# MATH 174A: FINAL EXAM
# THURSDAY, MARCH 22, 2007

There are three problems. Do all of them.

**Problem 1.** Consider the vector field $X(x,y) = (-y,x) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ in $\mathbb{R}^2$.

(1) Find all integral curves of $X$.
(2) Show that the equation $Xu = 0$, $u \in C^1(\mathbb{R}^2)$, with initial condition $u(0,y) = f(y)$, $y > 0$, $f$ given $C^1$ function vanishing near 0, has a unique solution.
(3) Can $Xu = 0$, $u \in C^1(\mathbb{R}^2)$, with initial condition $u(0,y) = f(y)$, $y \in \mathbb{R}$, $f$ given $C^1$ function vanishing near 0, be solved in general?

**Problem 2.** Consider the heat equation $u_t = ku_{\theta\theta}$ is a thin ring, where $u(t,\theta)$ is the temperature at time $t$, place $\theta$, and suppose that the initial temperature of the ring is $u(\theta,0) = \phi(\theta)$, $\theta \in \mathbb{S}^1$, $\phi$ is a given function in $C^1(\mathbb{S}^1)$.

(1) Using the separation of variables, show that the solution of the heat equation is of the form
$$\sum_{n\in\mathbb{Z}} A_n e^{-n^2kt}e^{in\theta},$$
and find $A_n$ in terms of $\phi$.
(2) The total heat energy in the ring at time $t$ is
$$Q(t) = \int_0^{2\pi} u(\theta,t)\,d\theta.$$
Using the PDE, show that $Q(t)$ is a constant (independent of $t$). (Hint: consider $dQ/dt$.)
(3) Find $Q$ in terms of the $A_n$. What does the conservation of heat energy, as in (2), correspond to in the series solution (1)?
(4) Find $\lim_{t\to+\infty} u(\theta,t)$ in terms of $\phi$.

**Problem 3.** Recall that $s(\mathbb{Z})$ is the set of rapidly decreasing bi-infinite sequences, i.e. sequences $a = \{a_n\}$ such that $\|a\|_k = \sup(1+|n|)^k|a_n| < \infty$ for all $k \geq 0$ integer, and we make $s(\mathbb{Z})$ into a complete metric space using these norms.

Recall also that a linear map $u : s(\mathbb{Z}) \to \mathbb{C}$ is continuous if and only if there exists $C > 0$ and $k$ such that $|u(a)| \leq C\|a\|_k$ for all $a \in s(\mathbb{Z})$, and an analogous statement holds for all metric spaces where the metric is constructed from a sequence of norms, e.g. $C^\infty(\mathbb{S}^1)$.

(1) Show that if $a = \{a_n\} \in s(\mathbb{Z})$, and $a^{(k)} \in s(\mathbb{Z})$ is the sequence with $a_n^{(k)} = 0$ if $|n| > k$, $a_n^{(k)} = a_n$ if $|n| \leq k$, then $\lim_{k\to\infty} a^{(k)} = a$ in $s(\mathbb{Z})$.
(2) Show that if $b = \{b_n\}$ is a polynomially bounded sequence, i.e. there exists $N$ and $C$ such that $|b_n| \leq C(1+|n|)^N$ for all $n \in \mathbb{Z}$, then $b$ defines a continuous linear functional on $s(\mathbb{Z})$ by $b(\{a_n\}) = \sum_{n\in\mathbb{Z}} b_n a_n$.
(3) Show that if $b(a) = 0$ for all $a \in s(\mathbb{Z})$ then $b = 0$.

(4) Conversely, suppose that $u : s(\mathbb{Z}) \to \mathbb{C}$ is a continuous linear map. Let $\epsilon^{(k)} \in s(\mathbb{Z})$ be the sequence whose $k$th entry is 1, and all others are 0: $\epsilon^{(k)}_n = 1$ if $k = n$, $\epsilon^{(k)}_n = 0$ if $k \neq n$. Let $b_n = u(\epsilon^{(n)})$. Show that $\{b_n\}$ is polynomially bounded, and $u(\{a_n\}) = \sum b_n a_n$ for all $a = \{a_n\} \in s(\mathbb{Z})$. (Hint: For the last claim, show it first if $a$ has only finitely many non-zero entries.)

Deduce that the set of continuous linear maps on $s(\mathbb{Z})$ can be identified with the set $s'(\mathbb{Z})$ of polynomially bounded sequences.

(5) Let $\mathcal{D}'(\mathbb{S}^1)$ be the set of continuous linear maps (distributions) $C^\infty(\mathbb{S}^1) \to \mathbb{C}$. Show that the Fourier series map $\mathcal{F} : C^\infty(\mathbb{S}^1) \to s(\mathbb{Z})$ extends to a map $\mathcal{F} : \mathcal{D}'(\mathbb{S}^1) \to s'(\mathbb{Z})$ by letting $(\mathcal{F}u)_n = (2\pi)^{-1} u(e_n)$, $e_n(x) = e^{-inx}$, $n \in \mathbb{Z}$. (That is, you must show that $\mathcal{F}$ indeed gives a map as stated, and if $u = \iota_f$, $f \in C^\infty(\mathbb{S}^1)$, i.e. $u(\phi) = \int_{\mathbb{S}^1} f(x)\phi(x)\,dx$, $\phi \in C^\infty(\mathbb{S}^1)$, then the above definition agrees with the standard one for the Fourier coefficients $\mathcal{F}f$ of $f$.)

(6) Find $\mathcal{F}\delta_\omega$, $\omega \in \mathbb{S}^1$. (Recall: $\delta_\omega(\phi) = \phi(\omega)$.)

(7) Recall that for $u \in \mathcal{D}'(\mathbb{S}^1)$, $\frac{du}{dx} \in \mathcal{D}'(\mathbb{S}^1)$ is given by $\frac{du}{dx}(\phi) = -u(\frac{d\phi}{dx})$, $\phi \in C^\infty(\mathbb{S}^1)$. Show that $(\mathcal{F}\frac{du}{dx})_n = in(\mathcal{F}u)_n$, $u \in \mathcal{D}'(\mathbb{S}^1)$.

(8) Show that $\mathcal{F} : \mathcal{D}'(\mathbb{S}^1) \to s'(\mathbb{Z})$ is injective, i.e. that if $\mathcal{F}u = 0$, i.e. $u(e_n) = 0$ for all $n$, then $u = 0$, by using that the Fourier series of a $C^\infty$ function converges to the function in $C^\infty$ (i.e. uniformly with all derivatives).

(9) Use this to show that if $u \in \mathcal{D}'(\mathbb{S}^1)$ and $\frac{du}{dx} = 0$, then $u$ is the distribution associated to a constant function.