MATH 172: PROBLEM SET 5
DUE 2:15PM, FRIDAY, FEBRUARY 13, 2015

Problem 1. Do Stein-Shakarchi, vol. 3, Ch. 2, Exercise 4.

Problem 2. Do Stein-Shakarchi, vol. 3, Ch. 2, Exercise 19.

Problem 3. Do Stein-Shakarchi, vol. 3, Ch. 2, Exercise 21.

Problem 4. Do Stein-Shakarchi, vol. 3, Ch. 2, Problem 4.

Problem 5. This is the compact version of vol. 3, Ch. 2, Exercise 21 above.

Suppose f, g are measurable functions on $[-\pi, \pi]^d$, and extend them to be 2π-periodic in each coordinate to define functions f_{per}, g_{per} on $\mathbb{R}^d$.

(i) Show that $(x, y) \mapsto f_{\mathrm{per}}(x - y)g_{\mathrm{per}}(y)$ is measurable on $[-\pi, \pi]^{2d}$.

(ii) Show that if f, g are integrable on $[-\pi, \pi]^d$ then $(x, y) \mapsto f_{\mathrm{per}}(x-y)g_{\mathrm{per}}(y)$ is integrable on $[-\pi, \pi]^{2d}$.

(iii) With f, g are integrable on $[-\pi, \pi]^{2d}$, let

$$(f * g)(x) = \int_{[-\pi,\pi]^d} f_{\mathrm{per}}(x - y)g_{\mathrm{per}}(y)\, dy.$$

Show that $f * g$ is well-defined for almost every x.

(iv) Show that $f * g$ is integrable on $[-\pi, \pi]^d$ if f, g are, and that

$$\|f * g\|_{L^1([-\pi,\pi]^d)} \leq \|f\|_{L^1([-\pi,\pi]^d)} \|g\|_{L^1([-\pi,\pi]^d)}$$

with equality if f, g are (a.e.) non-negative.

(v) Show that $f * g = g * f$ in L^1.

(vi) Show that if either f_{per} or g_{per} is C^k and $f, g \in L^1([-\pi, \pi]^d)$, then $(f * g)_{\mathrm{per}}$ is C^k.