ASYMPTOTIC BEHAVIOR OF GENERALIZED EIGENFUNCTIONS IN N-BODY SCATTERING

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Abstract. In this paper an asymptotic expansion is proved for locally (at infinity) outgoing functions on asymptotically Euclidian spaces. This is applied to N-body scattering where the two-body interactions are one-step polyhomogeneous symbols of order -1 or -2 (hence long-range and short-range respectively). The asymptotic behavior of the N-body resolvent applied to Schwartz functions is thereby deduced away from the singular set, where some of the potentials do not decay at infinity.

1. Introduction

In this paper we prove an asymptotic expansion for locally (at infinity) outgoing functions on asymptotically Euclidian spaces, and as an application we obtain the asymptotic behavior of the N-body resolvent applied to Schwartz functions away from the singular set. The N-body Hamiltonian is $H_V = \Delta + \sum_i V_i$, where Δ is the positive Laplacian on $\mathbb{R}^n$, and the V_i are real-valued functions on linear subspaces X^i of $\mathbb{R}^n$ (extended by orthogonal projection; see e.g. [1, 10] for a detailed description). We shall assume that V_i are one-step polyhomogeneous (classical) symbols of order -1 or -2 on X^i, so they are long-range or short-range respectively. Let $X_i = (X^i)^\perp$, and $C_i = \mathbb{S}^{n-1} \cap X_i$, $\mathbb{S}^{n-1}$ being the unit sphere in $\mathbb{R}^n$. Then the singular set is $\cup_i C_i$. We follow [8, 10] in our normalization of the (modified) resolvent: $R_V(\lambda) = (H_V - \lambda^2)^{-1}$ when λ is in the physical half-plane, $\operatorname{Im}\lambda < 0$. Thus, for $\sigma > 0$

$$(H_V - (\sigma \pm i0))^{-1} = R_V(\mp\sigma^{1/2}). \tag{1.1}$$

Date: May 30th, 1996.

Our main conclusion for the N-body problem is:

Theorem. *If $f \in \mathcal{S}(\mathbb{R}^n)$, $0 \neq \lambda \in \mathbb{R}$, and the V_i are of order -2 then $R_V(\lambda)f$ has a full asymptotic expansion*

$$R_V(\lambda)f(r\theta) \sim e^{-i\lambda r} r^{-(n-1)/2} \sum_{j\geq 0} a_j(\theta) r^{-j} \tag{1.2}$$

with $a_j \in \mathcal{C}^\infty(\mathbb{S}^{n-1} \setminus \cup_i C_i)$. If the V_i are of order -1, and $\alpha \in \mathcal{C}^\infty(\mathbb{S}^n_+ \setminus \cup_i C_i)$ is defined by $\alpha(\theta) = (2\lambda)^{-1} \sum_i \lim_{r\to\infty} rV_i(r\theta)$, then

$$R_V(\lambda)f(r\theta) \sim e^{-i\lambda r} r^{-(n-1)/2+i\alpha} \sum_{j=0}^{\infty} \sum_{s\leq 2j} a_{j,s}(\theta) r^{-j} (\log r)^s \tag{1.3}$$

with $a_{j,s} \in \mathcal{C}^\infty(\mathbb{S}^{n-1} \setminus \cup_i C_i)$.

Herbst and Skibsted have obtained the top term of this asymptotic expansion in [2] for more general potentials by using time-dependent radiation estimates. Isozaki had proved a closely related result in [3] in the case of three-body scattering, and the expansion at $\cup_i C_i$ was obtained in [10] in the same situation for Schwartz potentials. This theorem together with the resolvent estimates of [1] and the uniqueness theorem of [4] allows us to interpret the open part of the N-cluster to N-cluster scattering matrix, $S_V(\lambda)$, in Section 4 in a geometric way, i.e. by using the asymptotic expansion of the generalized eigenfunctions. In fact, the possibility of such an interpretation already follows from the results of Herbst and Skibsted, since it only depends on the leading term of the expansion.

We show that the local properties of outgoing functions are valid in a more general setting which we proceed to describe in some detail. Thus, let X be an n-dimensional compact manifold with boundary. Following Melrose's definition given in [7], we say that a Riemannian metric g in the interior of X is a scattering metric on X if

$$g = x^{-4}\, dx^2 + x^{-2} h \tag{1.4}$$

where x is a boundary defining function and h is a smooth symmetric 2-tensor on X with a non-degenerate restriction to the boundary ∂X. We shall denote this restriction by h as well. This situation arises naturally in Euclidian scattering if we consider the radial compactification SP of $\mathbb{R}^n$ into $\mathbb{S}^n_+$, and pull back the standard metric by SP^{-1}.

The Laplacian Δ of such a metric g is a scattering differential operator: $\Delta \in \mathrm{Diff}^2_{\mathrm{sc}}(X)$ (see [7]). The microlocalization of this algebra of differential operators is the corresponding pseudo-differential operator calculus $\Psi^{*,*}_{\mathrm{sc}}(X)$. This gives rise to the scattering wave front set, $\mathrm{WF}_{\mathrm{sc}}$, on the boundary $C_{\mathrm{sc}}X$ of the radial fiber-compactification ${}^{\mathrm{sc}}\bar{T}^*X$ of ${}^{\mathrm{sc}}T^*X$. A product decomposition $U = [0, \epsilon)_x \times \partial X$ of X near the boundary allows us to introduce coordinates (x, y, τ, μ) on ${}^{\mathrm{sc}}T^*_U X$ by writing a covector $v \in {}^{\mathrm{sc}}T^*_p X$ as $v = \tau(x^{-2}\,dx) + x^{-1}\mu$; here (y, μ) are coordinates on $T^*\partial X$. For $0 \neq \lambda \in \mathbb{R}$ the characteristic set $\Sigma_{\Delta-\lambda^2} \subset {}^{\mathrm{sc}}T^*_{\partial X}X$ of $\Delta - \lambda^2$ is $\tau^2 + |\mu|^2 = \lambda^2$; $|\mu| = h(y, \mu)^{1/2}$ is the metric length of μ. Since propagation of singularities takes place inside $\Sigma_{\Delta-\lambda^2}$, we are particularly interested in the structure of the Hamiltonian vector field, ${}^{\mathrm{sc}}H^{2,0}_{|\zeta|^2-\lambda^2}$ of the symbol $|\zeta|^2 - \lambda^2 = \tau^2 + |\mu|^2 - \lambda^2$ of $\Delta - \lambda^2$ (and its integral curves) inside it. In fact, there are two radial surfaces,

$$R^{\pm}_{\lambda} = \{(y, \pm\lambda, 0) : y \in \partial X\}, \tag{1.5}$$

at which ${}^{\mathrm{sc}}H^{2,0}_{|\zeta|^2-\lambda^2}$ vanishes, and all integral curves inside $\Sigma_{\Delta-\lambda^2}$ tend to R^+_λ as $t \to -\infty$, and to R^-_λ as $t \to \infty$ (here we use the notation of [10] and we take $\lambda > 0$). Thus, principal-type propagation takes places in $\Sigma_{\Delta-\lambda^2} \setminus (R^-_\lambda \cup R^+_\lambda)$. In [7] Melrose gives global results excluding the whole of $R^{\pm}_\lambda$ from $\mathrm{WF}_{\mathrm{sc}}$ under global assumptions on the absence of $\mathrm{WF}_{\mathrm{sc}}$ nearby. In this paper we make the simple observation that the arguments in [7] can be localized by using the characterization of the integral curves of ${}^{\mathrm{sc}}H^{2,0}_{|\zeta|^2-\lambda^2}$ obtained by Melrose and Zworski in [9]. As a straightforward application we deduce the weak asymptotics of the N-body resolvent away from the singular set by using the characterization of the resolvent by Gérard, Isozaki and Skibsted [1]. It should be noted that this asymptotic expansion for short-range N-body scattering can be proved directly by using the explicit free resolvent on $\mathbb{R}^n$,

$(\Delta-\lambda^2)^{-1}$, and iterative regularity arguments together with the resolvent estimates of [1].

I am very grateful to Richard Melrose for our lively and fruitful discussions and for his encouragement.

2. Local properties

We first construct a function on ${}^{\mathrm{sc}}T^*X$ whose quantization commutes with the Laplacian to top order. To do so, we shall consider the flow of ${}^{\mathrm{sc}}H^{2,0}_{|\zeta|^2-\lambda^2}$ inside ${}^{\mathrm{sc}}T^*_{\partial X}X$. Here we can use the results of Melrose and Zworski who analyzed the integral curves above the boundary in detail in [9]. In [9, Lemma 2] it is shown that the integral curves of ${}^{\mathrm{sc}}H^{2,0}_{\sqrt{g}}$ (which coincide with those of ${}^{\mathrm{sc}}H^{2,0}_{|\zeta|^2-\lambda^2}$) in ${}^{\mathrm{sc}}T^*_{\partial X}X$ are points of the form $(y,\tau,0)$, and curves of the form

$$\tau=\sigma\cos(s+s_0),\qquad \mu=\sigma\sin(s+s_0)\hat\mu,\qquad (y,\hat\mu)=\exp((s+s_0)H_{\frac12 h})(y',\hat\mu')$$

where $s_0\in[0,\pi]$, $s\in(-s_0,\pi-s_0)$, $\sigma>0$, $(y',\hat\mu')\in T^*\partial X$, $h(y',\hat\mu')=1$ and $ds/dt=\frac12|\mu|=\frac12 h(y,\mu)^{1/2}$. Note that as $t\to-\infty$, i.e. $s\to-s_0$ the integral curve through (y,τ,μ) will tend to $(y',\lambda,0)$ where

$$y'=p(y,\tau,\mu)=\pi_1(\exp(-\arccos(\tau/(\tau^2+|\mu|^2)^{1/2})H_{\frac12 h})(y,\mu/|\mu|)); \tag{2.1}$$

here $\pi_1:T^*\partial X\to\partial X$ is projection to the base. Now, p is certainly smooth when $\mu\neq0$. To see how it behaves when μ is small and $\tau>0$, note that there

$$\arccos(\tau/(\tau^2+|\mu|^2)^{1/2})=\arcsin(|\mu|/(\tau^2+|\mu|^2)^{1/2}). \tag{2.2}$$

Scaling the covector $\mu/|\mu|$ we find that

$$p(y,\tau,\mu)=\pi_1(\exp(-|\mu|^{-1}\arcsin(|\mu|/(\tau^2+|\mu|^2)^{1/2})H_{\frac12 h})(y,\mu)). \tag{2.3}$$

Since $r^{-1}\arcsin(r/(\tau^2+r^2)^{1/2})$ is a smooth function of τ and r in $\tau>0$ which is even in r, it follows that p is actually smooth in ${}^{\mathrm{sc}}T^*_{\partial X}X\setminus\{(y,\tau,0):\tau\leq0\}$. We restate this in the following lemma:

Lemma 2.1. *The map $p:{}^{sc}T^*_{\partial X}X\setminus\{(y,\tau,0):\tau\leq0\}\to\partial X$ is smooth.*

Remark 2.2. In the case of Euclidian scattering, i.e. $X = \mathbb{S}^n_+$, $\partial X = \mathbb{S}^{n-1}$, and g is the pull-back of the standard metric on $\mathbb{R}^n$ via the stereographic projection, p can be expressed very simply:

$$p(y,\tau,\mu) = (\tau^2 + |\mu|^2)^{-1/2}(\tau y - \mu) \in \mathbb{S}^{n-1}. \tag{2.4}$$

This corresponds to the fact that under the Legendre diffeomorphism integral curves of ${}^{\mathrm{sc}}H^{2,0}_{|\zeta|^2-\lambda^2}$ over ∂X become curves whose projection to the base is constant.

Let $U = [0,\epsilon) \times \partial X$ be a product decomposition of X near its boundary, $F : {}^{\mathrm{sc}}T^*_U X \to {}^{\mathrm{sc}}T^*_{\partial X} X$ be the corresponding fibration of ${}^{\mathrm{sc}}T^*_U X$.

Corollary 2.3. *If $\phi \in \mathcal{C}^\infty(\partial X)$, then $\tilde\phi = F^* p^* \phi \in \mathcal{C}^\infty({}^{sc}T^*_U X \setminus \{(x,y,\tau,0) : \tau \leq 0\})$ satisfies ${}^{sc}H^{2,0}_{|\zeta|^2-\lambda^2}\tilde\phi \in x\mathcal{C}^\infty({}^{sc}T^*_U X \setminus \{(x,y,\tau,0) : \tau \leq 0\})$.*

Proof. This is immediate since $\tilde\phi$ is constant along integral curves of ${}^{\mathrm{sc}}H^{2,0}_{|\zeta|^2-\lambda^2}$ in ∂X. □

We can now prove local properties of the scattering wave front set near $R^\pm_\lambda$; the following propositions are analogues of [7, Proposition 9-12]. In the following $\pi : {}^{\mathrm{sc}}\bar{T}^* X \to X$ is projection to the base.

Proposition 2.4. *If $\lambda \neq 0$, $s < -\frac{1}{2}$, $u \in \mathcal{C}^{-\infty}(X)$, $K \subset X$ is compact then*

$$R^\pm_\lambda \cap \pi^{-1}(K) \cap \mathrm{WF}^{*,s+1}_{sc}((\Delta - \lambda^2)u) = \emptyset, \tag{2.5}$$

$$R^\pm_\lambda \cap \mathrm{cl}(\mathrm{WF}^{*,s}_{sc}(u) \setminus R^\pm_\lambda) \cap \pi^{-1}(K) = \emptyset \tag{2.6}$$

imply that

$$\mathrm{WF}^{*,s}_{sc}(u) \cap R^\pm_\lambda \cap \pi^{-1}(K) = \emptyset. \tag{2.7}$$

Proof. We only need to modify Melrose's proof [7, Proposition 9] by inserting an additional factor; we follow that proof in detail for the convenience of the reader. For the sake of definiteness we take $\lambda > 0$ and R^+_λ above. For R^-_λ we would have to change the definition of p by mapping (y,τ,μ) to the end point (i.e as $t \to \infty$) of the integral curve through it, and if $\lambda < 0$ we need to reverse the signs in $R^\pm_\lambda$ in this proof. Thus, let $\phi \in \mathcal{C}^\infty(\partial X)$ be identically 1 in a neighborhood of K, supported sufficiently close to K, and let $\tilde\phi$ be as in the previous corollary. Let $s' \leq s$, and define

$$b_r = b\phi_3(\frac{x}{r}), \qquad b = x^{-s'-1/2}\tilde\phi(x,y,\tau,\mu)\phi_1(\frac{\tau^2+|\mu|^2}{\lambda^2})\phi_2(\frac{\tau}{|\mu|})\phi_4(x). \tag{2.8}$$

Here given $\delta > 0$ we take $\phi_1 \in \mathcal{C}^\infty(\mathbb{R})$ supported in $(1-\delta, 1+\delta)$, positive definite in the interior of this set. Let $\phi_2 \in \mathcal{C}^\infty(\mathbb{R})$ be supported in $[1/\delta, \infty)$, identically 1 in $[2/\delta, \infty)$, $\phi_3 \in \mathcal{C}^\infty(\mathbb{R})$ vanish in $(-\infty, 1)$ and identically 1 in $(2, \infty)$, and finally $\phi_4 \in \mathcal{C}^\infty(\mathbb{R})$ identically 1 near 0, supported in $(-\epsilon, \epsilon)$. Thus, we have only changed b_r by inserting the factor of $\tilde\phi$. Note that the support conditions on ϕ_1 and ϕ_2 insure that on these supports $\tilde\phi$ is smooth. In particular, b_r it is well defined and bounded in $\mathcal{A}^{(-\infty,-s'-1/2)}({}^{\mathrm{sc}}\bar T^*X)$.

Since ${}^{\mathrm{sc}}H^{2,0}_{|\zeta|^2-\lambda^2}\tilde\phi \in x\mathcal{C}^\infty({}^{\mathrm{sc}}\bar T^*X)$, $\tilde\phi$ only affects the commutator $[\Delta, B_r^2]$ by adding a term G'_r which is bounded in $\mathcal{A}^{(-\infty,-2s'+1)}({}^{\mathrm{sc}}\bar T^*X)$. More precisely,

$$(\Delta-\lambda^2)B_r^2 - B_r^2(\Delta-\lambda^2) = -2ixC_r \tag{2.9}$$

where the (principal) symbol c_r of C_r is

$$c_r = \tau((-2s'-1)\phi_3^2(\frac{x}{r}) + 2\frac{x}{r}(\phi_3'\phi_3)(\frac{x}{r}))b^2 + e_r, \tag{2.10}$$

$$e_r = -2x^{-2s'-1}\frac{\tau^2+|\mu|^2}{|\mu|}\phi_1^2(\frac{\tau^2+|\mu|^2}{\lambda^2})\phi_2'(\frac{\tau}{|\mu|})\phi_2(\frac{\tau}{|\mu|})\phi_3^2(\frac{x}{r})\phi_4^2(x). \tag{2.11}$$

Thus, with an appropriate choice of ϕ_3 the first two terms are squares of symbols and we find that

$$(\Delta-\lambda^2)B_r^2 - B_r^2(\Delta-\lambda^2) = -2i(A_r^2 + F_r^2 + G_r + G'_r) \tag{2.12}$$

where $A_r \in \Psi_{\mathrm{sc}}^{-\infty,-s'}(X)$ and $F_r \in \Psi_{\mathrm{sc}}^{-\infty,-s'}(X)$ are self-adjoint, A_r is elliptic on $R^\pm_\lambda \cap \pi^{-1}(K)$ (its symbol is in fact an elliptic multiple of $x^{1/2}b_r$), G_r is bounded of order $-2s'$ with essential support disjoint from $R^\pm_\lambda$ but included in a small neighborhood of $R_\lambda^\pm \cap \pi^{-1}(K)$, $G'_r \in \Psi_{\mathrm{sc}}^{-\infty,-2s'+1}(X)$. Now, all terms of (2.12) are order ∞ for $r > 0$, so we can pair with u which gives

$$-\operatorname{Im}\langle B_r^2 u, (\Delta - \lambda^2)u\rangle = \|A_r u\|^2 + \|F_r u\|^2 + \langle G_r u, u\rangle + \langle G'_r u, u\rangle. \tag{2.13}$$

From the Cauchy-Schwarz inequality

$$|\langle B_r^2 u, u\rangle| \leq \epsilon^2 \|x^{1/2} B_r u\|^2 + C\epsilon^{-2}\|x^{-1/2} B_r(\Delta - \lambda^2)u\|^2, \qquad \epsilon > 0. \tag{2.14}$$

Since the symbol of A_r is an elliptic multiple of $x^{1/2}b$ (2.13) and (2.14) give

$$\|x^{1/2} B_r u\|^2 \leq C(\|x^{-1/2} B_r(\Delta - \lambda^2)u\|^2 + |\langle G_r u, u\rangle| + |\langle G'_r u, u\rangle|) \tag{2.15}$$

Take S sufficiently small so that $\mathrm{WF}_{\mathrm{sc}}^{*,S}(u) \cap \pi^{-1}(K) \cap R_\lambda^\pm = \emptyset$. Then for $s' \leq S + 1/2$, $s' \leq s$ each term on the right hand side is bounded by assumption, so $x^{1/2} B_r u \in L^2(X)$, so $\mathrm{WF}_{\mathrm{sc}}^{*,s'}(u) \cap R_\lambda^\pm \cap \pi^{-1}(K) = \emptyset$, i.e. we have gained regularity. Applying this iteratively proves the proposition. □

Since this argument is based on inserting an additional factor which commutes with the Laplacian to top order, the same method also gives a localized version of the finer regularity result [7, Proposition 10].

Proposition 2.5. *If $\lambda \neq 0$, $s \geq -\frac{1}{2}$, $u \in \mathcal{C}^{-\infty}(X)$, $K \subset X$ compact, then*

$$R_\lambda^\pm \cap \mathrm{WF}_{sc}^{*,s+1}((\Delta - \lambda^2)u) \cap \pi^{-1}(K) = \emptyset, \tag{2.16}$$

$$R_\lambda^\pm \cap \mathrm{WF}_{sc}^{*,-1/2}(u) \cap \pi^{-1}(K) = \emptyset \tag{2.17}$$

imply that

$$\mathrm{WF}_{sc}^{*,s}(u) \cap R_\lambda^\pm \cap \pi^{-1}(K) = \emptyset. \tag{2.18}$$

We now proceed to prove a weaker, but local, version of [7, Proposition 11].

Proposition 2.6. *If $\lambda \neq 0$, $K \subset X$ compact, and $u \in \mathcal{C}^{-\infty}(X)$ satisfies*

$$\mathrm{WF}_{sc}^{*,-1/2}(u) \cap \pi^{-1}(K) = \emptyset, \tag{2.19}$$

$$\mathrm{WF}_{sc}^{M+2,s-1}(u) \cap \pi^{-1}(K) \subset R_\lambda^\pm, \tag{2.20}$$

$$\mathrm{WF}_{sc}^{M,s}((\Delta-\lambda^2)u)\cap\pi^{-1}(K)=\emptyset \tag{2.21}$$

for some $s\geq\frac{1}{2}$, *then*

$$\mathrm{WF}_{sc}^{M+2,s-1}(u)\cap\pi^{-1}(K)=\emptyset. \tag{2.22}$$

Proof. By Proposition 2.5

$$\mathrm{WF}_{\mathrm{sc}}^{*,s-1}(u)\cap R_\lambda^{\pm}\cap\pi^{-1}(K)=\emptyset, \tag{2.23}$$

and hence the conclusion follows. □

Remark 2.7. Since we only need to use the normal symbol of Δ in the previous propositions, Δ can be replaced by $\Delta+V$ in all of them if V satisfies $\phi V\in x\mathcal{C}^\infty(X)$ for some $\phi\in\mathcal{C}^\infty(X)$ which is identically 1 in a neighborhood of K.

We are now ready to prove that locally outgoing functions do in fact have an asymptotic expansion at ∂X.

Proposition 2.8. *If* $0\neq\lambda\in\mathbb{R}$, $K\subset X$ *compact,* $u\in\mathcal{C}^{-\infty}(X)$,

$$\mathrm{WF}_{sc}(u)\cap\pi^{-1}(K)\subset R_\lambda^+, \tag{2.24}$$

$$\mathrm{WF}_{sc}((\Delta-\lambda^2)u)\cap\pi^{-1}(K)=\emptyset \tag{2.25}$$

then $x^{-(n-1)/2}e^{i\lambda/x}u\in\mathcal{C}^\infty(U')$ *for a sufficiently small neighborhood* U' *of* K.

Proof. This proposition can be proved in exactly the same way as the corresponding global result [7, Proposition 12]. Namely, we commute powers of the special extension $\tilde{\Delta}_0$ of the boundary Laplacian Δ_0 through $\Delta-\lambda^2$. Of course, we have to use the local results given above instead of the global ones. The only additional ingredient is noting that the regularity and interpolation estimates in the proof of [7, Proposition 12] only use the pseudodifferential operator calculus on the fibers, ∂X, of the local product decomposition. Since this calculus is local, that proof goes through in this case too.

More explicitly, we show the first step of that proof by using cut-offs. Let $f = (\Delta - \lambda^2)u$. First note that by Proposition 2.4 $\mathrm{WF}_{\mathrm{sc}}^{*,-1/2-\delta}(u) \cap \pi^{-1}(K) = \emptyset$ for all $\delta > 0$. Commuting $\tilde{\Delta}_0$ through $\Delta - \lambda^2$ and using that

$$[\tilde{\Delta}_0, \Delta] \in x^2 \operatorname{Diff}_{\mathrm{c}}^3(X) \cap x^3 \operatorname{Diff}_{\mathrm{b}}^3(X) \subset \operatorname{Diff}_{\mathrm{sc}}^3(X) \tag{2.26}$$

we conclude that $\mathrm{WF}_{\mathrm{sc}}^{*,-1/2-\delta}((\Delta - \lambda^2)\tilde{\Delta}_0 u) \cap \pi^{-1}(K) = \emptyset$. Then Proposition 2.4 implies that $\mathrm{WF}_{\mathrm{sc}}^{*,-3/2-\delta}(\tilde{\Delta}_0 u) \cap \pi^{-1}(K) = \emptyset$. This means that for $\phi_1 \in \mathcal{C}^\infty(X)$ identically 1 on a neighborhood of K, supported sufficiently close to K,

$$\phi_1 \tilde{\Delta}_0 u \in H_{\mathrm{sc}}^{\infty,-3/2-\delta}(X). \tag{2.27}$$

The interpolation estimates on the fibers, ∂X, of the local product decomposition allows us to conclude that with $\phi_2 \in \mathcal{C}^\infty(X)$ supported in a sufficiently small neighborhood of K

$$\phi_2 \operatorname{Diff}_{\mathrm{c}}^2(X) u \subset H_{\mathrm{sc}}^{\infty,-3/2-\delta}(X), \qquad \phi_2 \operatorname{Diff}_{\mathrm{c}}^1(X) u \subset H_{\mathrm{sc}}^{\infty,-1-\delta}(X). \tag{2.28}$$

Using the identity $(\Delta - \lambda^2)(\tilde{\Delta}_0 u) = [\Delta, \tilde{\Delta}_0]u + \tilde{\Delta}_0 f$ and (2.26) we conclude that

$$\phi_2 (\Delta - \lambda^2)(\tilde{\Delta}_0 u) \in H_{\mathrm{sc}}^{\infty,1/2-\delta}(X), \tag{2.29}$$

so by Proposition 2.4

$$\mathrm{WF}_{\mathrm{sc}}^{*,-1/2-\delta}(\tilde{\Delta}_0 u) \cap \pi^{-1}(K) = \emptyset, \tag{2.30}$$

which is an improvement over (2.28). Just as in [7], we can iterate this argument to conclude that for all k

$$\mathrm{WF}_{\mathrm{sc}}^{*,-1/2-\delta}(\tilde{\Delta}_0^k u) \cap \pi^{-1}(K) = \emptyset, \tag{2.31}$$

i.e. for $\phi \in \mathcal{C}^\infty(X)$ supported sufficiently close to K

$$\phi \tilde{\Delta}_0^k u \in H_{\mathrm{sc}}^{\infty,-1/2-\delta}(X). \tag{2.32}$$

This means that u is a $\mathcal{C}^\infty$ function on a neighborhood $\tilde{U}$ of $K \cap \partial X$ in ∂X with values in $x^{(n-1)/2-\delta} L_b^2([0,\epsilon))$; L_b^2 is the L^2 space with respect to dx/x. We can conclude as in [7] that with $v = x^{-(n-1)/2}u$

$$v = e^{-i\lambda/x}a_0 + v'; \qquad a_0 \in \mathcal{C}^\infty(\tilde{U}),\ v' \in \mathcal{C}^\infty(\tilde{U}; x^{1-\delta}L^2_b([0,\epsilon))). \tag{2.33}$$

We can certainly construct on outgoing formal solution u' with leading coefficient equal to a_0 in a neighborhood of $K \cap \partial X$, and then with $u'' = u - u'$ we have

$$\mathrm{WF}^{*,0}_{\mathrm{sc}}(u'') \cap \pi^{-1}(K) = \emptyset,\ \mathrm{WF}_{\mathrm{sc}}(u'') \cap \pi^{-1}(K) \subset R^+_\lambda, \tag{2.34}$$

$$\mathrm{WF}^{*,*}_{\mathrm{sc}}((\Delta - \lambda^2)u'') \cap \pi^{-1}(K) = \emptyset. \tag{2.35}$$

Hence Proposition 2.6 proves that $\mathrm{WF}^{*,*}_{\mathrm{sc}}(u'') \cap \pi^{-1}(K) = \emptyset$, so for $\phi \in \mathcal{C}^\infty(X)$ supported sufficiently close to K, $\phi u'' \in \dot{\mathcal{C}}^\infty(X)$. Thus, u has an asymptotic expansion near K as claimed. □

Remark 2.9. This proposition holds with essentially the same proof if we replace Δ by $\Delta + V$ if $\phi V \in x^2\mathcal{C}^\infty(X)$ for some $\phi \in \mathcal{C}^\infty(X)$, $\phi \equiv 1$ in a neighborhood of K, just as the corresponding one does in [7].

Melrose has also proved the corresponding global proposition for long range potentials $V \in x\mathcal{C}^\infty(X)$ [5]. His proof can be modified to give a local result. To state this, introduce the index set (with $\mathbb{N}$ being the set of the non-negative integers)

$$\mathcal{K} = \{(m,p) : m, p \in \mathbb{N},\ p \le 2m\}. \tag{2.36}$$

For a description of the space $\mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X)$ of polyhomogeneous conormal distributions to the boundary see [6]. Essentially $u \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X)$ means that u has an asymptotic expansion in $x^m(\log x)^p$, $p \le 2m$, $m \to \infty$, with smooth coefficients on ∂X.

Proposition 2.10. *If* $0 \neq \lambda \in \mathbb{R}$, $K \subset X$ *compact,* $V = xV'$, $V' \in \mathcal{C}^\infty(X)$ *real valued,* $\alpha = (2\lambda)^{-1}V'|_{\partial X}$, $u \in \mathcal{C}^{-\infty}(X)$,

$$\mathrm{WF}_{sc}(u) \cap \pi^{-1}(K) \subset R^+_\lambda, \tag{2.37}$$

$$\mathrm{WF}_{sc}((\Delta + V - \lambda^2)u) \cap \pi^{-1}(K) = \emptyset \tag{2.38}$$

then $x^{i\alpha-(n-1)/2}e^{i\lambda/x}u \in \mathcal{A}^{\mathcal{K}}_{phg}(U')$ *for a sufficiently small neighborhood* U' *of* K. *Thus, for* $\tilde{U}$ *a small neighborhood of* $K\cap\partial X$ *in* ∂X

$$u \sim e^{-i\lambda/x}\sum_{j=0}^{\infty}\sum_{r\leq 2j} x^{(n-1)/2-i\alpha+j}(\log x)^r a_{j,r}(\omega), \qquad a_{j,r}\in\mathcal{C}^\infty(\tilde{U}). \tag{2.39}$$

Proof. The additional problem arises because $[V,\tilde{\Delta}_0] \in x\,\mathrm{Diff}^1_{\mathrm{c}}(X) \subset \mathrm{Diff}^1_{\mathrm{sc}}(X)$ only. This is sufficient to show (2.27), hence (2.28). Now, however,

$$(\Delta+V-\lambda^2)(\tilde{\Delta}_0 u) = [\Delta+V,\tilde{\Delta}_0]u + \tilde{\Delta}_0 f, \qquad f=(\Delta+V-\lambda^2)u, \tag{2.40}$$

only implies that for all $\delta>0$

$$\phi_2(\Delta+V-\lambda^2)(\tilde{\Delta}_0 u) \in H^{\infty,-\delta}_{\mathrm{sc}}(X) \tag{2.41}$$

where $\phi_2\in\mathcal{C}^\infty(X)$ is supported in a sufficiently small neighborhood of K as beforehand. Thus, (2.30) is replaced by

$$\mathrm{WF}^{*,-1-\delta}_{\mathrm{sc}}(\tilde{\Delta}_0 u)\cap\pi^{-1}(K)=\emptyset. \tag{2.42}$$

This result is not strong enough to carry out the iteration of Proposition 2.8, so we have to improve it further. Notice that (2.42) is in fact an improvement over (2.27), and correspondingly we can replace (2.28) by

$$\phi_3\,\mathrm{Diff}^2_{\mathrm{c}}(X)u \subset H^{\infty,-1-\delta}_{\mathrm{sc}}(X), \qquad \phi_3\,\mathrm{Diff}^1_{\mathrm{c}}(X)u\subset H^{\infty,-3/4-\delta}_{\mathrm{sc}}(X) \tag{2.43}$$

where $\phi_3\in\mathcal{C}^\infty(X)$ is supported sufficiently close to K. Hence, (2.41) is replaced by

$$\phi_3(\Delta+V-\lambda^2)(\tilde{\Delta}_0 u)\in H^{\infty,1/4-\delta}_{\mathrm{sc}}(X), \tag{2.44}$$

so

$$\mathrm{WF}^{*,-3/4-\delta}_{\mathrm{sc}}(\tilde{\Delta}_0 u)\cap\pi^{-1}(K)=\emptyset. \tag{2.45}$$

We can iterate this procedure to conclude that

$$\mathrm{WF}^{*,-1/2-\delta-\delta'}_{\mathrm{sc}}(\tilde{\Delta}_0 u)\cap\pi^{-1}(K)=\emptyset \tag{2.46}$$

for all $\delta'>0$, so as $\delta>0$ was arbitrary, (2.30) holds for all $\delta>0$.

We now iterate this argument as in Proposition 2.8. Note that

$$[V,\tilde{\Delta}_0^k]\in x\,\mathrm{Diff}_{\mathrm{c}}^{2k-1}(X). \tag{2.47}$$

Hence, if we have shown that $\mathrm{Diff}_{\mathrm{c}}^{2(k-1)}(X)u\subset H_{\mathrm{sc}}^{\infty,-1/2-\delta}(X)$, then

$$(\Delta+V-\lambda^2)(\tilde{\Delta}_0^k u)=[\Delta+V,\tilde{\Delta}_0^k]u+\tilde{\Delta}_0^k f\in H_{\mathrm{sc}}^{\infty,-1/2-\delta}(X), \tag{2.48}$$

so $\tilde{\Delta}_0^k u\in H_{\mathrm{sc}}^{\infty,-3/2-\delta}(X)$. We can then repeat the previous iteration to conclude that (2.32) holds in this case too. Hence u is a $\mathcal{C}^\infty$ function on a neighborhood $\tilde{U}$ of $K\cap\partial X$ in ∂X with values in $x^{(n-1)/2-\delta}L^2_b([0,\epsilon))$ for all $\delta>0$.

Let $v=x^{-(n-1)/2}u$. Then

$$(x^2D_x+\lambda-\alpha x)(x^2D_x-\lambda+\alpha x)-((x^2D_x)^2+V-\lambda^2)\in x^2\mathcal{C}^\infty(X), \tag{2.49}$$

so as in the proof of [7, Proposition 12] we can conclude that

$$(x^2D_x+\lambda-\alpha x)(x^2D_x-\lambda+\alpha x)v\in\mathcal{C}^\infty(\tilde{U};x^{2-\delta}L^2_b([0,\epsilon))). \tag{2.50}$$

The first factor is elliptic on $\mathrm{WF}_{\mathrm{sc}}(v)$, so we can remove it. Since

$$(x^2D_x-\lambda+\alpha x)e^{-i\lambda/x}x^{-i\alpha}=0, \tag{2.51}$$

it follows by integration that

$$v=e^{-i\lambda/x}x^{-i\alpha}a_0+v';\qquad a_0\in\mathcal{C}^\infty(\tilde{U}),\ v'\in\mathcal{C}^\infty(\tilde{U};x^{1-\delta}L^2_b([0,\epsilon))). \tag{2.52}$$

Given $a_{0,0}\in\mathcal{C}^\infty(\partial X)$ Melrose has constructed approximate eigenfunctions (i.e. ones satisfying $(\Delta+V-\lambda^2)u'\in\dot{\mathcal{C}}^\infty(X)$) as in (2.39) but with $a_{j,r}\in\mathcal{C}^\infty(\partial X)$. This construction is also local, and it is based on the observation that

$$\begin{aligned}&(\Delta+V-\lambda^2)((\log x)^r x^{-i\alpha+p}e^{-i\lambda/x}b)\\&\qquad=x^{-i\alpha+p+1}e^{-i\lambda/x}(\sum_{s\le r}(\log x)^s g_s+x\sum_{s\le r+2}(\log x)^s g'_s)\end{aligned} \tag{2.53}$$

for $b\in\mathcal{C}^\infty(X)$, where $g_s,\ g'_s\in\mathcal{C}^\infty(X)$, $g_r=-i\lambda(2p-n+1)b$, which allows us to solve away the error terms iteratively if we start with $p=(n-1)/2$, $r=0$. Then the formal expansion as in (2.39) can be summed asymptotically to $u'\in\mathcal{C}^{-\infty}(X)$ (see [6]).

Now choose $a_{0,0} \in \mathcal{C}^\infty(\partial X)$ such that $a_{0,0} = a_0$ in a neighborhood of K. Considering the difference $u'' = u - u'$ shows as in Proposition 2.8 that $\phi u'' \in \dot{\mathcal{C}}^\infty(X)$ when $\phi \in \mathcal{C}^\infty(X)$ is supported sufficiently close to K, so u indeed has an asymptotic expansion near K. □

3. Weak N-body asymptotics

We now briefly recall the setting of N-body scattering. Our basic space is $\mathbb{R}^n$, and we consider perturbations of the (standard positive) Laplacian Δ on it. More precisely, let X_i, $i = 1, ..., M$ be linear subspaces, $X^i = X_i^\perp$, and denote the corresponding orthogonal projections by π_i and π^i respectively. The interaction potentials are then of the form $(\pi^i)^* V_i$, i.e. they are functions on X^i. We now define two particularly nice class of interaction potentials. Let $n_i = \dim X^i$, and consider the stereographic compactification SP_{n_i} of $\mathbb{R}^{n_i}$ into $\mathbb{S}^{n_i}_+$. Identifying X^i with $\mathbb{R}^{n_i}$ by a choice of a basis, we say that V_i is short-range if $V_i \in \mathrm{SP}^*_{n_i} \rho_i^2 \mathcal{C}^\infty(\mathbb{S}^{n_i}_+)$, and V_i is long-range if $V_i \in \mathrm{SP}^*_{n_i} \rho_i \mathcal{C}^\infty(\mathbb{S}^{n_i}_+)$. Here ρ_i is a defining function of $\mathbb{S}^{n_i-1} = \partial \mathbb{S}^{n_i}_+$. This definition is independent of the choice of basis, and it means simply that V_i is a one-step polyhomogeneous (classical) symbol on X^i of order -2 or -1 in the two cases respectively. The perturbed Laplacian is

$$H_V = \Delta + V, \qquad V = \sum_{i=1}^{M} (\pi^i)^* V_i. \tag{3.1}$$

We shall drop $(\pi^i)^*$ from now on, and consider V_i as a function on $\mathbb{R}^n$.

In fact, we proceed to compactify $\mathbb{R}^n$ into $X = \mathbb{S}^n_+$ to arrive in the framework of the previous section. We let $\bar{X}_i = \mathrm{cl}(\mathrm{SP}(X_i))$, $C_i = \mathbb{S}^{n-1} \cap \bar{X}_i$ where $\mathbb{S}^{n-1} = \partial \mathbb{S}^n_+$. The standard boundary defining function x of $\mathbb{S}^n_+$ is $x = (\mathrm{SP}^{-1})^* |w|^{-1}$ near $\mathbb{S}^{n-1}$; w being the coordinates on $\mathbb{R}^n$. Note that for short-range potentials

$$(\mathrm{SP}^{-1})^* V_i \in x^2 \mathcal{C}^\infty(\mathbb{S}^n_+ \setminus \cup_i C_i), \tag{3.2}$$

while for long-range ones

$$(\mathrm{SP}^{-1})^* V_i \in x \mathcal{C}^\infty(\mathbb{S}^n_+ \setminus \cup_i C_i). \tag{3.3}$$

We now prove a simple consequence of the local result of Section 2 and of the microlocal characterization of the resolvent by Gérard, Isozaki and Skibsted [1] and obtain the asymptotic behavior of $R_V(\lambda)f$ away from the singular set, $\cup_i C_i$, where $f \in \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$. First we introduce some notation if (at least some of) the V_i are long-range, so $V_i = xV_i'$ on $\mathbb{S}^n_+ \setminus \cup_i C_i$. Namely we let

$$\alpha = (2\lambda)^{-1} \sum_i V_i'|_{\mathbb{S}^{n-1} \setminus \cup_i C_i} \in \mathcal{C}^\infty(\mathbb{S}^{n-1} \setminus \cup_i C_i). \tag{3.4}$$

Then we have the following theorem:

Theorem 3.1. *If $\lambda \in \mathbb{R} \setminus \{0\}$, $f \in \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$ and the V_i are short-range then*

$$e^{i\lambda/x} x^{-(n-1)/2} R_V(\lambda) f \in \mathcal{C}^\infty(\mathbb{S}^n_+ \setminus \cup_i C_i). \tag{3.5}$$

If the V_i are long-range, $\mathcal{K}$ as in (2.36)*, then*

$$e^{i\lambda/x} x^{i\alpha-(n-1)/2} R_V(\lambda) f \in \mathcal{A}^{\mathcal{K}}_{phg}(\mathbb{S}^n_+ \setminus \cup_i C_i). \tag{3.6}$$

Proof. It follows from [1] that

$$\mathrm{WF}_{\mathrm{sc}}(R_V(\lambda)f) \cap \{(y,\tau,\mu) : \tau < \lambda\} = \emptyset. \tag{3.7}$$

In fact, following Isozaki [4], we let $\mathcal{R}^k_\pm(a)$ to be the set of $\mathcal{C}^\infty$ functions p on $T^*\mathbb{R}^n = \mathbb{R}^n_w \times \mathbb{R}^n_\xi$ satisfying

$$|\partial_w^r \partial_\xi^s p(w,\xi)| \leq C_{r,s} \langle w\rangle^{-r} \langle \xi\rangle^{-k} \tag{3.8}$$

for $0 \leq r \leq k$, $0 \leq s \leq k$ and on $\operatorname{supp} p$

$$\inf_{w,\xi} \pm \frac{w \cdot \xi}{\langle w\rangle} > a \tag{3.9}$$

with the $+$ sign corresponding to $\mathcal{R}^k_+(a)$ and the $-$ sign to $\mathcal{R}^k_-(a)$. Note that if $p \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+)$ has compact support in the ξ variable, then (3.8) is automatically satisfied for all k. Introducing the usual coordinates on ${}^{\mathrm{sc}}T^*\mathbb{S}^n_+$ near ${}^{\mathrm{sc}}T^*_{\mathbb{S}^{n-1}}\mathbb{S}^n_+$, we have

$$\tau = -\frac{w \cdot \xi}{|w|}. \tag{3.10}$$

Thus, (3.9) states in this case that there exists $\epsilon > 0$ such that $\pm\tau < -a - \epsilon$ on $\operatorname{supp} p$.

Now let

$$p(x, y, \tau, \mu) = \phi_1(x)\phi_2(\tau, \mu) \tag{3.11}$$

where $\phi_1 \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near 0, and it is supported sufficiently close to 0, $\phi_2 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ supported in $\tau < \lambda$. Thus, $p \in \mathcal{R}_+^k(-\lambda)$. It is shown in [1], see also [4, Theorem 1.2], that for all $s > -1/2$, $t > 1$ there exists $k = k(s)$ such that

$$PR_V(\lambda) \in \mathcal{B}(H^{0,s+t}, H^{0,s}) \tag{3.12}$$

for all $P \in \mathcal{R}_+^k(-\lambda)$. Here we adopted the notation $H^{k,l} = H_{\mathrm{sc}}^{k,l}(\mathbb{S}_+^n)$; i.e. $H^{k,l}$ is just the (image under $(\mathrm{SP}^{-1})^*$ of the) weighted Sobolev space $\langle w\rangle^{-k}H^l(\mathbb{R}^n)$. This allows us to conclude that if P is the (Weyl) quantization of p then $PR_V(\lambda)f \in H^{0,s}$ for all s. Taking into account that $P \in \Psi_{\mathrm{sc}}^{-\infty,0}(\mathbb{S}_+^n)$, it follows that $PR_V(\lambda)f \in \dot{\mathcal{C}}^\infty(\mathbb{S}_+^n)$. Since p was an arbitrary symbol subject to the support condition in τ, (3.7) follows.

On the other hand, $\phi(\Delta + V - \lambda^2) \in \Psi_{\mathrm{sc}}^{2,0}(\mathbb{S}_+^n)$ is elliptic in

$$\{(y, \tau, \mu) : \tau \geq \lambda, y \in K\} \setminus \{(y, \lambda, 0) : y \in K\}$$

where $\phi \in \mathcal{C}^\infty(X)$ vanishes near $\cup_i C_i$, and is identically 1 on K, $K \subset \mathbb{S}_+^n \setminus \cup_i C_i$ is compact. Moreover,

$$\phi(\Delta + V - \lambda^2)R_V(\lambda)f = \phi f \in \dot{\mathcal{C}}^\infty(X), \tag{3.13}$$

so we conclude that

$$\mathrm{WF}_{\mathrm{sc}}(R_V(\lambda)f) \cap \pi^{-1}(\mathbb{S}^{n-1} \setminus \cup_i C_i) \subset R_\lambda^+. \tag{3.14}$$

Thus for short-range V_i we can apply Proposition 2.8 (and take into account the remark following it), while for long-range V_i we use Proposition 2.10 to prove the theorem. □

4. The scattering matrix

Theorem 3.1 allows us to describe the open part of the N-cluster to N-cluster scattering matrix in terms of the asymptotic behavior of certain generalized eigenfunctions. We restrict ourselves to short-range potentials to simplify the statements; the long-range case is completely analogous.

Theorem 4.1. *Let $\lambda > 0$, $0 < \epsilon < \lambda$, and suppose that k is sufficiently large. Then for $a_0 \in \mathcal{C}_c^\infty(\mathbb{S}^{n-1} \setminus \cup_i C_i)$ there exists a unique $u \in H^{0,-1}$ such that*

$$(H_V - \lambda^2)u = 0, \qquad u = u_+ + u_-, \tag{4.1}$$

$$v_- = e^{-i\lambda/x} x^{-(n-1)/2} u_- \in \mathcal{C}^\infty(\mathbb{S}^n_+), \qquad v_-|_{\mathbb{S}^{n-1}} = a_0, \tag{4.2}$$

and $P_+ u_+ \in L^2$ for all $P_+ \in \mathcal{R}^k_+(-\epsilon)$. Moreover, we have

$$u_+ = e^{-i\lambda/x} x^{(n-1)/2} v_+, \qquad v_+ \in \mathcal{C}^\infty(\mathbb{S}^n_+ \setminus \cup_i C_i), \tag{4.3}$$

and there exists $f \in \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$ such that $u_- = R_V(-\lambda)f$, $u_+ = -R_V(\lambda)f$.

Proof. First we discuss uniqueness. So suppose that u and u' satisfy the assumptions of the theorem. Then $(H_V - \lambda^2)(u - u') = 0$, and

$$u_- - u'_- = e^{i\lambda/x} x^{(n-1)/2}(v_- - v'_-) \in L^2 \tag{4.4}$$

since $v_-|_{\mathbb{S}^{n-1}} = v'_-|_{\mathbb{S}^{n-1}}$. Thus, for sufficiently large k $P_+(u_- - u'_-) \in L^2$, since then $P_+ \in \mathcal{R}^k_+(-\epsilon)$ is bounded on L^2. Therefore,

$$P_+(u - u') = P_+(u_- - u'_-) + P_+ u_+ - P_+ u'_+ \in L^2. \tag{4.5}$$

Hence, by Isozaki's uniqueness theorem [4, Theorem 1.3] $u = u'$ proving the uniqueness claim.

Now given $a_0 \in \mathcal{C}_c^\infty(\mathbb{S}^{n-1} \setminus \cup_i C_i)$ we can construct $u_- = e^{i\lambda/x} x^{(n-1)/2} v_-$, $v_- \in \mathcal{C}^\infty(\mathbb{S}^n_+)$, $v_-|_{\mathbb{S}^{n-1}} = a_0$ with the property that $(H_V - \lambda^2)u_- \in \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$ by an iterative argument as in [7, Proposition 12] since that construction is local and a_0 is supported away from $\cup_i C_i$, so near $\operatorname{supp} a_0$ V is smooth. Let $f = (H_V - \lambda^2)u_-$,

$u_+ = -R_V(\lambda)f$, and finally $u = u_- + u_+$. Then $(H_V - \lambda^2)u = 0$, and $P_+u_+ \in L^2$ by the resolvent estimates of [1], see also [4, Theorem 1.2], proving the existence of such generalized eigenfunctions. The asymptotic expansion of u_+ now follows from Theorem 3.1. Finally note that for $P_- \in \mathcal{R}^k_-(\epsilon)$, $P_-R_V(-\lambda)f \in L^2$ by the resolvent estimates, and

$$(x^2D_x + \lambda)u_- \in L^2, \tag{4.6}$$

so by Isozaki's result [3, Lemma 1.4] $P_-u_- \in L^2$. Thus,

$$P_-(u_- - R_V(-\lambda)f) \in L^2, \qquad (H_V - \lambda^2)(u_- - R_V(-\lambda)f) = 0. \tag{4.7}$$

It follows from the uniqueness theorem of [4] that $u_- = R_V(-\lambda)f$, completing the proof of the theorem. □

This theorem allows us to define the open part of the (absolute) scattering matrix as

$$S_V(\lambda) : \mathcal{C}^\infty_c(\mathbb{S}^{n-1} \setminus \cup_i C_i) \to \mathcal{C}^\infty(\mathbb{S}^{n-1} \setminus \cup_i C_i), \tag{4.8}$$

$$S_V(\lambda)(a_0) = v_+|_{\mathbb{S}^{n-1}\setminus\cup_i C_i}. \tag{4.9}$$

The more customary relative scattering matrix can be obtained from this by the antipodal reflection on $\mathbb{S}^{n-1}$; see [7, 8].

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