MATH 174A: PROBLEM SET 1
DUE THURSDAY, JANUARY 18, 2007

Problem 1. Suppose that $(V, \|.\|)$ is a normed vector space, $A : \mathbb{R}^n \to V$ is a linear map. Let $\|.\|_2$ denote the standard norm on $\mathbb{R}^n$. Show that there is $C > 0$ such that $\|Av\| \leq C\|v\|_2$ for all $v \in \mathbb{R}^n$. (Hint: write $v = \sum_{j=1}^{n} a_j e_j$, where e_j is the jth unit vector.)

Problem 2. Suppose that $(V_1, \|.\|_1)$ and $(V_2, \|.\|_2)$ are normed vector spaces, and $A : V_1 \to V_2$ is linear. Show that the following are equivalent:
  (1) A is continuous at some point,
  (2) A is continuous,
  (3) A is bounded, i.e. there is $C > 0$ such that $\|Av\|_2 \leq C\|v\|_1$ for all $v \in V_1$.
We denote by $\mathcal{L}(V_1, V_2)$ the set of bounded linear operators from V_1 to V_2.

Problem 3. With V_1 and V_2 as above, $A \in \mathcal{L}(V_1, V_2)$, let
$$\|A\| = \|A\|_{\mathcal{L}(V_1, V_2)} = \sup_{\|v\|_1 \leq 1} \|Av\|.$$
Show that $\|.\|$ is a norm on $\mathcal{L}(V_1, V_2)$, and if $V_1 = V_2$, it satisfies $\|AB\| \leq \|A\|\|B\|$. (Note: if $V_1 \neq \{0\}$, $\|A\| = \sup_{\|v\|_1=1} \|Av\|$.)

Problem 4. Do Ex.I.1.1. on p.6 of Taylor's book.

Problem 5. Do Ex.I.1.6. on p.7 of Taylor's book. (Extra hint: Show that $\frac{1}{h} \int_0^h (\frac{\partial f}{\partial y}(x, y + s) - \frac{\partial f}{\partial y}(x, y)) \, dx \to 0$ as $h \to 0$.)

Problem 6. Do Ex.I.1.7. on p.7 of Taylor's book.

Problem 7. Do Ex.I.2.1. on p.11 of Taylor's book.

Problem 8. Do Ex.I.2.4. on p.11 of Taylor's book.

Problem 9. Consider the ODE $y' = \sqrt{|y|}$, $y(0) = 0$. Show that the solution of this ODE is not unique on any interval including 0. Why does this not contradict the existence and uniqueness theorem for ODEs?