MATH 174A: MIDTERM
DUE AT 4PM ON MONDAY, FEBRUARY 12, 2007

Problem 1. The purpose of this problem is to see how our results so far help us solve first order PDEs. Suppose O is an open subset of $\mathbb{R}^n$, and consider a scalar semilinear PDE on O, i.e. suppose that X is a C^1 vector field on O (so $X : O \to \mathbb{R}^n$), $c : O \times \mathbb{R} \to \mathbb{R}$ is C^1, and consider the PDE

$$X(x)u(x) = c(x, u(x)),\ u \in C^1(O; \mathbb{R}).$$

Suppose also that M is a hypersurface such that for $p \in M \cap O$, $X(p)$ is not in the tangent space of M, and we want $u|_M = w$, where w is a given C^1 function.

(1) Show that if γ is an integral curve of X and $v = u \circ \gamma$ satisfies $v'(t) = c(\gamma(t), v(t))$, then the PDE is satisfied at $x = \gamma(t)$. Show also conversely, if the PDE holds, then the ODE $v'(t) = c(\gamma(t), v(t))$ must be satisfied for all integral curves.
(2) Show that the PDE has a unique C^1 solution u near M. (Hint: 'straighten out' the vector field X.)
(3) Show that if X and c are C^k, then u is also C^k.
(4) Solve the following PDE explicitly:

$$u_x + xu_y = u^2,\ u(0, y) = \sin y.$$

Show that while the solution exists for $|x|$ small, it does not extend to all of $\mathbb{R}^2$. (Hint: You need not straighten out the vector field. Find the integral curves directly.)

Problem 2. Suppose M is a compact k-dimensional surface in $\mathbb{R}^n$, and h is a C^∞ function on $\mathbb{R}^n$. Then

$$\nabla h = (\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n})$$

is a C^∞ vector field on $\mathbb{R}^n$. For $p \in M$, let $X(p) \in T_pM$ be the orthogonal projection of $\nabla h(p)$ to T_pM, i.e. $\nabla h(p) - X(p) \in (T_pM)^\perp$. Let $\mathcal{F}_X^t$ denote the flow of X at time t.

(1) Show that for $p \in M$ and $v \in T_pM$, $X(p) \cdot v = Dh(p)v$, where $\cdot$ on the left hand side is the standard inner product on $\mathbb{R}^n$.
(2) Show that for all $p \in M$, $h(\mathcal{F}_X^t(p))$ is an increasing function of t.
(3) We say that a point $q \in M$ is a critical point of $h|_M$ if $Dh(q)v = 0$ for $v \in T_qM$. Show that the critical points of $h|_M$ are exactly the critical points of X (i.e. the points q where $X(q) = 0$).
(4) Suppose that there are finitely many critical points of $h|_M$. Show that for all $p \in M$, $\lim_{t\to+\infty} \mathcal{F}_X^t(p)$ exists (and similarly for $t \to -\infty$) and is a critical point of h. (Hint: consider $\gamma_n(t) = \mathcal{F}_X^{t+T_n}(p)$. For each n, γ_n is an integral curve of X. What can you say about these if $\mathcal{F}_X^{T_n}(p)$ converges?)
(5) If $M = \mathbb{S}^{n-1}$ is the unit sphere given by $M = \{x \in \mathbb{R}^n : \ \|x\| = 1\}$ and $h(x) = x_n$, find $\lim_{t\to+\infty} \mathcal{F}_X^t(p)$ and $\lim_{t\to-\infty} \mathcal{F}_X^t(p)$ for all $p \in M$.

(6) (Extra credit only!) If $q \in M$ then in Problem Set 2 you have constructed a vector field $\tilde{X}$ on $U \subset \mathbb{R}^k$ open and a smooth map $\Psi : U \to M$ such that $\Psi(q') = q$ for some q', $D\Psi(q')$ is injective, and integral curves of $\tilde{X}$ map to integral curves of X under Ψ. (With the push-forward notation we introduced in class, $X = \Psi_* \tilde{X}$.) Show that if q is a critical point of $h|_M$ then $\tilde{X}(x) = A(x - q') + O(\|x - q'\|^2)$ where $A = D\tilde{X}(q')$ is called the linearization of $\tilde{X}$. Show also that A is symmetric with respect to the inner product induced by Ψ, i.e. $(Au, v) = (u, Av)$, with $(u, v) = \langle \Psi_* u, \Psi_* v \rangle$, where $\langle ., . \rangle$ is the standard inner product on $\mathbb{R}^n$.

(7) (Extra credit only!) Show that if q is a local minimum, resp. maximum, of h then $(Au, u) \geq 0$, resp. $(Au, u) \leq 0$ for all $u \in \mathbb{R}^k$. If these inequalities are strict, the local minima/maxima are called non-degenerate. Use this to conclude that $\|x - q'\|^2$ is increasing, resp. decreasing, in some neighborhood of q' along integral curves of $\tilde{X}$.

(8) (Extra credit only!) Suppose now that $h|_M$ has only two critical points, q_1, q_2, and they are non-degenerate. Show that $M \setminus \{q_1, q_2\}$ is diffeomorphic to the cylinder $\mathbb{R} \times \mathbb{S}^{k-1}$, and M itself is homeomorphic to $\mathbb{S}^k$.

The flow of $-X$ is called the gradient flow of h; the flow of X is thus the time-reversed gradient flow.

Note: if you prefer to have X defined near M, rather than just on M, near $p_0 \in M$ you can take $\Phi : \mathbb{R}^n \to \mathbb{R}^{n-k}$, as in the definition of a surface, such that $D\Phi(p_0)$ is surjective and M is the set of points with $\Phi = 0$. Then the surfaces $M_c = \{x : \Phi(x) = c\}$, $c \in \mathbb{R}^{n-k}$, can be used (locally) to define X: $X(q)$ is the orthogonal projection of ∇h to $T_q M_c$, with $c = \Phi(q)$.

Problem 3. Suppose V is a finite dimensional real vector space, $T^*V = V \times V^*$ its cotangent bundle. For $p \in V$, let ω_p be the symplectic form on $T_p T^*V$, and let $J_p : T_p T^*V \to T_p^* T^*V$ be the isomorphism induced by ω_p as in Problem Set 4.

(1) If $e_1, \dots, e_n$ is a basis of V, $f_1, \dots, f_n$ is the dual basis of V^*, then

$$(e_1, 0), \dots, (e_n, 0), (0, f_1), \dots, (0, f_n)$$

is a basis of $T^*V = V \times V^*$; one often writes the corresponding coordinates on T^*V as $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$. Find the matrix of J_p explicitly with respect to this basis.

(2) Suppose that $f : T^*V \to \mathbb{R}$ is a C^∞ function. For $p \in T^*V$, define a vector $H_f(p) \in T_p T^*V$ by $H_f(p) = J_p^{-1} df(p)$, where $df(p) \in T_p^* T^*V$ is the differential of f at p. Show that

$$\omega_p(v, H_f(p)) = df(p)v$$

for all $v \in T_p T^*V$, and $H_f : T^*V \to TT^*V$, $H_f : p \mapsto H_f(p)$, is a C^∞ map.

(3) Find $H_f(p)$ explicitly in the basis described in (1).

(4) A bicharacteristic γ is an integral curve of H_f, i.e. a C^1 map $\gamma : I \to T^*V$ such that $\gamma'(t) = H_f(\gamma(t))$ for all $t \in I$. Write down this equation explicitly as an ODE using the basis described in (1).

(5) Show that bicharacteristics are C^∞. Show also that f is constant along bicharacteristics γ, i.e. $f(\gamma(t_1)) = f(\gamma(t_2))$ for all $t_1, t_2 \in I$.

(6) Suppose that f is such that $f(p) \to +\infty$ as $\|p\| \to \infty$ – here $\|.\|$ is some norm on T^*V (recall that all norms are equivalent on T^*V). Show that the

integral curves γ can be defined on all of $\mathbb{R}$ (i.e. any integral curve can be extended to another one that is defined on all of $\mathbb{R}$).

(7) Now suppose $V = \mathbb{R}^n$ with the standard inner product, $f((x,\xi)) = \frac{1}{2}(a\|x\|^2 + \|\xi\|^2)$, $a > 0$. Find the bicharacteristics explicitly.

In classical mechanics, one regards V as the configuration space, usually corresponding to positions of particles, T^*V as the phase space, with the V^* factor corresponding to momenta, f is the energy function on T^*V, and bicharacteristics are trajectories of particles, and the corresponding ODE is Hamilton's equations of motion. The constancy of f along bicharacteristics is the conservation of energy. If $f(p) \to +\infty$ as $\|p\| \to \infty$, one calls f confining. The example is the harmonic oscillator; $\frac{1}{2}\|\xi\|^2$ is the kinetic energy and $\frac{a}{2}\|x\|^2$ is the potential energy.

Note that H_f and the bicharacteristics are defined without reference to a basis – so they do not depend on choices of a basis. In other words, the dynamics looks the same no matter what basis you choose. In fact, more is true: even diffeomorphisms 'preserve the dynamics', i.e. it is not affected by changes of coordinates – provided one relates f in the two settings to each other the correct way.