MATH 174A: PROBLEM SET 2
DUE THURSDAY, JANUARY 25, 2007

Problem 1. (Cf. Taylor, I.1.4) Recall that a map F between complex vector spaces is called differentiable (in the complex sense) at x if there is a *complex linear* map L such that $F(x+y) = F(x) + Ly + R(x,y)$ with $\lim_{\|y\|\to 0} \frac{\|R(x,y)\|}{\|y\|} = 0$. If F is differentiable as a complex map, with DF continuous, one calls F holomorphic, or complex analytic.

Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ by writing $z \in \mathbb{C}^n$ as $z = x + iy$, $x, y \in \mathbb{R}^n$. Multiplication by i on $\mathbb{C}^n$ becomes multiplication by a matrix J_n on $\mathbb{R}^{2n}$,

$$J_n = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix},$$

where I is the identity operator on $\mathbb{R}^n$.

Show that for $O \subset \mathbb{C}^n$ open, $F : O \to \mathbb{C}^m$ is holomorphic if and only if F is C^1 as a real map (i.e. identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, $\mathbb{C}^m$ with $\mathbb{R}^{2m}$), and $DF \in \mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^{2m})$, regarded as a real linear map, satisfies

$$J_m DF(z) = DF(z) J_n$$

for all $z \in O$.

Write out $DF(z)$ as a block matrix corresponding to $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$, and similarly with $\mathbb{R}^{2m}$, writing $F = u + iv$, $u, v : O \to \mathbb{R}^m$, and deduce the Cauchy-Riemann equations

$$D_x u = D_y v, \ D_y u = -D_x v$$

hold if and only if F is holomorphic.

Problem 2. Show that all norms on a finite dimensional vector space V are equivalent, i.e. if $\|.\|_1$ and $\|.\|_2$ are norms on V then there is a constant $C > 0$ such that $\|v\|_1 \leq C\|v\|_2$ and $\|v\|_2 \leq C\|v\|_1$ for all $v \in V$. (Hint: You may assume V is real as every complex vector space is also a real vector space. Use a basis to reduce the question to norms on $\mathbb{R}^n$. Now use the fact that the unit sphere in $\mathbb{R}^n$ is compact in the Euclidean topology, and show that all norms on $\mathbb{R}^n$ are equivalent to the Euclidean norm.)

Problem 3. Do Taylor I.2.3.

For this problem, take the following extrinsic definition of a *smooth surface* in $\mathbb{R}^n$: A smooth k-dimensional surface M ($0 \leq k \leq n$) is a subset of $\mathbb{R}^n$, with the relative topology, such that for each $p \in M$ there is a neighborhood O of p in $\mathbb{R}^n$ and a smooth map: $\Phi : O \to \mathbb{R}^{n-k}$ such that

$$M \cap O = \Phi^{-1}(\{0\}) = \{x \in O : \ \Phi(x) = 0\}$$

and $D\Phi(p) : \mathbb{R}^n \to \mathbb{R}^k$ is a surjective linear map.

A *smooth hypersurface* is a smooth $n-1$-dimensional surface in M, in which case Φ is simply a real valued function, and the assumption is that $D\Phi(p) \neq 0$.

The meaning of *smooth* depends on the context, it usually means either C^1 or C^∞. (C^1 suffices here.)

Also use the following definition of tangency. First, a curve in M is a smooth map $\gamma : I \to \mathbb{R}^n$ and $\gamma(t) \in M$ for $t \in I$; here I is an interval. A vector $V \in \mathbb{R}^n$ is *tangent* to M at $p \in M$ if there is a curve $\gamma : I \to M$, I a neighborhood of 0, $\gamma(0) = p$, such that $\gamma'(0) = V$. A vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is tangent to M if for all $x \in M$, $F(x)$ is tangent to M at x.

In order to do the 'straightening out' of the hint, use the implicit function theorem by breaking up the standard coordinate functions on $\mathbb{R}^n$ into two groups (which you may then rearrange), (x, z), $x \in \mathbb{R}^k$, $z \in \mathbb{R}^{n-k}$, such that $D_z\Phi(p)$ is invertible, so on a neighborhood $O' \subset O$ of p, $M \cap O' = \{(x, f(x)) : x \in U\}$, $U \subset \mathbb{R}^k$ open, f smooth.

Now define a vector field $\tilde{F} : U \to \mathbb{R}^k$ such that solutions of $x'(t) = \tilde{F}(x(t))$ lift to solve the original ODE, i.e. letting $y(t) = (x(t), f(x(t)))$, $y'(t) = F(y(t))$.

Problem 4. Do Taylor I.3.1.

Problem 5. Do Taylor I.3.4.

Problem 6. Do Taylor I.4.3.

Problem 7. Do Taylor I.4.4.