MATH 174A: PROBLEM SET 8
DUE THURSDAY, MARCH 15, 2007

Problem 1. Do Taylor 3.3.15. (You can assume 3.3.14. We have discussed the relevant part it in class.)

Problem 2. Do Taylor 3.3.16.

Problem 3. Do Taylor 3.4.1.

Problem 4. Do Taylor 3.4.2.

Problem 5. Suppose that $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$, is a constant coefficient differential operator. We say that P is elliptic if the polynomial $P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ has no zeros $\xi \in \mathbb{R}^n \setminus \{0\}$.

(1) Show that if P is elliptic then there exists $c > 0$ such that $|P_m(\xi)| \geq c|\xi|^m$, $\xi \in \mathbb{R}^n \setminus \{0\}$.

(2) Show that if P is non-zero on $\mathbb{R}^n$, then it is elliptic, and the PDE $P(D)u = f$, $f \in \mathcal{S}'(\mathbb{R}^n)$ given, has a unique solution $u \in \mathcal{S}'(\mathbb{R}^n)$. (Hint: show that $|P(\xi)| \geq c(1 + |\xi|)^m$ for some $c > 0$.) Show also that if $f \in \mathcal{S}(\mathbb{R}^n)$ then $u \in \mathcal{S}(\mathbb{R}^n)$.

(3) Show that if P is elliptic, $u \in \mathcal{S}'(\mathbb{R}^n)$, $P(D)u = f$, and $f \in \mathcal{S}(\mathbb{R}^n)$ then $u \in C^\infty(\mathbb{R}^n)$. (Hint: Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a 'bump function': $\chi(\xi) = 1$ if $|\xi| < R$, $\chi(\xi) = 0$ if $|\xi| > 2R$. Choose $R > 0$ appropriately, and write $\mathcal{F}u = \chi\mathcal{F}u + (1 - \chi)\mathcal{F}u$.)

It is a little harder, but not hard, to prove *elliptic regularity*, namely that if P is elliptic, $P(D)u = f$, $f \in C^\infty(\mathbb{R}^n)$, then $u \in C^\infty(\mathbb{R}^n)$.

Problem 6. Suppose that (X, d) is a metric space.

(1) Suppose that $f : [0,\infty) \to [0,\infty)$ is C^1 (continuously differentiable), $f(0) = 0$, $f'(0) > 0$, $f'(x) \geq 0$ for all x, and f' is decreasing (i.e. $x \leq y$ implies $f'(x) \geq f'(y)$). Show that $f \circ d : X \times X \to [0,\infty)$ is a metric on X. (Hint: show that f is increasing, and $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0,\infty)$.)

(2) Suppose d and d' are metrics on X. One says that the topology generated by d' is weaker than the topology of d if every d'-open set is d-open. If d' and d have the same open sets, they are called equivalent.

Show that the topology generated by d' is weaker than the topology generated by d if and only if given $\epsilon > 0$ and $x \in X$ there is $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d'(x, y) < \epsilon.$$

Use this to show that with f as in (1), d and $f \circ d$ generate the same topology.

(3) Conclude that if d is a metric on X, then so is $d' = \frac{d}{1+d}$, and these two metrics generate the same topology. Note that $d'(x, y) < 1$ for all $x, y \in X$.

Problem 7. Let $\rho_1, \rho_2, \ldots,$ be metrics on X with $\rho_j \leq 1$. Let

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \rho_j(x, y). \tag{1}$$

(1) Show that d is a metric on X.
(2) Show that a sequence $\{x_n\}$ converges to some $x \in X$ with respect to d if and only if it converges with respect to ρ_j for every j, i.e. if and only if given j and $\epsilon > 0$ there is N such that $n \geq N$ implies $\rho_j(x_n, x) < \epsilon$.
(3) Now suppose that X is a vector space and each d_j is a translation invariant metric, i.e. $d_j(x+z, y+z) = d_j(x, y)$ for all $x, y, z \in X$. Let ρ_j be translation invariant metrics equivalent to d_j with $\rho_j < 1$. Show that a sequence $\{x_n\}$ is Cauchy with respect to d if and only if it is Cauchy with respect to every d_j.
(4) Now suppose that $X_1, X_2, \ldots$ are vector spaces, $X_1 \supset X_2 \supset \ldots$ and $X = \cap_{k=1}^{\infty} X_k$. Let d_k be translation invariant metrics on X_k, and suppose that the inclusion maps $\iota_k : X_k \to X_{k-1}$ are all continuous. Show that if (X_k, d_k) is complete for every k then (X, d) is complete.
(5) Let $C^\infty(\mathbb{S}^1)$ denote the set of complex valued infinitely differentiable functions on $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Let d_k be the metric given by the C^k norm:

$$\|f\|_{C^k} = \sum_{m=0}^{k} \sup\{|f^{(m)}(x)| : \ x \in \mathbb{S}^1\}.$$

Let d be the corresponding metric on $C^\infty(\mathbb{S}^1)$. Show that $C^\infty(\mathbb{S}^1)$ is a complete metric space in which sequences $\{x_n\}$ converge, resp. are Cauchy, if and only if they converge, resp. are Cauchy, in every C^k. (Thus, convergence of a sequence $\{f_n\}$ is just the uniform convergence of all derivatives $\{f_n^{(k)}\}$.)