MATH 174A: PROBLEM SET 4
DUE WEDNESDAY, FEBRUARY 7, 2007

Problem 1. Suppose V is a finite dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Its dual V^* is the vector space $\mathcal{L}(V, \mathbb{F})$ of linear maps from V to $\mathbb{F}$. The elements of V^* are called linear functionals on V.

(1) Show that V^* is finite dimensional, $\dim V^* = \dim V$, and in fact if $e_1, \ldots, e_n$ is a basis of V then the linear functionals $f_1, \ldots, f_n$ defined by

$$f_j(e_k) = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k, \end{cases}$$

and extended to V by linearity:

$$f_j(\sum_k a_k e_k) = \sum_k a_k f_j(e_k) = a_j,$$

are a basis of V^*. (Hint: Suppose that $f = \sum_j b_j f_j$ and find the $b_j's$. Now just *define* the b_j by the resulting formula, and show that they work.) $\{f_1, \ldots, f_n\}$ is called the basis *dual* to $\{e_1, \ldots, e_n\}$.

(2) If V is real (i.e. $\mathbb{F} = \mathbb{R}$) and has an inner product, there is a natural map $\iota \in \mathcal{L}(V, V^*)$, namely $\iota(v)(w) = (v, w)$, where $(.,.)$ on the right hand side is the inner product. (There's an analogous map if $\mathbb{F} = \mathbb{C}$, but it is conjugate linear.) Show that ι is a bijection, hence an isomorphism of vector spaces. Thus, *given an inner product* V can be identfied with V^*, but the identification depends on the *choice of the inner product*.

(3) For $v \in V$, consider the map $j \in \mathcal{L}(V, \mathcal{L}(\mathbb{F}, V))$ given as follows: $j(v) \in \mathcal{L}(\mathbb{F}, V)$ is the map $j(v)a = av$, $a \in \mathbb{F}$. Show that j is a bijection from V to $\mathcal{L}(\mathbb{F}, V)$, hence V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic.

Problem 2. If V, W are finite dimensional vector spaces over $\mathbb{R}$, $O \subset V$, and $F : O \to W$ is a C^1 map, we have defined its derivative $DF(p)$ at $p \in O$ as an element of $\mathcal{L}(V, W)$.

(1) If V is a vector space, $\gamma : I \to V$ a C^1 curve with $\gamma(0) = p$, show that $D\gamma(0)$ can be naturally identified with an element $\gamma'(0)$ of V.

(2) Show that the tangent space T_pV of V at p, defined as the set of vectors v in V for which there is a curve γ with $\gamma(0) = p$ and $\gamma'(0) = v$ is all of V. We define the tangent bundle of V as the disjoint union of the T_pV, $p \in V$, i.e. since $T_pV = V$, as $TV = V \times V$.

(3) If $O \subset V$, $p \in O$, $f : O \to \mathbb{R}$, then $Df(p) \in \mathcal{L}(V, \mathbb{R}) = V^*$. One usually writes $df(p) = DF(p)$. Show that the cotangent space T_p^*V of V at p, defined as the set of elements α of V^* for which there is a C^1 function f defined near p with $df(p) = \alpha$, is all of V^*. We define the cotangent bundle of V as the disjoint union of the T_p^*V, $p \in V$, i.e. as $T_p^*V = V^*$, as $T^*V = V \times V^*$. (Note that TV can be identified with T^*V if one is given an inner product, but the identification depends on the inner product.)

(4) Notice that T^*V itself is a vector space: $T^*V = V \oplus V^*$. Write elements of T^*V as $w = (v, \alpha)$. We define a map $\Omega : T^*V \times T^*V \to \mathbb{R}$ by

$$\Omega((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_1(v_2) - \alpha_2(v_1).$$

Show that Ω is bilinear, i.e.

$$\Omega(cw_1, w_2) = c\Omega(w_1, w_2) = \Omega(w_1, cw_2),\ c \in \mathbb{R},\ w_1, w_2 \in T^*V,$$

$$\Omega(w_1 + w_2, w_3) = \Omega(w_1, w_3) + \Omega(w_2, w_3),\ w_1, w_2, w_3 \in T^*V,$$

with similar additivity in the second slot, Ω is antisymmetric, i.e.

$$\Omega(w_1, w_2) = -\Omega(w_2, w_1),\ w_1, w_2 \in T^*V,$$

and is non-degenerate, i.e. for $w_1 \in T^*V$ non-zero, there is $w_2 \in T^*V$ such that $\Omega(w_1, w_2) \neq 0$.

(5) Note that Ω (indeed, any bilinear form on $T^*V \times T^*V$) defines a map $J : T^*V \to (T^*V)^*$ as follows: for $w \in T^*V$, $J(w)w' = \Omega(w', w)$. Show that this map is an isomorphism using that Ω is non-degenerate.

For each $p \in T^*V$, T_pT^*V can be identified with T^*V, hence one obtains a non-degenerate bilinear antisymmetric map ω_p on T_pT^*V. It is called the *symplectic form.*

Problem 3. Do Taylor I.3.5. (Assume that if $F : O \to \mathbb{C}^m$ is holomorphic, where $O \subset \mathbb{C}^n$ is open, and $p \in O$, then the Taylor series of F converges in a neighborhood of p. We will prove this in the second half of the course.)

Problem 4. Do Taylor I.6.1.

Problem 5. Do Taylor I.6.2.

Problem 6. Do Taylor I.7.1.

Problem 7. Do Taylor I.7.2.

Problem 8. Do Taylor I.7.3.

Problem 9. Do Taylor I.7.9, assuming that $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ has n real linearly independent eigenvectors, and each eigenvalue is negative. (Hint: show that A is self-adjoint with respect to *some* inner product $\langle ., . \rangle$ on $\mathbb{R}^n$, i.e. $\langle Au, v \rangle = \langle u, Av \rangle$ for $u, v \in \mathbb{R}^n$.)