LECTURES ON THE PROPAGATION OF SINGULARITIES IN MANY-BODY SCATTERING, AARHUS, NOVEMBER 5-7, 1998

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Let $\mathcal{X}$ be a (finite) family of linear subspaces X_a, $a \in I$, of $\mathbb{R}^n$, and let X^a be the orthocomplement of X_a. Let π_a and π^a denote the orthogonal projections to X_a and X^a respectively. A many-body Hamiltonian is an operator of the form

$$H = \Delta + \sum_{a \in I} (\pi^a)^* V_a; \tag{0.1}$$

here Δ is the positive Laplacian on $\mathbb{R}^n$ and the V_a are real-valued functions on X^a in an appropriate class. For example, they can be polyhomogeneous symbols of order -1 which includes examples such as the Coulomb potential if its singularity at the origin is removed. Thus H is a self-adjoint operator on $L^2 = L^2(\mathbb{R}^n)$. The study of many-body scattering is an effort to understand the properties of this operator, H, in detail. We refer to [17] for the definitions of the basic objects in scattering theory; or see [23] for the definitions and the connection to the geometric point of view taken in these lectures.

Certain fundamental properties of H, especially global ones, such as its spectrum, resolvent estimates, the existence of wave operators and asymptotic completeness, are well understood [16, 17, 3, 4, 10, 5, 18, 2, 25]. Local, and hence microlocal results, are more rare. Thus, while the structure of the 2-cluster to 2-cluster S-matrix as well as that of the 2-cluster to 3-cluster S-matrix in 3-body scattering has been understood for some years now (they have smooth kernels, a *global* statement), see the works of Isozaki, Bommier and Skibsted [8, 19, 1], results involving other clusters are of more recent origin, starting with my paper [21] on the structure of the free-to-free (3-cluster to 3-cluster) S-matrix in 3-body scattering (the wave front relation is given by the broken geodesic flow on the sphere at 'infinity', a *microlocal* statement).

The purpose of these lectures is to state and explain these microlocal results, at least giving an outline of their proofs. In particular, the similarity to traditional microlocal analysis (often in singular settings) will be emphasized. The similarities between certain features of scattering theory and the study of hyperbolic PDE's were first emphasized by Melrose in his geometric generalization of the two-body problem [12], followed by the FIO-type construction of the Poisson operator by Melrose and Zworski in the same setting [15], though it had been known for some time that the role of the base and cotangent variables are reversed when passing from the standard pseudo-differential operator (ps.d.o.) algebra on $\mathbb{R}^n$ to ps.d.o's of interest in scattering (see especially [9, 5, 6, 26] or [7, Chapter XXX]).

In our geometrically more complicated setting we introduce a wave front set, $\mathrm{WF}_{\mathrm{Sc}}$, at spatial infinity (i.e. at the boundary $\partial\mathbb{S}^n_+$ of the radial compactification of $\mathbb{R}^n$ into a hemisphere $\mathbb{S}^n_+$), and we analyze propagation of 'singularities' (i.e. of

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$\mathrm{WF}_{\mathrm{Sc}}(u)$) for solutions of $(H-\lambda)u=0$, $\lambda\in\mathbb{R}$. This is closely related to the results on the structure of the S-matrices. The analogy in traditional microlocal analysis in this case is given by the wave equation on spaces with boundaries and the reflection of singularities of its solutions [13, 14], [7, Chapter XXIV], transmission problems, and more generally reflected singularities of solutions of the wave equation on spaces with corners ω [11]. To see the analogy in the geometry, we need to consider the intersection C_a of $\partial\mathbb{S}^n_+$ with the closure of the image of X_a (under the compactification); V_a will be singular at C_a (it is not even continuous there) since it decays in all directions away from C_a but not at C_a (unless it vanishes identically). The role of the C_a then corresponds to that of the boundary faces (hypersurfaces or corners) of ω.

From the technical point of view, these results are based on microlocal positive commutator estimates. That is, we will consider operators whose commutator with H is, to top order, positive in the part of the phase space where we wish to conclude that u is 'nice' (i.e. that the region is disjoint from $\mathrm{WF}_{\mathrm{Sc}}(u)$). Such microlocal positive commutator techniques were used, for example, in the proofs of the reflected singularities results of [13, 7] since they are very amenable to complicated geometric settings. Positive commutator results have also played a major role in many-body scattering starting with the work of Mourre [16], Perry, Sigal and Simon [17], Froese and Herbst [4], Jensen [9], Gérard, Isozaki and Skibsted [5] and Wang [24]. However, these results involved *globally* positive commutators (to 'top order', i.e. modulo compact operators in this case). Thus, in a sense these lectures will combine fundamental tools of traditional microlocal analysis and many-body scattering to prove microlocal results in the latter setting.

The lecture plan more or less follows a revised version of my PhD thesis [20], but there will be simplifications and improved results taken from my forthcoming manuscript [22].

1. Lecture 1: Introduce the compactified spaces, describe a geometric generalization of Euclidian many-body scattering, define the basic objects (Poisson operator, S-matrices), and state the results on the propagation of singularities of generalized eigenfunctions of many-body Hamiltonians (that is, tempered distributional solutions of $(H-\lambda)u=0$ and the structure of the S-matrices.
2. Lecture 2: Introduce the scattering calculus $\Psi^{m,l}_{\mathrm{sc}}$, the associated wave front set $\mathrm{WF}_{\mathrm{sc}}(u)$ (hopefully drawing on Richard Melrose's lectures) and its many-body version $\Psi^{m,l}_{\mathrm{Sc}}$ and $\mathrm{WF}_{\mathrm{Sc}}(u)$. Describe the positive commutator proof for the propagation of singularities in the scattering calculus (2-body type scattering) and its analogies to traditional microlocal analysis (e.g. propagation of singularities for solutions of the wave equation). Discuss global positive commutator estimates such as the Mourre estimate.
3. Lecture 3: Prove the propagation of singularities (i.e. of $\mathrm{WF}_{\mathrm{Sc}}(u)$) for solutions of $(H-\lambda)u=0$ where H is a many-body operator satisfying certain conditions, and show how closely related results imply the corresponding structure theorems for the S-matrices.

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