MATH 174A: PROBLEM SET 6
DUE THURSDAY, MARCH 1, 2007

**Problem 1.** For $f, g \in C(\mathbb{S}^1)$, let
$$f * g = \frac{1}{2\pi} \int_{\mathbb{S}^1} f(\theta - \omega) g(\omega) \, d\omega.$$

(1) Show that $f * g \in C(\mathbb{S}^1)$ and $\|f * g\|_\infty \leq \|f\|_\infty \|g\|_\infty$, where $\|f\|_\infty = \sup\{|f(\theta)| : \theta \in \mathbb{S}^1\}$.

(2) Show that $f * g = g * f$.

(3) Show that in fact $\|f * g\|_\infty \leq \|f\|_\infty \|g\|_1$, where $\|g\|_1 = \frac{1}{2\pi} \int |g(\theta)| \, d\theta$. Use this to show that convolution extends to a continuous (i.e. bounded) bilinear map
$$* : C(\mathbb{S}^1) \times L^1(\mathbb{S}^1) \to C(\mathbb{S}^1).$$

(4) Show that for $f, g \in C(\mathbb{S}^1)$, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. Use this to show that convolution extends to a bilinear map
$$* : L^1(\mathbb{S}^1) \times L^1(\mathbb{S}^1) \to L^1(\mathbb{S}^1).$$

(5) Show that if $f \in C^1(\mathbb{S}^1)$ and $g \in L^1(\mathbb{S}^1)$ then $f * g \in C^1(\mathbb{S}^1)$, $(f * g)' = f' * g$. (Hint: Assume $g \in C(\mathbb{S}^1)$, and show $f * g \in C^1(\mathbb{S}^1)$, $\|f * g\|_{C^1} \leq \|f\|_{C^1} \|g\|_{L^1}$, with $\|f\|_{C^k} = \sum_{j=0}^{k} \|f^{(j)}\|_\infty$.)

(6) Show more generally that if $f \in C^k(\mathbb{S}^1)$ and $g \in L^1(\mathbb{S}^1)$ then $f * g \in C^k(\mathbb{S}^1)$, and $(f * g)^{(k)} = f^{(k)} * g$.

(7) Show that $\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g)$.

**Problem 2.** Show that $L^2(\mathbb{S}^1) \subset L^1(\mathbb{S}^1)$. That is, show that the identity map $i : C(\mathbb{S}^1) \to C(\mathbb{S}^1)$ extends to an injective continuous linear map $\iota : L^2(\mathbb{S}^1) \to L^1(\mathbb{S}^1)$.

Hint: Show that there is $C > 0$ such that for all $f \in C(\mathbb{S}^1)$, $\|f\|_1 \leq C\|f\|_2$, and use this to conclude that we get a bounded linear extension $\iota : L^2(\mathbb{S}^1) \to L^1(\mathbb{S}^1)$. Here $\|.\|_2$ is the $L^2$-norm. Now suppose $\iota(f) = 0$, $f \in L^2(\mathbb{S}^1)$. Let $\{f_n\}$, $f_n \in C(\mathbb{S}^1)$ be a Cauchy sequence in $L^2$, $f_n \to f$. Show that there is $\phi \in C(\mathbb{S}^1)$ such that $(f, \phi)_{L^2} \neq 0$ (hint: $C(\mathbb{S}^1)$ is dense in $L^2(\mathbb{S}^1)$), hence $\lim_{n \to \infty} (f_n, \phi)_{L^2} \neq 0$. But now show that $|(f_n, \phi)| \leq C\|f_n\|_1$.

**Problem 3.** Show that if $x \in \mathbb{R}^n$ and $\delta > 0$ then there exists $\phi \in C^\infty(\mathbb{R}^n)$ such that $\phi \geq 0$, $\phi(x) > 0$ and if $\|y - x\| \geq \delta$ then $\phi(y) = 0$. (Hint: Show first that the function $\chi : \mathbb{R} \to \mathbb{R}$ defined by $\chi(t) = 0$ for $t \leq 0$, $\chi(t) = e^{-1/t}$, $t > 0$, is $C^\infty$.)

**Problem 4.** Do Taylor 3.2.1.

**Problem 5.** Do Taylor 3.2.2.

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