MATH 174A: PROBLEM SET 5
DUE THURSDAY, FEBRUARY 22, 2007

Problem 1. Suppose that $I = [a,b]$ is an interval. Let $\|f\|_1 = \int_a^b |f(x)|\,dx$ for $f \in C(I;\mathbb{R})$ (i.e. f is a continuous real-valued function on I), and let $L^1(I)$ denote the completion of $C(I;\mathbb{R})$ with respect to $\|.\|_1$.

(1) Show that the Riemann integral $\int : C(I;\mathbb{R}) \to \mathbb{R}$, $\int f = \int_a^b f(x)\,dx$, extends to a bounded linear map $\int : L^1(I) \to \mathbb{R}$ (which would be called the Lebesgue integral).

(2) Suppose $f \in L^1(I)$, i.e. f is the equivalence class of a Cauchy sequence $\{f_n\}$ of continuous functions. We say that $f \geq 0$ if there exists a Cauchy sequence $\{f_n\}$ representing f such that $f_n \geq 0$ or all n. Show that if $f \geq 0$ and $g \geq 0$ then $f + g \geq 0$, and if $c \in \mathbb{R}$, $c \geq 0$ then $cf \geq 0$. Show also that if $f \geq 0$ then $\int f \geq 0$. (Thus, it makes sense to say whether an element of $L^1(I)$ is non-negative on I. We also say $f \geq g$ if $f - g \geq 0$, so it makes sense to say that an element of $L^1(I)$ is greater than another element of $L^1(I)$.)

(3) Show that if $f \geq 0$ and $-f \geq 0$ then $f = 0$. Use this to conclude that $\geq$ is a partial order on $L^1(I)$, i.e. $f \geq g$ and $g \geq h$ implies $f \geq h$, and $f \geq g$ and $g \geq f$ implies $f = g$.

(4) Suppose that $\{f^{(k)}\}_{k=1}^{\infty}$ is a sequence in $L^1(I)$ and $f^{(k)} \to f$ in $L^1(I)$. Show that if $f^{(k)} \geq 0$ for all k then $f \geq 0$. (Hint: Consider $\tilde{f}^{(k)} \in C(I;\mathbb{R})$ such that $\tilde{f}^{(k)} \geq 0$ and $\tilde{f}^{(k)}$ is close to $f^{(k)}$.)

(5) Suppose $x \in I$. Show that there is no continuous linear map $E_x : L^1(I) \to \mathbb{R}$ such that $E_x(f) = f(x)$ for all $f \in C(I;\mathbb{R})$. That is, evaluating continuous functions at x cannot be extended in a reasonable manner to $L^1(I)$, i.e. elements of $L^1(I)$ do not have values at any given point. (Hint: If such a map existed, and $f \in L^1(I)$ were represented by a Cauchy sequence $\{f_n\}$, what would $E_x(f)$ be? Now find different Cauchy sequences representing the same f.)

Problem 2. Suppose that V is an inner product space, D is a linear subspace, and $A : D \to V$ is a linear operator (not necessarily continuous). We say that A is symmetric if $(Au, v) = (u, Av)$ for all $u, v \in V$. We say that $v \in D$ is an eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$ if $Av = \lambda v$.

(1) Show that if $A : D \to V$ is symmetric, then all eigenvalues of A are real, and all eigenvectors corresponding to different eigenvalues are orthogonal, i.e. $Av = \lambda v$, $Aw = \mu w$, $\lambda \neq \mu$ implies $(v, w) = 0$.

(2) Let $A = \frac{1}{i}\frac{d}{dx}$ on $D = \{f \in C^1([0, 2\pi]) : \ f(0) = f(2\pi)\}$, with $V = C([0, 2\pi])$. Show that A is symmetric, and the functions e^{inx} are orthogonal to each other on $[0, 2\pi]$.

(3) Let $V = C([a,b])$, and let D be a subspace of $C^2([a,b])$. Under what conditions on D is A, given by $Af = -f''$, symmetric? (Hint: Calculate $(Af, g) - (f, Ag)$.)

(4) Show that the functions $\sin nx$, $n \geq 1$ integer, are orthogonal to each other on $[0, \pi]$.
(5) Show that the functions $\sin(n + \frac{1}{2})x$, $n = 0, 1, 2, \dots$ are orthogonal to each other on $[0, \pi]$.

Problem 3. (cf. Taylor 3.1.5, 3.1.6) Suppose f is a 2π-periodic C^1 function on $\mathbb{R}$, i.e. $f \in C^1_p([0, 2\pi])$. Let $c_n = (f, e^{inx})$, $b_n = (f', e^{inx})$, where

$$(f, g) = (2\pi)^{-1} \int_0^{2\pi} f(x)\,\overline{g(x)}\,dx.$$

(1) Prove that $\sum |b_n|^2 < \infty$ and conclude that $\sum n^2 |c_n|^2 < \infty$.
(2) Prove that $\sum |c_n| < \infty$.
(3) Prove that $\sum_{n=-M}^{M} c_n e^{inx}$ is uniformly convergent as $M \to \infty$.
(4) Let $S_N(f)(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$. Prove that

$$(S_N f)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta + x) \frac{\sin(N + \frac{1}{2})x}{\sin(x/2)}\,dx.$$

The 2π-periodic function

$$D_N(x) = \frac{\sin(N + \frac{1}{2})x}{\sin(x/2)} = \sum_{n=-N}^{N} e^{inx}$$

is known as the Dirichlet kernel.

(5) Using that $\frac{1}{2\pi} \int_0^{2\pi} D_N(x)\,dx = 1$, show that for $f \in C^1_p([0, 2\pi])$

$$f(\theta) - S_N f(\theta) = \left(\sqrt{2} \sin(N + \frac{1}{2})x, g_\theta \right),$$

$g_\theta(x) = \frac{f(\theta+x) - f(\theta)}{\sqrt{2}\sin(x/2)}$, $g_\theta \in C^0([0, 2\pi])$.

(6) Using that $\sqrt{2}\,\sin(N + \frac{1}{2})x$, $N = 0, 1, 2, \dots$ is an orthonormal set in $L^2([0, 2\pi])$, show that $S_N f(\theta) \to f(\theta)$, and conclude that the Fourier series of $f \in C^1_p([0, 2\pi])$ converges uniformly to f.

Problem 4. (cf. Taylor 3.1.8.) Suppose now that f is piecewise C^1 and 2π-periodic on $\mathbb{R}$, i.e. there exist a finite number of points $x_j \in [0, 2\pi)$, $j = 1, 2, \dots, n$, such that $f(x_j\pm) = \lim_{x \to x_j \pm} f(x)$ and $\lim_{x \to x_j \pm} f'(x)$ exist, but are not necessarily equal to each other, and away from

$$S = \{x_j + 2\pi k : \ j = 1, \dots, n, \ k \in \mathbb{Z}\},$$

f is C^1.

Show that at each $x \notin S$, the Fourier series converges to $f(x)$, while at each x_j, the Fourier series converges to $\frac{1}{2}\left(f(x_j+) + f(x_j-)\right)$.

Hint: use that $\sqrt{2}\,\sin(N + \frac{1}{2})x$, $N = 0, 1, 2, \dots$ is an orthonormal set in $L^2([0, \pi])$ with inner product

$$(f, g)_{L^2([0,\pi])} = \pi^{-1} \int_0^{\pi} f(x)\,\overline{g(x)}\,dx.$$