MATH 174A: PROBLEM SET 7
DUE THURSDAY, MARCH 8, 2007

Problem 1. Use separation of variables to solve the Dirichlet problem for the Laplacian: $\Delta u = 0$ in Ω, $u|_{\partial\Omega} = f$ given, where $\Omega = \{z \in \mathbb{R}^2 : \ a < |z| < b\}$, $a, b > 0$, is an annulus. (Hint: remember that there are two linearly independent solutions of the radial ODE. Keep them both. Also let f_a, resp. f_b be f at the two boundary circles, so $u(z) = f_a(z)$, if $|z| = a$, etc.)

Problem 2. The Fourier sine series of a function f on $[0, \ell]$, $\ell > 0$, is the expansion

$$\sum_{n=1}^{\infty} a_n e_n,\ e_n(x) = \sin(n\pi x/\ell),\ a_n = (f, e_n)_{[0,\ell]},\ (f, g)_{[0,\ell]} = \frac{2}{\ell}\int_0^{\ell} f\,\overline{g}\,dx.$$

(1) Show that if $f \in C^1([0, \ell])$ and satisfies homogeneous Dirichlet boundary conditions, i.e. $f(0) = 0$, $f(\ell) = 0$, then the Fourier sine series converges to f uniformly.
(2) Suppose f is piecewise C^1 on $[0, \ell]$. What does its Fourier sine series converge to pointwise?

Hint: if $f \in C^1([0, \ell])$, extend f to be an odd 2ℓ-periodic function F on $\mathbb{R}$, let $g(x) = F(\pi x/\ell)$, and use the results from $\mathbb{S}^1$. Notice that the Fourier sine series of f just becomes the standard Fourier series of g!

Note: if $f \in L^2([0, \ell])$, then the Fourier sine series of f converges to f in L^2, but you do not need to prove this.

Problem 3. (1) Prove the following maximum principle for the heat equation $u_t = k\Delta u$ on $D = \Omega \times (0, T)_t$, $\Omega \subset \mathbb{R}^n$ bounded, open, $T > 0$, $k > 0$:
If $u \in C^2(D) \cap C(\overline{D})$, then $\max_{\overline{D}} u$ is attained either on $\partial\Omega \times [0, T]$ or on $\overline{\Omega} \times \{0\}$.
(2) Use this to state and prove a uniqueness and stability result for solutions of the heat equation: $u_t = k\Delta u$ on $\Omega \times (0, \infty)$, $u(x, t) = h(x, t)$ given if $x \in \partial\Omega$, $u(x, 0) = \phi(x)$ given, $x \in \Omega$.

Problem 4. Solve the heat equation on the interval, representing a rod whose ends are kept at temperature 0, and whose initial temperature is ϕ:

$$u_t = ku_{xx},\ (x, t) \in (0, \ell) \times (0, \infty),\ k > 0,$$
$$u(0, t) = 0,\ u(\ell, t) = 0,$$
$$u(x, 0) = \phi(x),$$

$\phi \in C^1([0, \ell])$, $\phi(0) = 0 = \phi(\ell)$, by separating variables and using Fourier sine series in x. Make sure that you prove that your series solutions actually solves the PDE and satisfies the boundary and initial conditions.

Show also that the solution is C^∞ for $t > 0$.

Problem 5. (Taylor 3.3.1) Let $C_0(\mathbb{R}^n)$ denote the space of continuous functions v on $\mathbb{R}^n$ such that $v(\xi) \to 0$ as $|\xi| \to \infty$. Show that the Fourier transform satisfies $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$. (Hint: $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, and prove that if f_j is a sequence in $\mathcal{S}(\mathbb{R}^n)$ which converges uniformly, the limit f is in $C_0(\mathbb{R}^n)$.) This is the Riemann-Lebesgue lemma.