MATH 172: PROBLEM SET 7
DUE WEDNESDAY, MARCH 4, 2015

Problem 1. (i) Let K_P be the one-dimensional Poisson kernel:

$$K_P(r,\omega) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\omega+r^2}.$$

Show that for $f \in L^1([-\pi,\pi])$, $f * K_P \to f$ in L^1 as $r \nearrow 1$.

(ii) Show that 2π-periodic C^∞ functions are dense in $L^1([-\pi,\pi]^d)$.

Problem 2. Consider the wave equation on a ring of length 2ℓ. We let x be the arclength variable along the ring, $x \in [-\ell,\ell]$. We would like to understand wave propagation along the ring, so consider the wave equation with *periodic boundary conditions*:

$$u_{tt} = c^2 u_{xx},\ u(-\ell,t) = u(\ell,t),\ u_x(-\ell,t) = u_x(\ell,t).$$

(i) Show that if the initial conditions are $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$, with ϕ,ψ 2ℓ-periodic C^{k+1}, $k \geq 2$, functions then there is a solution $u \in C^k(\mathbb{R}_x \times \mathbb{R}_t)$, 2ℓ-periodic in x of the wave equation satisfying the initial and boundary conditions.

(ii) Show that if the initial conditions are $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$, with ϕ,ψ 2ℓ-periodic C^3, then there is at most one $C^3(\mathbb{R}_x \times \mathbb{R}_t)$, 2ℓ-periodic in x, solution of the wave equation.

(iii) Find the solution with initial condition

$$u(x,0) = 0,\ u_t(x,0) = \cos(2\pi x/\ell) - \sin(\pi x/\ell),\ x \in [-\ell,\ell].$$

Problem 3. Consider the (real-valued) heat equation on a rod of length ℓ with insulated ends and $k > 0$:

$$u_t = k u_{xx},\ u_x(0,t) = 0 = u_x(\ell,t).$$

(i) Show that if the initial condition is $u(x,0) = \phi(x)$, $\phi \in L^2([0,\ell])$, then there is a solution $u \in C^\infty([0,\ell] \times (0,\infty))$ such that the family of function $U(t)(x) = u(x,t)$, depending on the parameter $t > 0$, tends to ϕ in L^2 as $t \to 0$.

(ii) Show that if ϕ is C^1, then the convergence of $U(t)$ to ϕ is uniform.

(iii) Show that if $u \in C^2([0,\ell] \times (0,\infty))$ satisfying the PDE and the boundary condition, then $E(t) = \int_0^\ell u(x,t)^2$ is monotone decreasing. (Hint: what is $E'(t)$?)

(iv) Show that if $\phi \in L^2([0,\ell])$, then there is a unique solution $u \in C^2([0,\ell] \times (0,\infty))$ of the heat equation satisfying the boundary conditions and with $U(t) \to \phi$ in L^2 as $t \to 0$.

Problem 4. Do Stein-Shakarchi, vol. 3, Ch. 2, Exercise 22.

Problem 5. Do Stein-Shakarchi, vol. 3, Ch. 2, Problem 1.

Problem 6. Suppose that $f \in L^1(\mathbb{R}^n)$. Throughout this problem, $a \in \mathbb{R}^n$.

(i) Let $f_a(x) = f(x-a)$. Show that $(\mathcal{F}f_a)(\xi) = e^{-ia\cdot\xi}(\mathcal{F}f)(\xi)$.

(ii) Let $g_a(x) = e^{ix\cdot a}f(x)$. Show that $(\mathcal{F}g_a)(\xi) = (\mathcal{F}f)(\xi - a)$.

(iii) Show that $(\mathcal{F}^{-1}f_a)(x) = e^{ia\cdot x}(\mathcal{F}^{-1}f)(x)$.

(iv) Show that $(\mathcal{F}^{-1}g_a)(x) = (\mathcal{F}^{-1}f)(x+a)$.

Problem 7. Use part (i) of Problem 6 to show that $(\mathcal{F}(\partial_j f))(\xi) = i\xi_j(\mathcal{F}f)(\xi)$ if f is C^1 and $|x|^N f$, $|x|^N \partial_j f$ are bounded for some $N > n$.