MATH 172: PROBLEM SET 8
DUE FRIDAY, MARCH 13, 2015, 2:15PM
(BEGINNING OF LECTURE)

Problem 1. Find the Fourier transform on $\mathbb{R}$ of the following functions:

(i) $f(x) = \chi_{[-a,a]}$, $a > 0$.
(ii) $f(x) = \chi_{[0,\infty)} e^{-ax}$, where $a > 0$.
(iii) $f(x) = |x|^n e^{-a|x|}$, where $a > 0$, and $n \geq 0$ integer.
(iv) $f(x) = (1 + x^2)^{-1}$. (Hint: use that if $f = \mathcal{F}^{-1} g$ with $g \in L^1(\mathbb{R})$ then $g = \mathcal{F} f$ by the Fourier inversion formula. Rewrite $(1 + x^2)^{-1}$ as partial fractions (factor the denominator).)

Problem 2. Find the Fourier transform on $\mathbb{R}^3$ of the function $f(x) = |x|^n e^{-a|x|}$, where $a > 0$ and $n \geq -1$ integer. (Hint: express the integral in the Fourier transform in polar coordinates.)

Problem 3.

(i) On $\mathbb{R}^3$, find the Fourier transform of the function $g(x) = |x|^{-1}$. (Hint: to do this efficiently, consider $g(x)$ as the limit of $g_a(x) = e^{-a|x|}|x|^{-1}$, and use your result from the previous problem.)
(ii) Solve $\Delta u = f$ on $\mathbb{R}^3$, where $f \in \mathcal{S}(\mathbb{R}^3)$, writing your answer as a convolution.

Problem 4. Show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then there is an integer $m \geq 0$ and $C > 0$ such that for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$|u(\phi)| \leq C \|\phi\|_m$$

where

$$\|\phi\|_m = \sum_{|\alpha|+|\beta| \leq m} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta \phi|.$$

Hints: This relies on the continuity of u as a map $u : \mathcal{S} \to \mathbb{C}$. So suppose for the sake of contradiction that no such m and C exist; in particular for an integer $j > 0$, $m = j$ and $C = j$ do not work, i.e. there exists $\phi_j \in \mathcal{S}$ such that

$$|u(\phi_j)| > j \|\phi_j\|_j.$$

Note that ϕ_j cannot be 0 (for then $u(\phi_j)$ would vanish by linearity). Let $\psi_j = \frac{1}{j\|\phi_j\|_j} \phi_j$, so $\psi_j \in \mathcal{S}$, $\|\psi_j\|_j = \frac{1}{j}$ and

$$|u(\psi_j)| > j \|\psi_j\|_j = 1.$$

Now show that $\psi_j \to 0$ in $\mathcal{S}$ as $j \to \infty$, and use this to get a contradiction with the continuity of u.

Problem 5. Suppose that $f \in L^1(\mathbb{R}^n)$, $f \geq 0$, $\|f\|_{L^1} = 1$. Show that $\sup |\mathcal{F} f| = 1$, and it is attained exactly at 0. (*Hint:* For the 'only attained at 0' part, consider first $\operatorname{Re} \mathcal{F} f$ and show this is < 1 away from 0; to show the general statement, write $\mathcal{F} f(\xi) = r e^{i\theta}$, $r \geq 0$, $\theta \in \mathbb{R}$, and show that $r = \operatorname{Re}(e^{-i\theta} \mathcal{F} f(\xi)) < 1$ for $\xi \neq 0$.)

Problem 6. A model of probability theory is the following. One has a non-negative function $f \in L^1(\mathbb{R})$ (the probability density) with $\int f = 1$, and one identifies events with measurable subsets E of $\mathbb{R}$. Thus, one says that the probability of an event E is $\int_E f$; this is a real number between 0 and 1. For instance numbers on the real line represent a measurement, so $\int_E f$ corresponds to the probability that the measured value lies in the set E. If $xf \in L^1$ as well, the expected value of the measurement is then $\overline{x} = \int_{\mathbb{R}} xf(x)\,dx$, also called the first moment. One can always subtract $\overline{x}$ from x and recenter to have 0 expectation: $\tilde{x} = x - \overline{x}$ satisfies $\int \tilde{x} f(\tilde{x} + \overline{x})\,d\tilde{x} = 0$.

(i) If one does N *independent* measurements, the probability that the ordered collection of measurements $(x_1, \ldots, x_N)$ take value in a measurable set $E \subset \mathbb{R}^N$ is

$$\int_E f(x_1) \ldots f(x_N)\,dx_1 \ldots dx_N.$$

Correspondingly, the probability that the sum $s_N = \sum_{j=1}^N x_j$ is in a measurable set $A \subset \mathbb{R}$ is, with $E = \{(x_1, \ldots, x_N) : \; x_1 + \ldots + x_N \in A\}$,

$$\int_E f(x_1) \ldots f(x_N)\,dx_1 \ldots dx_N.$$

Show that

$$\int_E f(x_1) \ldots f(x_N)\,dx_1 \ldots dx_N = \int_A F_N(x)\,dx$$

where $F_N = f * f * \ldots * f$ (N factors).

(ii) More generally, we consider F_N as a distribution, i.e. we consider $\int F_N \phi$, $\phi \in \mathcal{S}(\mathbb{R})$. Using the Fourier transform, show that $F_N \to 0$ in tempered distributions, and use this to show that for A compact, $\int_A F_N \to 0$. (*Hint:* dominate $0 \le \int_A F_N$ by $\int F_N \phi$ for appropriate non-negative ϕ, using that $F_N \ge 0$.) This says that for any fixed compact A the probability of s_N being in A goes to 0. (This is expected: the sum should get larger with N.)

(iii) Let $S_N = s_N/N$ be the average of the measurements. Show that the probability of S_N being in a measurable set B, which is the same as s_N being in $A = NB = \{Nx : \; x \in B\}$, is

$$\int_B G_N(x)\,dx, \; G_N(x) = NF_N(Nx).$$

Again, consider G_N as a distribution. Assuming $f \in \mathcal{S}(\mathbb{R})$ for simplicity, and that $\int xf(x)\,dx = 0$, using the Fourier transform show that $G_N \to \delta_0$ in tempered distributions, and use this to show that if B is closed, disjoint from 0, then $\int_B G_N \to 0$ as $N \to \infty$. This is the law of large numbers: the average measurement converges to 0 in this precise sense. (*Hint:* use the terms, up to the quadratic one, of the Taylor series of $\mathcal{F}f$ at 0, but convert your expansion into one in terms of an exponential (with related Taylor series). Then show the claim first if B is compact, excluding 0, and also that if $B = [-R, R]$, $R > 0$ then $\int_B G_N \to 1$.)

(iv) One may wonder precisely how this convergence takes place, still assuming $f \in \mathcal{S}(\mathbb{R})$ for simplicity, with $\int xf(x)\,dx = 0$. In this case, the variance is the second moment, $\sigma^2 = \int x^2 f(x)\,dx$.

To do so, one focuses in what happens near 0 and considers $\tilde{S}_N = s_N/\sqrt{N} = \sqrt{N} S_N$. Show that the probability of $\tilde{S}_N$ being in a measurable set B is

$$\int_B H_N(x)\,dx, \; H_N(x) = \sqrt{N} F_N(\sqrt{N} x).$$

Show now that H_N converges to the tempered distribution given by $\Phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)}$ in tempered distributions. (Note that Φ has variance σ^2: it is the centered normalized Gaussian with this variance.) This is a version of the central limit theorem.