MATH 174A: PROBLEM SET 3
DUE THURSDAY, FEBRUARY 1, 2007

Problem 1. (Cf. Taylor I.1.3.) Let $M_{n\times n}(\mathbb{C})$ denote the set of $n \times n$ complex matrices. Suppose $A \in M_{n\times n}(\mathbb{C})$ is invertible. Using

$$\det(A + tB) = (\det A)\det(I + tA^{-1}B)$$

show that

$$D\det(A)B = (\det A)\operatorname{Tr}(A^{-1}B).$$

(Hint: you have already shown $D\det(I)B = \operatorname{Tr} B$.)

Note: this shows that $\mathrm{SL}_n(\mathbb{C})$ defined as the set of matrices $A \in M_{n\times n}$ with $\det A = 1$ is a C^∞, indeed holomorphic, (hyper)surface in $M_{n\times n} = \mathbb{C}^{n^2}$: take $B = A$ to conclude that $D\det(A)$ is surjective. (The same calculation using $M_{n\times n}(\mathbb{R})$ shows that $\mathrm{SL}_n(\mathbb{R})$ is real analytic.)

Problem 2. Let $O_n(\mathbb{R})$ denote the set of matrices $A \in M_{n\times n}(\mathbb{R})$ with the property that $AA^t = I$; here A^t denotes the transpose of A (i.e. the ij entry of A^t is the ji entry of A). Let $S_{n\times n}(\mathbb{R})$ denote the set of symmetric matrices, i.e. matrices A such that $A^t = A$. Note that $S_{n\times n}$ can be identified with $\mathbb{R}^{\frac{n(n+1)}{2}}$ as for symmetric matrices A, the below diagonal entries are determined by the above diagonal entries.

Consider the map $F : M_{n\times n} \to S_{n\times n}$ given by $F(A) = AA^t$. Show that

$$(DF)(A)B = AB^t + BA^t,$$

and show that for $A \in O_n(\mathbb{R})$, $DF(A) : M_{n\times n} \to S_{n\times n}$ is surjective.

Use this to show that $O_n(\mathbb{R})$ is a compact surface in $M_{n\times n}$ of dimension $\frac{n(n-1)}{2}$. $O_n(\mathbb{R})$ is called the orthogonal group on $\mathbb{R}^n$.

Problem 3. Suppose that M is a smooth k-dimensional surface in $\mathbb{R}^n$. Show that for each $p \in M$, the set of vectors tangent to M at p form a k-dimensional linear subspace of $\mathbb{R}^n$. (Hint: Use the straightening out of the previous problem set.) We denote this by T_pM.

Show also that if on a neighborhood O of p in $\mathbb{R}^n$, M is defined by $\Phi = 0$, then T_pM is the nullspace of $D\Phi(p) : \mathbb{R}^n \to \mathbb{R}^{n-k}$.

Use this to conclude that the disjoint union of tangent spaces T_pM, $p \in M$, is a $2k$-dimensional surface in $\mathbb{R}^{2n}$: consider the set

$$TM = \{(p, v) \in \mathbb{R}^{2n} :\ p \in M,\ v \in T_pM\},$$

and show that the map

$$F : O \times \mathbb{R}^n \to \mathbb{R}^{2(n-k)},\ F(x, v) = (\Phi(x), D\Phi(x)v)$$

defines TM on $O \times \mathbb{R}^n$. (That is, TM is given by $F = 0$, and DF is surjective on TM.) TM is called the tangent bundle of M.

Note: Let $(.,.)$ denote the standard inner product on $\mathbb{R}^n$. Every $v \in \mathbb{R}^n$ defines a linear map $\iota(v) : \mathbb{R}^n \to \mathbb{R}$ by $\iota(v)w = (v, w)$. Conversely, for every linear map $A : \mathbb{R}^n \to \mathbb{R}$ there is a vector $v \in \mathbb{R}^n$ such that for all $w \in \mathbb{R}^n$, $(v, w) = Aw$, i.e. ι is surjective. (ι is also injective.)

Now, if $\Phi = (\Phi_1, \dots \Phi_{n-k})$, then $D\Phi_j(p)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$, $\nabla\Phi_j(p)$ denotes the image of $D\Phi_j(p)$ under ι^{-1}; it is of course just $(\partial_1\Phi_j, \dots, \partial_n\Phi_j)$.

Thus, we can reinterpret the result above: T_pM is the orthocomplement of the span of $\nabla\Phi_1(p), \dots, \nabla\Phi_{n-k}(p)$.

Problem 4. Do Taylor I.4.9.

Problem 5. Do Taylor I.5.1. (Use the alternative hint.)

Problem 6. Do Taylor I.5.2.