MATH 172: PROBLEM SET 6 (EXPANDED, EXTENDED)
DUE WEDNESDAY, FEBRUARY 25, 2015

Problem 1. Do Stein-Shakarchi, vol. 3, Ch. 4, Exercise 5.

Problem 2. Do Stein-Shakarchi, vol. 3, Ch. 4, Exercise 6.

Problem 3. Do Stein-Shakarchi, vol. 3, Ch. 4, Exercise 7.

Problem 4. Do Stein-Shakarchi, vol. 3, Ch. 4, Exercise 8. Here a unitary map is an invertible isometry $U : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces, so U, U^{-1} are linear maps preserving norms (and thus inner products by the parallelogram identity).

Problem 5. Let $\phi(x) = |x|$ on $[-\pi, \pi]$. Let

$$f(x) = a_0 + a_1 \cos x + a_2 \cos(2x) + b_1 \sin x + b_2 \sin(2x),$$

With what choice of the coefficients a_j and b_j is the $L^2([-\pi, \pi])$ error $\|f - \phi\| = \|f - \phi\|_{L^2}$ minimal?

Problem 6. Do Stein-Shakarchi, vol. 3, Ch. 4, Problem 1.

Problem 7. Let $\phi(x) = x(\ell - x)$ on $[0, \ell]$.

(i) Find the Fourier sine series of ϕ, and state what it converges to for $x \in \mathbb{R}$.
(ii) Find the Fourier cosine series of ϕ, and state what it converges to for $x \in \mathbb{R}$.
(iii) Compare the decay rates of the coefficients of the two series as $n \to \infty$. Why do the coeffients decay faster in one of the cases?

Problem 8. For both of the following functions prove or disprove whether the Fourier sine series converges uniformly:

(i) $\phi(x) = x$ on $[0, \ell]$,
(ii) $\phi(x) = x(\ell - x)^2$ on $[0, \ell]$.

You do not need to compute the respective Fourier series.

Problem 9. Suppose $\phi \in L^1([-\pi, \pi])$, and define the Fourier coefficients of ϕ by extending the L^2-definition:

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \phi(x)\, dx.$$

(i) Show that the sequence $\{C_n\}_{n=-\infty}^{\infty}$ is bounded, and $\sup |C_n| \leq \frac{1}{2\pi} \|\phi\|_{L^1}$ in fact.
(ii) Show that $\lim_{|n| \to \infty} C_n = 0$.

Problem 10. For the purposes of this problem, define $L^1([0, 1])$ and $L^2([0, 1])$ as the completions of $C([0, 1])$ in the L^1, resp. L^2-norm (using the Riemann integral).

Without using the measure theoretic concepts we developed, show that $L^2([0, 1]) \subset L^1([0, 1])$, i.e. the identity map $C([0, 1]) \to C([0, 1])$ has a unique continuous extension to a map $\iota : L^2([0, 1]) \to L^1([0, 1])$ (show both existence and uniqueness), and this map is injective.

(Hint: For the injectivity, for $f \in L^2([0, 1])$ show that there is $\phi \in C([0, 1])$ such that $\langle f, \phi \rangle \neq 0$, and then show that the map $L^2 \ni g \mapsto \langle g, \phi \rangle \in \mathbb{C}$ extends continuously to a map $L^1 \to \mathbb{C}$.)