$$

and

$$(\mathcal{F}(e^{i\theta}\phi))_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)\,e^{-i(n-1)\theta}\,d\theta = (\mathcal{F}\phi)_{n-1}.$$

Similarly, with

$$\mathcal{F}^{-1} : s(\mathbb{Z}) \to \mathcal{C}^\infty(\mathbb{S}^1),\ \mathcal{F}^{-1}\{c_n\}(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta},$$

we have

$$\mathcal{F}^{-1}\{inc_n\} = \frac{d}{d\theta}\mathcal{F}^{-1}\{c_n\},$$

while

$$\mathcal{F}^{-1}\{c_{n-1}\} = \sum_{n=-\infty}^{\infty} c_{n-1}e^{in\theta} = \sum_{n=-\infty}^{\infty} c_n e^{i(n+1)\theta} = e^{i\theta}\mathcal{F}^{-1}\{c_n\}.$$

Thus, the map $T = \mathcal{F}^{-1}\mathcal{F} : \mathcal{C}^\infty(\mathbb{S}^1) \to \mathcal{C}^\infty(\mathbb{S}^1)$ satisfies

$$T\frac{d}{d\theta} = \frac{d}{d\theta}T,\ Te^{i\theta} = e^{i\theta}T.$$

We have a lemma:

Lemma 0.1. *Suppose $T : \mathcal{C}^\infty(\mathbb{S}^1) \to \mathcal{C}^\infty(\mathbb{S}^1)$ is linear, and commutes with $e^{i\theta}$ and $\frac{d}{d\theta}$. Then T is a scalar multiple of the identity map, i.e. there exists $c \in \mathbb{C}$ such that $Tf = cf$ for all $f \in \mathcal{C}^\infty(\mathbb{S}^1)$.*

Proof. Let $\omega \in \mathbb{R}$. We show first that if $\phi(\omega) = 0$ and $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$ then $(T\phi)(\omega) = 0$. Indeed, if we let

$$\phi_1(\theta) = \phi(\theta)/(e^{i\theta} - e^{i\omega})$$

then by Taylor's theorem (or L'Hopital's rule) ϕ_1 is $\mathcal{C}^\infty$, and it is 2π-periodic as both the denominator and the numerator are. Thus, $\phi = (e^{i\theta} - e^{i\omega})\phi_1$, and hence

$$T\phi = (e^{i\theta} - e^{i\omega})(T\phi_1),$$

where we used that T is linear and commutes with multiplication by $e^{i\theta}$. Substituting in $\theta = \omega$ yields $(T\phi)(\omega) = 0$ indeed.

Thus, fix $\omega \in \mathbb{R}$, and let $g \in \mathcal{C}^\infty(\mathbb{S}^1)$ be the function $g \equiv 1$. Let $c(\omega) = (Tg)(\omega)$; thus $c \in \mathcal{C}^\infty(\mathbb{S}^1)$. We claim that for $f \in \mathcal{C}^\infty(\mathbb{S}^1)$,

$$(Tf)(\omega) = c(\omega)f(\omega).$$

Indeed, let $\phi(\theta) = f(\theta) - f(\omega)g(\theta)$, so $\phi(\omega) = f(\omega) - f(\omega)g(\omega) = 0$. Thus, $0 = (T\phi)(\omega) = (Tf)(\omega) - f(\omega)(Tg)(\omega) = (Tf)(\omega) - c(\omega)f(\omega)$, proving our claim.

We have not used that T commutes with ∂_θ so far. But

$$\begin{aligned} c(\omega)(\partial_\theta f)(\omega) &= T(\partial_\theta f)(\omega) = \partial_\theta(Tf)|_{\theta=\omega} = \partial_\theta(c(\theta)f(\theta))|_{\theta=\omega} \\ &= (\partial_\theta c)(\omega)f(\omega) + c(\omega)(\partial_\theta f)(\omega). \end{aligned}$$

Comparing the two sides, and taking f such that f never vanishes, yields

$$(\partial_\theta c)(\omega) = 0$$

for all ω. Thus, c is a constant, proving the lemma. □

The constant c can be evaluated by applying T to a single function, e.g. to $f \equiv 1$, which yields $(\mathcal{F}f)_n = 0$ if $n \neq 0$, $(\mathcal{F}f)_0 = 1$, hence $Tf = f$, so T is indeed the identity map.

Now, if $\phi \in L^2([-\pi, \pi])$, we take ϕ_n whose 2π-periodic extensions are $\mathcal{C}^\infty$, which converge to ϕ in L^2. Since $T\phi_n = \phi_n$, and $T : L^2([-\pi, \pi]) \to L^2([-\pi, \pi])$ is continuous, we deduce that

$$T\phi = \lim_{n\to\infty} T\phi_n = \lim_{n\to\infty} \phi_n = \phi,$$

i.e. the Fourier series converges in L^2 to ϕ.

If $\phi \in C^1(\mathbb{S}^1)$, then we have seen that the Fourier series of ϕ converges uniformly to some function ψ, and as $C^1(\mathbb{S}^1) \subset L^2(\mathbb{S}^1)$, it converges in L^2 to ϕ. But uniform convergence implies L^2 convergence, so in fact the Fourier series converges in L^2 to ψ, so $\psi = \phi$, and thus we conclude that the Fourier series of ϕ converges uniformly to ϕ.

As we could reduce all the Fourier series we have considered to the full Fourier series on $[-\pi, \pi]$, we deduce the following result:

Theorem 0.2. *The following orthogonal sets of functions are complete in the respective vector spaces V, and thus the corresponding Fourier series of any $\phi \in V$ converges in L^2 to ϕ.*

(i) *In $V = L^2([0,\ell])$, with the standard inner product,*

$$f_n(x) = \sin\left(\frac{n\pi x}{\ell}\right), \; n \geq 1, \; n \in \mathbb{Z}.$$

(ii) *In $V = L^2([0,\ell])$, with the standard inner product,*

$$f_0(x) = 1, \; f_n(x) = \cos\left(\frac{n\pi x}{\ell}\right), \; n \geq 1, \; n \in \mathbb{Z}.$$

(iii) *In $V = L^2([-\ell,\ell])$, with the standard inner product,*

$$f_n(x) = e^{in\pi x/\ell}, \; n \in \mathbb{Z}.$$

(iv) *In $V = L^2([-\ell,\ell])$, with the standard inner product,*

$$f_0(x) = 1, \; f_n(x) = \cos\left(\frac{n\pi x}{\ell}\right), \; n \geq 1, \; n \in \mathbb{Z},$$
$$g_n(x) = \sin\left(\frac{n\pi x}{\ell}\right), \; n \geq 1, \; n \in \mathbb{Z}.$$

Moreover, if the appropriate extension, namely

(i) *odd, 2ℓ-periodic,*
(ii) *even, 2ℓ-periodic,*
(iii) *2ℓ-periodic,*
(iv) *2ℓ-periodic,*

of ϕ is C^1, then the corresponding Fourier series converges to ϕ uniformly, and if the appropriate extension is C^∞, then the convergence is in C^∞.

We discuss a more traditional proof, using the Dirichlet kernel, in the Appendix.

We also connect these results to solving the Dirichlet problem for the Laplacian on the disk of radius R,

$$D = \mathbb{B}^2_R = \{x \in \mathbb{R}^2 : \; |x| < R\},$$

namely

$$\Delta u = 0, \; x \in D,$$
$$u|_{\partial D} = h,$$

with h a given function on ∂D. The general separated solution was

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)), \tag{3}$$

and using orthogonality, we have deduced the coefficients must be

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\theta)\, d\theta,$$
$$A_n = \frac{1}{\pi R^n} \int_0^{2\pi} h(\theta)\, \cos(n\theta)\, d\theta,$$
$$B_n = \frac{1}{\pi R^n} \int_0^{2\pi} h(\theta)\, \sin(n\theta)\, d\theta.$$

Now, if h is C^1, then we have seen that its Fourier series converges uniformly to h, i.e. $u(R,\theta) = h(\theta)$. Also, if h is C^2, then $|A_n|, |B_n| \leq C/R^n n^2$ for $n \geq 1$. This suffices to conclude that for $r < R$,

$$|r^n(A_n \cos(n\theta) + B_n \sin(n\theta))| \leq \frac{2C}{n^2}\left(\frac{r}{R}\right)^n.$$

By the Weierstrass M-test, the series (3) converges uniformly on the closed disk, i.e. for $r \in [0,R]$, and in addition the term-by-term m-times differentiated series still converges uniformly in $[0,\rho]_r$ for any $\rho < R$ since the factors of n^m this differentiation gives are counterbalanced by $\left(\frac{\rho}{R}\right)^n$, and $\sum_{n=1}^{\infty} n^m \alpha^n$ converges whenever $m \in \mathbb{R}$ and $\alpha \in (0,1)$, so u is a $\mathcal{C}^\infty$ function of r and θ for $r \in [0,R)$. A simple modification of our argument for uniform convergence of the Fourier series of C^1 functions would in fact extend this conclusion to h merely C^1. As $v = r^n \cos(n\theta)$ and $v = r^n \sin(n\theta)$ solve $\Delta v = 0$ for $r \in (0,R)$, u itself satisfies $\Delta u = 0$ for $r \in (0,R)$ – the origin may a priori be a problem, since polar coordinates are singular there. That this is not the case follows from $r^n e^{in\theta} = (re^{i\theta})^n = (x+iy)^n$, so taking real and imaginary parts shows that $r^n\cos(n\theta)$ and $r^n \sin(n\theta)$ are simply polynomials in D, and in particular the term-by-term differentiated series, differentiated with respect to x or y, actually still converges in $|x|^2+|y|^2 < R$, uniformly in $|x|^2+|y|^2 \leq \rho$, $\rho < R$, so in fact u is $\mathcal{C}^\infty$ at the origin as well, and solves $\Delta u = 0$ for each summand does so.

Thus, for at least $h \in C^2$ (and as a simple argument shows, $h \in C^1$ in fact), we have solved the PDE: we found $u \in C^2(D) \cap C^0(\overline{D})$ solving our problem. It turns out that one can write the solution in a more explicit form, which enables one to obtain a better result. Namely, for $r \leq \rho$, where $\rho < R$, in view of the absolute convergence of the series (with plenty of room left, so one could even differentiate arbitrarily many times and have absolute and uniform convergence),

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\omega)\,d\omega \\ &\quad + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \frac{1}{\pi}\int_{-\pi}^{\pi} h(\omega)\left(\cos(n\omega)\cos(n\theta)+\sin(n\omega)\sin(n\theta)\right)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\omega)\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left(\cos(n\omega)\cos(n\theta)+\sin(n\omega)\sin(n\theta)\right)\right)\,d\omega. \end{aligned}$$

Now, denoting $\frac{1}{2\pi}$ times the term in parantheses by K,

$$\begin{aligned} 2\pi K(r,\theta,\omega) &= 1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left(\cos(n\omega)\cos(n\theta)+\sin(n\omega)\sin(n\theta)\right) \\ &= 1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos(n(\theta-\omega)) \\ &= 1+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left(e^{in(\theta-\omega)}+e^{-in(\theta-\omega)}\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n e^{in(\theta-\omega)} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n e^{-in(\theta-\omega)}. \end{aligned} \tag{4}$$

Both sums are those of geometric series, so the summation yields

$$
\begin{aligned}
2\pi K(r,\theta,\omega) &= \frac{1}{1-\left(\frac{r}{R}\right)e^{i(\theta-\omega)}} + \left(\frac{r}{R}\right)e^{-i(\theta-\omega)}\frac{1}{1-\left(\frac{r}{R}\right)e^{-i(\theta-\omega)}} \\
&= \frac{1-\left(\frac{r}{R}\right)e^{-i(\theta-\omega)}}{1-2\left(\frac{r}{R}\right)\cos(\theta-\omega)+\left(\frac{r}{R}\right)^2} + \frac{\left(\frac{r}{R}\right)e^{-i(\theta-\omega)}-\left(\frac{r}{R}\right)^2}{1-2\left(\frac{r}{R}\right)\cos(\theta-\omega)+\left(\frac{r}{R}\right)^2} \\
&= \frac{1-\left(\frac{r}{R}\right)^2}{1-2\left(\frac{r}{R}\right)\cos(\theta-\omega)+\left(\frac{r}{R}\right)^2},
\end{aligned}
$$

where the second equality was bringing the two terms to common denominator. We thus deduce that for $r < R$,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\omega)\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\omega)+r^2}\,d\omega.$$

Rewriting in terms of Euclidean coordinates, if $y = R\omega$, $x = r\theta$, then the numerator is $R^2 - |x|^2$, while the denominator is $|x-y|^2$, so altogether we have

$$u(x) = \frac{1}{2\pi R}\int_{\partial D} h(y)\frac{R^2-|x|^2}{|x-y|^2}\,dS(y). \tag{5}$$

This is called the *Poisson formula*. In particular,

$$u(0) = \frac{1}{2\pi R}\int_{\partial D} h(y)\,dS(y),$$

i.e. since the circumference of D is $2\pi R$, the value of u in the center is the *average value of* h, which is called the *mean value property* of solutions of $\Delta u = 0$.

Now, the Poisson formula would give an independent (i.e. new, not relying on Lemma 0.1) way of proving that the Fourier series of h sums to h. Namely, one can show directly, using (5), that for $h \in C^1$ the Fourier series, which we already know converges uniformly to some function, converges to h. Indeed, in view of the uniform convergence of the series (3) on the closed disk, at $r = R$ it converges to

$$
\begin{aligned}
\lim_{r\to R-} u(r,\theta) &= \lim_{r\to R-}\int_{-\pi}^{\pi} h(\omega)K_P(r,\theta-\omega)\,d\omega, \\
K_P(r,\psi) &= \frac{1}{2\pi}\,\frac{R^2-r^2}{R^2-2Rr\cos\psi+r^2}.
\end{aligned}
$$

We claim that this limit is $h(\theta)$. Using the series definition of K, (4), and noting that

$$K(r,\theta,\omega) = K_P(r,\theta-\omega),$$

so

$$K_P(r,\psi) = \frac{1}{2\pi}\Big(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos(n\psi)\Big)$$

one sees that by the orthogonality of cosines to the constant function 1, for each $r < R$,

$$\int_{-\pi}^{\pi} K_P(\psi)\,d\psi = 1,$$

and K_P is 2π-periodic in ψ, so

$$\begin{aligned} u(r,\theta)-h(\theta) &= \int_{-\pi}^{\pi} K_P(r,\theta-\omega)\,(h(\omega)-h(\theta))\,d\omega \\ &= \int_{-\pi}^{\pi} K_P(r,\omega)\,(h(\theta-\omega)-h(\theta))\,d\omega, \end{aligned}$$

where the last equality involved a change of variables, but in view of the 2π-periodicity, the domain of integration did not need to change (i.e. the integral over $[-\pi,\pi]$ is the same as that over $[\theta-\pi,\theta+\pi]$). Now, the advantage of the Poisson kernel, K_P, over the Dirichlet kernel discussed in the Appendix is that for $r<R$ one has $K_P\geq 0$, directly from its definition. Moreover, $K_P(r,\psi)\to 0$ as $r\to R$ uniformly outside $[-\delta,\delta]$ for any $\delta>0$. Thus,

$$\begin{aligned} &\int_{-\pi}^{\pi} K_P(r,\omega)\,(h(\theta-\omega)-h(\theta))\,d\omega \\ &= \int_{-\delta}^{\delta} K_P(r,\omega)\,(h(\theta-\omega)-h(\theta))\,d\omega + \int_{[-\pi,\pi]\setminus[-\delta,\delta]} K_P(r,\omega)\,(h(\theta-\omega)-h(\theta))\,d\omega. \end{aligned}$$

Now even if h is merely continuous, given any $\epsilon>0$, choosing δ sufficiently small, $|h(\theta-\omega)-h(\theta)|<\epsilon$ for all ω with $|\omega|\leq\delta$, and thus *for any* $r\in(0,R)$, the absolute value of the first term is

$$\leq \epsilon\int_{-\delta}^{\delta} K_P(r,\omega)\,d\omega \leq \epsilon\int_{-\pi}^{\pi} K_P(r,\omega)\,d\omega = \epsilon.$$

On the other hand, by the uniform convergence of $K_P(r,\omega)$ to 0 outside $(-\delta,\delta)$, the second term also goes to 0 as $r\to R$, in particular is $<\epsilon$ in absolute value if r is sufficiently large. Thus, even if just h is continuous, $u(r,\theta)\to h(\theta)$ uniformly as $r\to R-$. In particular, if h is C^1, the partial sums of the Fourier series converge to h uniformly, giving the promised proof.

We formalize the properties of K_P for a general purpose result.

Proposition 0.3. *Suppose that for each $N\in\mathbb{N}$, K_N is a 2π-periodic measurable function on $\mathbb{R}$, and*

(i) *$K_N\in L^1([-\pi,\pi])$, and there is $M>0$ such that $\|K_N\|_{L^1([-\pi,\pi])}\leq M$ for all N.*
(ii) *$\int_{[-\pi,\pi]} K_N = 1$,*
(iii) *For all $\delta>0$, $K_N\chi_{[-\pi,\pi]\setminus(-\delta,\delta)}\to 0$ in L^1 as $N\to\infty$.*

Then for $h\in C^0_{\mathrm{per}}(\mathbb{R})=C^0(\mathbb{S}^1)$,

$$h_N(\theta)=\int_{-\pi}^{\pi} K_N(\theta-\omega)h(\omega)\,d\omega=(K_N*h)(\theta)$$

converges to h uniformly.

A family K_N satisfying (i)-(iii) is called a family of good kernels.

Note that (ii) implies the bound of (i) (with $M=1$) if $K_N\geq 0$ pointwise.

Note also that one can replace $N\in\mathbb{N}$ by another parameter set, such as $N\in(0,1]$, and let $N\to 0$, etc., without any changes to the result.

Proof. As before,

$$
\begin{aligned}
h_N(\theta) - h(\theta) &= \int_{-\pi}^{\pi} K_N(\theta-\omega)\,(h(\omega)-h(\theta))\,d\omega \\
&= \int_{-\pi}^{\pi} K_N(\omega)\,(h(\theta-\omega)-h(\theta))\,d\omega \\
&= \int_{-\delta}^{\delta} K_N(\omega)\,(h(\theta-\omega)-h(\theta))\,d\omega \\
&\qquad + \int_{[-\pi,\pi]\setminus[-\delta,\delta]} K_N(\omega)\,(h(\theta-\omega)-h(\theta))\,d\omega.
\end{aligned} \tag{6}
$$

Now as h is continuous on $[-2\pi, 2\pi]$, it is uniformly continuous there, so given any $\epsilon > 0$, choosing δ sufficiently small,

$$|h(\theta-\omega) - h(\theta)| < \frac{\epsilon}{2M}$$

for all ω with $|\omega| \leq \delta$. Thus *for any* N, the absolute value of the first term of (6) is

$$\leq \frac{\epsilon}{2M}\int_{-\delta}^{\delta} |K_N(\omega)|\,d\omega \leq \frac{\epsilon}{2M}\|K_N\|_{L^1([-\pi,\pi])} \leq \frac{\epsilon}{2}.$$

On the other hand, the absolute value of the second term is

$$\leq 2\|K_N\|_{L^1([-\pi,\pi]\setminus(-\delta,\delta))} \sup|h|,$$

so by (iii), it goes to 0 as $N \to \infty$, so in particular is $< \epsilon/2$ if N is sufficiently large, say $N > N_0$. Summing up, we see that $N > N_0$ implies that $\sup|h - h_N| < \epsilon$, proving the uniform convergence. □

Note that the standard definition of convergence of a series, via partial sums, is a kind of a regularization. If all terms of a series $\sum u_n$ are bounded, a different regularization is to consider $\sum r^n u_n$ for $r < 1$, when the series converges by the Weierstrass M-test, and let $r \to 1$. This is a better behaved regularization than the standard definition, and is called *Abel summability.* Thus, for h merely continuous, the Fourier series is Abel summable to h, but it need not converge uniformly to h (there are actual counterexamples).

Appendix A. The Dirichlet kernel

We now consider a more classical proof of the convergence of the Fourier series. So suppose now that ϕ is a function on $[-\pi, \pi]$ whose 2π-periodic extension, Φ, is C^1. Then the Fourier series of ϕ is given by (1), with $\ell = \pi$. We work out partial sums of the series,

$$S_N(\theta) = \sum_{n=-N}^{N} C_n e^{in\theta},\ C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\omega)\,e^{-in\omega}\,d\omega.$$

Thus,

$$\begin{aligned} S_N(\theta) &= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\omega)\, e^{in(\theta-\omega)}\, d\omega \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(\theta-\omega)} \phi(\omega)\, d\omega \\ &= \int_{-\pi}^{\pi} K_N(\theta-\omega)\phi(\omega)\, d\omega,\ K_N(\psi) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{in\psi}. \end{aligned}$$

Now, the sum for K_N is a geometric series, so using the summation formula for a geometric series,

$$\begin{aligned} K_N(\psi) &= \frac{1}{2\pi} \sum_{n=-N}^{N} e^{in\psi} = \frac{1}{2\pi} e^{-iN\psi} \frac{1-e^{i(2N+1)\psi}}{1-e^{i\psi}} \\ &= \frac{1}{2\pi} \frac{e^{i(N+1/2)\psi} - e^{-i(N+1/2)\psi}}{e^{i\psi/2} - e^{-i\psi/2}} = \frac{1}{2\pi} \frac{\sin\left((N+1/2)\psi\right)}{\sin\left(\psi/2\right)}. \end{aligned} \tag{7}$$

Note that the right hand side has the form $0/0$ at $\psi = 2k\pi$, $k \in \mathbb{Z}$, but by L'Hopital's rule it is continuous, with value $K_N(0) = \frac{1}{2\pi}(2N+1)$, and it is indeed $\mathcal{C}^\infty$ at these points. Moreover, K_N has the additional property, which is clear from the series formula since $e^{in\psi}$ is orthogonal to 1 in $L^2([-\pi,\pi])$ for $n \neq 0$,

$$\int_{-\pi}^{\pi} K_N(\psi)\, d\psi = \langle K_N, 1\rangle = \frac{1}{2\pi}\langle 1, 1\rangle = 1$$

for all N, and, as all the summands in the sum are 2π-periodic,

$$K_N(\psi + 2\pi) = K_N(\psi).$$

One calls K_N the *Dirichlet kernel*. Now, we need to analyze the difference

$$\begin{aligned} S_N(\theta) - \phi(\theta) &= \int_{-\pi}^{\pi} K_N(\theta-\omega)\phi(\omega)\, d\omega - \phi(\theta) \int_{-\pi}^{\pi} K_N(\theta-\omega)\, d\omega \\ &= \int_{-\pi}^{\pi} K_N(\theta-\omega)(\phi(\omega) - \phi(\theta))\, d\omega. \end{aligned}$$

It is convenient to rewrite the integral by a change of variables. In view of the 2π-periodicity of the integrand, we do not need to change the limits of the integral:

$$S_N(\theta) - \phi(\theta) = \int_{-\pi}^{\pi} K_N(\psi)(\Phi(\theta-\psi) - \Phi(\theta))\, d\psi.$$

Note that this integral, much like the one before, is very similar to a convolution on $\mathbb{R}$; the difference is that we merely integrate over an interval of length 2π. Since the integrand is 2π-periodic, this is best thought of as the convolution of K_N and the function $u_\theta(\psi) = \Phi(\psi) - \Phi(\theta)$ on the circle, $\mathbb{S}^1$.

Now, roughly, the K_N are approximations to δ_0. However, this is in a weaker sense than e.g. $\chi_N(x) = N\chi(x/N)$, where $\chi \geq 0$ vanishes outside $[-1,1]$, or indeed the Poisson kernel K_P above, is such, since $\int_{-\pi}^{\pi} |K_N(\psi)|\, d\psi$ is *not* bounded by a fixed constant C (independent of N): it is the oscillatory nature of K_N that makes the analysis more involved.

Writing out $K_N(\psi)$ as in (7), an orthogonality argument can be used to deduce the decay of $S_N(\theta) - \phi(\theta)$ as $N \to \infty$. Namely, suppose that Φ is C^1. Then by Taylor's theorem, the function

$$v_\theta(\psi) = \frac{\Phi(\theta - \psi) - \Phi(\theta)}{\sin(\psi/2)}$$

is continuous. Thus,

$$S_N(\theta) - \phi(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\left((N + 1/2)\psi\right) v_\theta(\psi)\, d\psi.$$

Now, the functions

$$Z_n(\psi) = \sin\left((n + 1/2)\psi\right),\ n \geq 0,\ n \in \mathbb{Z},$$

are orthogonal to each other on $[-\pi, \pi]$ (and indeed on $[0, \pi]$), since they are eigenfunctions of the symmetric operator $-\frac{d^2}{d\theta^2}$ with Neumann boundary condition at $-\ell$ and ℓ (as $\cos((n + 1/2)\pi) = 0$) with distinct eigenvalues $(n + 1/2)^2$. Moreover,

$$\int_{-\pi}^{\pi} Z_n(\psi)^2\, d\psi = \pi.$$

Thus,

$$S_N(\theta) - \phi(\theta) = \frac{1}{2\sqrt{\pi}} \frac{\langle v_\theta, Z_N \rangle}{\|Z_N\|_{L^2}^2} \|Z_N\|_{L^2}.$$

But we have shown that for *any* orthogonal set $\{x_n\}_{n=0}^{\infty}$ in an inner product space V, and any $v \in V$ with generalized Fourier coefficients c_n, $\sum |c_n|^2 \|x_n\|^2$ converges, and $c_n = \frac{\langle v, x_n \rangle}{\|x_n\|^2}$. Thus,

$$|S_N(\theta) - \phi(\theta)|^2 = \frac{1}{4\pi} |c_N|^2 \|Z_N\|_{L^2}^2,\ c_N = \frac{\langle v_\theta, Z_N \rangle}{\|Z_N\|_{L^2}^2}.$$

But as $\sum_N |c_N|^2 \|Z_N\|_{L^2}^2$ converges, its terms go to 0, i.e. $|c_N|^2 \|Z_N\|_{L^2}^2 \to 0$ as $N \to \infty$, so we deduce that $|S_N(\theta) - \phi(\theta)| \to 0$ as $N \to \infty$, proving the desired convergence.

Since we already know that for functions ϕ whose 2π-periodic extensions are C^1 the Fourier series converges uniformly (and thus pointwise) to some function, and we just showed pointwise convergence of the series to ϕ, we deduce that for such ϕ the Fourier series converges uniformly to ϕ.

Appendix B. Fejér's kernel

We now discuss another kernel that is better behaved than the Dirichlet kernel. This is the Fejér kernel, given by averaging the Dirichlet kernel K_N over N.

Namely, let

$$\begin{aligned}
\tilde K_{M+1}(\psi) &= \frac{1}{M+1}\sum_{N=0}^{M} K_N(\psi) = \sum_{N=0}^{M}\frac{1}{2\pi}\frac{e^{i(N+1/2)\psi}-e^{-i(N+1/2)\psi}}{e^{i\psi/2}-e^{-i\psi/2}}\\
&= \frac{1}{2\pi(M+1)}\frac{1}{e^{i\psi/2}-e^{-i\psi/2}}\Big(e^{i\psi/2}\frac{1-e^{i(M+1)\psi}}{1-e^{i\psi}} - e^{-i\psi/2}\frac{1-e^{-i(M+1)\psi}}{1-e^{-i\psi}}\Big)\\
&= \frac{1}{2\pi(M+1)}\frac{1}{e^{i\psi/2}-e^{-i\psi/2}}\Big(\frac{1-e^{i(M+1)\psi}}{e^{-i\psi/2}-e^{i\psi/2}} - \frac{1-e^{-i(M+1)\psi}}{e^{i\psi/2}-e^{-i\psi/2}}\Big)\\
&= \frac{1}{2\pi(M+1)}\frac{-1}{(e^{i\psi/2}-e^{-i\psi/2})^2}\Big(1-e^{i(M+1)\psi}+1-e^{-i(M+1)\psi}\Big)\\
&= \frac{1}{2\pi(M+1)}\frac{1}{4\sin^2(\psi/2)}(2-2(\cos(M+1)\psi))\\
&= \frac{1}{2\pi(M+1)}\frac{\sin^2((M+\frac{1}{2})\psi)}{\sin^2(\psi/2)}.
\end{aligned}$$

Since $\tilde K_M$ is an average of the K_N, each of which has integral 1, $\int_{-\pi}^{\pi}\tilde K_M(\psi)\,d\psi = 1$, just like the Dirichlet kernel. However, it has the significant advantage that$K_M \geq 0$ pointwise, and this $\int_{-\pi}^{\pi}|\tilde K_M(\psi)|\,d\psi = 1$ as well, just like for the Poisson kernel. Furthermore, for any $\delta > 0$, $\tilde K_M \to 0$ uniformly on $[-\pi,\pi]\setminus(-\delta,\delta)$, just as for the Poisson kernel, since $0 \leq \sin^2((M+\frac{1}{2})\psi) \leq 1$, $\sin^2(\psi/2)$ is bounded away from 0 there, and $M \to \infty$. Therefore Proposition 0.3 gives, for $h \in C^0_{\mathrm{per}}(\mathbb{R}) = C^0(\mathbb{S}^1)$, the uniform convergence of

$$s_M(\theta) = \int \tilde K_M(\theta-\omega)h(\omega)\,d\omega$$

to h as $M \to \infty$.

Note that at the level of the partial sums S_N of the Fourier series, $s_M = \frac{1}{M}\sum_{N=0}^{M-1} S_N$, so what we have shown is that although the partial sums do not necessarily converge uniformly for a merely continuous h, the *averaged* partial sums do. As for Abel summation, this is a better regularization of the sum of an infinite series than the standard definition; this is called *Cesàro summability.*