EXPONENTIAL DECAY OF EIGENFUNCTIONS IN MANY-BODY TYPE SCATTERING WITH SECOND ORDER PERTURBATIONS

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ABSTRACT. We show the exponential decay of eigenfunctions of second-order geometric many-body type Hamiltonians at non-threshold energies. Moreover, in the case of first order and small second order perturbations we show that there are no eigenfunctions with positive energy.

1. Introduction and results

In this paper we show that L^2-eigenfunctions of elliptic second order many-body type perturbations H of the Laplacian with non-threshold eigenvalues λ decay exponentially at a rate given by the distance of λ to the next threshold above it. If there are no positive thresholds, this implies the super-exponential decay of eigenfunctions at positive energies. We also show a unique continuation theorem at infinity, namely that for first-order and small second-order perturbations of the Laplacian, super-exponential decay of an eigenfunction ψ implies that ψ is identically 0. In particular, for these perturbations, an inductive argument shows that such Hamiltonians have no positive eigenvalues. These generalize results of [18], where only potential scattering was considered, although already in a geometric setting, and the pioneering work of Froese and Herbst [3], in which they considered many-body potential scattering in Euclidean space.

The methods are closely related to both those of Froese and Herbst and of the two-body type unique continuation theorems discussed in [11] and [8, Theorem 17.2.8]. However, the geometric nature of the problem forces a systematic treatment of various 'error terms', and in particular the use of a very stable argument. In particular, we emphasize throughout that for the exponential decay results only the indicial operators of H, which are non-commutative analogues of the usual principal symbol, affect the arguments, hence H can be generalized a great deal more. In addition, for unique continuation result only the indicial operators of H and its symbol in a high-energy sense (as in 'ellipticity with a parameter', or after rescaling, as in semiclassical problems) play a role. This explains, in particular, the first order (or small second-order) hypothesis on the perturbations for the unique continuation theorem, and raises the question to whether this theorem also holds under non-trapping conditions on the metric near infinity, or even more generally. The key estimates arise from a positive commutator estimate for the

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conjugated Hamiltonian, which is closely related to Hörmander's solvability condition for PDE's [10, 9, 2]; see [22] for a recent discussion, including the relationship to numerical computation.

Before stating the results precisely, recall from [14] that if $\bar{X}$ is a manifold with boundary and x is a boundary defining function on $\bar{X}$, a scattering metric g_0 is a Riemannian metric on $X = \bar{X}^\circ$ which is of the form $g_0 = x^{-4}\,dx^2 + x^{-2}h$ near $\partial\bar{X}$, where h is a symmetric 2-cotensor that restricts to a metric on $\partial\bar{X}$. Let Δ_{g_0} be the Laplacian of this metric. This is a typical element of $\mathrm{Diff}_{\mathrm{sc}}(\bar{X})$, the algebra of scattering differential operators. The latter is generated, over $\mathcal{C}^\infty(\bar{X})$, by the vector fields $\mathcal{V}_{\mathrm{sc}}(\bar{X}) = x\mathcal{V}_{\mathrm{b}}(\bar{X})$; $\mathcal{V}_{\mathrm{b}}(\bar{X})$ being the Lie algebra of $\mathcal{C}^\infty$ vector fields on $\bar{X}$ that are tangent to $\partial\bar{X}$.

In this paper we consider many-body type Hamiltonians. That is, let $\mathcal{C}$ be a cleanly intersecting family of closed embedded submanifolds of $\partial\bar{X}$ which is closed under intersections and which includes $C_0 = \partial\bar{X}$ (the latter only for convenient notation). As shown in [19], one can resolve $\mathcal{C}$ by blowing these up inductively, starting with the submanifold of the lowest dimension. The resulting space $[\bar{X};\mathcal{C}]$ is a manifold with corners, and the blow-down map $\beta : [\bar{X};\mathcal{C}] \to \bar{X}$ is smooth. Then $\mathrm{Diff}_{\mathrm{Sc}}(\bar{X},\mathcal{C})$ is similar to $\mathrm{Diff}_{\mathrm{sc}}(\bar{X})$, but with coefficients that are in $\mathcal{C}^\infty([\bar{X};\mathcal{C}])$: $\mathrm{Diff}_{\mathrm{Sc}}(\bar{X},\mathcal{C}) = \mathcal{C}^\infty([\bar{X};\mathcal{C}])\otimes_{\mathcal{C}^\infty(\bar{X})}\mathrm{Diff}_{\mathrm{sc}}(\bar{X})$. More generally, if E, F are vector bundles over $\bar{X}$, we can consider differential operators mapping smooth sections of β^*E to smooth sections of β^*F, denoted by $\mathrm{Diff}_{\mathrm{Sc}}(\bar{X},\mathcal{C};E,F)$, or simply $\mathrm{Diff}_{\mathrm{Sc}}(\bar{X},\mathcal{C};E)$ if $E = F$. The vector fields in $\mathrm{Diff}_{\mathrm{Sc}}(\bar{X};\mathcal{C})$ form exactly the set of all smooth sections of a vector bundle, denoted by ${}^{\mathrm{Sc}}T[\bar{X};\mathcal{C}]$ over $[\bar{X};\mathcal{C}]$, namely the pull-back of ${}^{\mathrm{sc}}T\bar{X}$ by β. The dual bundle is denoted ${}^{\mathrm{Sc}}T^*[\bar{X};\mathcal{C}]$.

It may help the reader if we explain why the Euclidean setting is a particular example of this setup. Indeed, the reader may be interested in the Euclidean magnetic and metric scattering specifically; if so, all the arguments given below can be translated into Euclidean notation as follows. There X is a vector space with a metric g_0, which can hence by identified with $\mathbb{R}^n$. Moreover, $\bar{X}$ is the radial (or geodesic) compactification of $\mathbb{R}^n$ to a ball. Explicitly, this arises by considering 'inverse' polar coordinates, and writing $w \in X$ as $w = r\omega = x^{-1}\omega$, $\omega \in \mathbb{S}^{n-1}$, so $x = |w|^{-1}$, e.g. in $|w| \geq 1$. In particular, $\partial\bar{X}$ is given by $x = 0$, i.e. it is just $\mathbb{S}^{n-1}$. The metric g_0 then has the form $dr^2 + r^{-2}h_0 = x^{-4}dx^2 + x^2h_0$, where h_0 is the standard metric on $\mathbb{S}^{n-1}$, so $(\bar{X}, g_0)$ fits exactly into this framework. Moreover, in the many-body setting, one is given a collection $\mathcal{X} = \{X_a :\ a \in I\}$ of linear subspaces of X. The corresponding cleanly intersecting family is given by $\mathcal{C} = \{C_a :\ a \in I\}$, where $C_a = \bar{X}_a \cap \partial\bar{X}$, and $\bar{X}_a$ is the closure of X_a in $\bar{X}$. Thus, C_a can also be thought of as the intersection of the unit sphere in $\mathbb{R}^n$ with X_a. Then ${}^{\mathrm{Sc}}T[\bar{X};\mathcal{C}]$, ${}^{\mathrm{Sc}}T^*[\bar{X};\mathcal{C}]$ are trivial vector bundles over $[\bar{X};\mathcal{C}]$; namely ${}^{\mathrm{Sc}}T^*[\bar{X};\mathcal{C}] = [\bar{X};\mathcal{C}] \times X^*$, X^* being the dual vector space of X.

We can now describe the operators H we consider in this paper. First, we assume that $H = \Delta \otimes \mathrm{Id}_E + V$ where $\Delta = \Delta_g$ is the Laplacian of a metric g such that

$$g \in \mathcal{C}^\infty([\bar{X};\mathcal{C}]; {}^{\mathrm{Sc}}T^*[\bar{X};\mathcal{C}] \otimes {}^{\mathrm{Sc}}T^*[\bar{X};\mathcal{C}])$$

is symmetric, $g - g_0$ vanishes at the free face, i.e. the lift of C_0, and $V \in \mathrm{Diff}^1_{\mathrm{Sc}}(\bar{X};\mathcal{C};E)$ is formally self-adjoint and vanishes at the free face.

Now, g_0 induces an orthogonal decomposition of ${}^{\mathrm{sc}}T^*\bar{X}$ at each $C_a \in \mathcal{C}$, which, with the Euclidean notation corresponds to the decomposition

$$T^*_{(w_a,w^a)}X_0 = T^*_{w_a}X_a \oplus T^*_{w^a}X^a;$$

X^a being the orthocomplement of X_a. In the geometric setting, X^a is simply short hand for the fibers $\beta_a^{-1}(y_a)$, $y_a \in C_a$ of the front face of $\beta_a : [\bar{X}; C_a] \to \bar{X}$, while a neighborhood of infinity in the radial compactification $\bar{X}_a$ of X_a stands for $C_a \times [0,\epsilon)_x$, see [19]. This allows us to define the indicial operators of differential operators $A \in \mathrm{Diff}^m_{\mathrm{Sc}}(\bar{X};\mathcal{C})$ invariantly (even in the geometric setting), at a cluster a, as a family of operators $\hat{A}_a(y_a,\xi_a)$, $y_a \in C_a$, $\xi_a \in {}^{\mathrm{sc}}T^*_{y_a}\bar{X}_a$, on functions on X^a, by freezing the coefficients of A at y_a and replacing derivatives $D_{(w_a)_j}$ by $(\xi_a)_j$. (Technically the a-indicial operators are defined on a blow-up of C_a, i.e. the above definition is valid for $y_a \in C_{a,\mathrm{reg}}$, i.e. for y_a away from all C_b which do not satisfy $C_b \supset C_a$; see [19] for the detailed setup.) For example, the a-indicial operator of $\Delta_{g_0} - \lambda$ is $|\xi_a|^2_{y_a} + \Delta_{X^a}(y_a) - \lambda$; here $\Delta_{X^a}(y_a)$ is, for each $y_a \in C_a$, a translation invariant operator on the fibers of the front face of $\beta_a : [\bar{X}; C_a] \to \bar{X}$.

We also assume that restricted to a front face, $g - g_0$ is a section of $T^*X^a \otimes T^*X^a$, i.e. depends on the interaction variables only over C_a, and that for each $y_a \in C_a$, $\hat{V}_a(y_a,\xi_a) \in \mathrm{Diff}^1(X^a;E)$ is independent of ξ_a. In other words, we assume that all indicial operators of H are pointwise in C_a 'product-type', i.e. have the form

$$\hat{H}_a(\xi_a) = |\xi_a|^2_{y_a} \otimes \mathrm{Id}_E + H^a(y_a), \tag{1.1}$$

where $H^a(y_a) \in \mathrm{Diff}^2(X^a;E)$, and y_a is the variable along C_a.

This product assumption is sufficient for all of our results, provided that on the complement of the set of thresholds, a Mourre-type global positive commutator estimate (local only in the spectrum of H) holds. If $H^a(y_a)$ has L^2-eigenvalues, the existence of such a Mourre estimate at various energies certainly depends on the behavior of the eigenvalues as a function of y_a, as shown by a related problem involving scattering by potentials of degree zero, [6, 7, 5]. So we assume in this paper that either $H^a(y_a)$ has no L^2 eigenvalues for any a, or in a neighborhood of C_a, $\bar{X}$ has the structure of (the radial compactification of) a conic slice of $X_a \times X^a$, over which E is trivial, and the indicial operators satisfy

$$\hat{H}_a(\xi_a) = |\xi_a|^2 \otimes \mathrm{Id}_E + H^a, \tag{1.2}$$

so H^a is independent of ξ_a, and in particular of its projection y_a to C_a. Here H^a is called the subsystem Hamiltonian for the subsystem a; it is also a many-body Hamiltonian (but one corresponding to fewer particles in actual many-body scattering!).

Examples of such Hamiltonians include the Laplacian of metric perturbations of g_0 in the Euclidean setting, both on functions, and more generally on forms, and the square of associated self-adjoint Dirac operators. Namely, let X_a be the collision planes, and let $g^a \in \mathcal{C}^\infty_c(X^a; T^*X^a \otimes T^*X^a)$, $a \neq 0$, be symmetric. Then g^a can also be regarded as a section of $T^*X \times T^*X$, and $g = g_0 + \sum_a g^a$ satisfies these criteria provided that it is positive definite (i.e. a metric). Of course, the compact support of the g^a can be replaced by first order decay at infinity as a section of ${}^{\mathrm{sc}}T^*\bar{X}^a \otimes {}^{\mathrm{sc}}T^*\bar{X}^a$, i.e. relative to the translation-invariant basis $dw^a_j \otimes dw^a_k$, $j,k = 1,\ldots \dim X^a$, of $\mathcal{C}^\infty(X^a; T^*X^a \otimes T^*X^a)$.

We prove the following results. Let Λ denote the thresholds of H, i.e. the set of the L^2 eigenvalues of all of its subsystem Hamiltonians H^a; this is a closed countable subset of $\mathbb{R}$. (Note that 0 is an eigenvalue of H^0, hence it is a threshold of H; if no non-trivial subsystem has an L^2-eigenvalue then $\Lambda = \{0\}$.) The following theorem states that non-threshold eigenfunctions of H decay exponentially at a rate given by the distance of the eigenvalue from the nearest threshold above it. This result also explains why the unique continuation theorem is considered separately, namely why super-exponential decay assumptions are natural there (unlike the Schwartz assumptions of [8, Theorem 17.2.8] in the two-body type setting).

Theorem. [*cf.* [18, Proposition B.2] *and Froese and Herbst,* [3, Theorem 2.1]] *Let $\lambda \in \mathbb{R} \setminus \Lambda$, and suppose that $\psi \in L^2_{sc}(\bar{X})$ satisfies $H\psi = \lambda\psi$. Then $e^{\alpha/x}\psi \in L^2_{sc}(\bar{X})$ for all $\alpha \in \mathbb{R}$ such that $[\lambda, \lambda + \alpha^2] \cap \Lambda = \emptyset$, i.e.*

$$\sup\{\lambda + \alpha^2 : \ e^{\alpha/x}\psi \in L^2_{sc}(\bar{X})\} \geq \inf\{\lambda' \in \Lambda : \ \lambda' > \lambda\}.$$

The estimates leading to this theorem are uniform, and in fact yield, as observed by Perry [17] in the Euclidean many-body potential scattering, that eigenvalues cannot accummulate at thresholds from above, hence the following corollary.

Corollary. *The thresholds $\lambda \in \Lambda$ are isolated from above, i.e. for $\lambda \in \Lambda$ there exists $\lambda' > \lambda$ such that $(\lambda, \lambda') \cap (\Lambda \cup \operatorname{spec}_{pp}(H)) = \emptyset$.*

The unique continuation theorem at infinity is the following.

Theorem. [*cf.* [18, Proposition B.3] *and Froese and Herbst,* [3, Theorem 3.1]] *Let $\lambda \in \mathbb{R}$ and let d denote a metric giving the usual topology on $\mathcal{C}^\infty$ sections of ${}^{Sc}T^*[X;\mathcal{C}] \otimes {}^{Sc}T^*[X;\mathcal{C}]$. There exists $\epsilon > 0$ such that if $d(g, g_0) < \epsilon$ and $H\psi = \lambda\psi$, $\exp(\alpha/x)\psi \in L^2_{sc}(\bar{X})$ for all α, then $\psi = 0$.*

As an immediate corollary we deduce the absence of positive eigenvalues for first order perturbation and small second order perturbations of Δ_{g_0}.

Theorem. *Let $\lambda > 0$, g is close to g_0 in a $\mathcal{C}^\infty$ sense. Suppose that $H\psi = \lambda\psi$, $\psi \in L^2_{sc}(\bar{X})$. Then $\psi = 0$.*

Proof. One proceeds inductively, showing that H^a does not have any positive eigenvalues, starting with $a = 0$, when this is certainly true. So suppose that for all b such that $X^b \subsetneq X^a$, H^b does not have any positive eigenvalues. Then the set of thresholds for H^a is disjoint from $(0, +\infty)$, so by the first theorem any eigenfunction with a positive eigenvalue decays super-exponentially, and then by the second theorem it vanishes. This completes the inductive step. $\square$

Remark. The last result in particular applies to $H = \Delta_g$ on functions even if g restricted to the front face of $[\bar{X}; C_a] \to \bar{X}$ is any smooth section of $T^*X^a \otimes T^*X^a$, i.e. only (1.1) holds (rather than (1.2)) to show that H has no L^2 eigenfunctions at all. Indeed, proceeding inductively as in the proof, we may assume that for all b such that $C_a \subsetneq C_b$, $\hat{H}_b(\xi_b)$ does not have any L^2-eigenvalues. Thus, the Mourre-type estimate is valid, hence $\hat{H}_a(\xi_a)$ has no positive eigenvalues. But $\hat{H}_a(\xi_a) \geq 0$, so it cannot have negative energy bound states, and by elliptic regularity, any L^2 element ψ of its nullspace would be in $H^\infty_{\mathrm{sc}}(\bar{X}^a)$ thus $\Delta_g = (d+\delta)^2 = \delta d + d\delta$ shows that $d\psi = 0$, hence $\psi = 0$. Thus, $\hat{H}_a(\xi_a)$ has no L^2-eigenvalues, completing the inductive step.

The rough idea of the proof of the two main results is to conjugate by exponential weights e^F, where F is a symbol of order 1, for example $F = \alpha/x$. If ψ is an eigenfunction of H of eigenvalue λ, then $\psi_F = e^F\psi$ solves

$$P\psi_F = 0 \text{ where } P = H(F) - \lambda = e^F H e^{-F} - \lambda.$$

Now $\operatorname{Re} P$ is given by $H - \alpha^2 - \lambda$, while $\operatorname{Im} P$ is given by $-2\alpha(x^2 D_x)$, modulo $x \operatorname{Diff}_{\mathrm{Sc}}(\bar{X}, \mathcal{C})$. By elliptic regularity, using $P\psi_F = 0$, $\|\psi_F\|_{x^p H^k_{\mathrm{sc}}(\bar{X})}$ is bounded by $C_{k,p}\|\psi_F\|_{x^p L^2_{\mathrm{sc}}(\bar{X})}$, so the order of various differential operators can be neglected, while the weight is important. Since

$$P^*P = (\operatorname{Re} P)^2 + (\operatorname{Im} P)^2 + i(\operatorname{Re} P \operatorname{Im} P - \operatorname{Im} P \operatorname{Re} P),$$

so

$$0 = (\psi_F, P^*P\psi_F) = \|\operatorname{Re} P\psi_F\|^2 + \|\operatorname{Im} P\psi_F\|^2 + (\psi_F, i[\operatorname{Re} P, \operatorname{Im} P]\psi_F). \tag{1.3}$$

Now, $[\operatorname{Re} P, \operatorname{Im} P] \in x \operatorname{Diff}^2_{\mathrm{sc}}(\bar{X}, \mathcal{C})$, i.e. has an extra order of vanishing, which shows that

$$\|\operatorname{Re} P\psi_F\| \le C_1\|x^{1/2}\psi_F\|, \ \|\operatorname{Im} P\psi_F\| \le C_1\|x^{1/2}\psi_F\|.$$

Due to the extra factor of $x^{1/2}$, this can be interpreted roughly as ψ_F being close to being in the nullspace of both $\operatorname{Re} P$ and of $\operatorname{Im} P$, hence both of $H - \lambda - \alpha^2$ and $x^2 D_x$.

If, moreover, $(\psi_F, i[\operatorname{Re} P, \operatorname{Im} P]\psi_F)$ is positive, modulo terms involving $\operatorname{Re} P$ and $\operatorname{Im} P$ (which can be absorbed in the squares in (1.3)), and terms of the form $(\psi_F, R\psi_F)$, $R \in x^2 \operatorname{Diff}_{\mathrm{Sc}}(\bar{X}, \mathcal{C})$, which are thus bounded by $C_2\|x\psi_F\|^2$, then the factor x (which has an extra $x^{1/2}$ compared to $\|x^{1/2}\psi_F\|$) yields easily a bound for $\|x^{1/2}\psi_F\|$ in terms of $\|\psi\|$. This gives estimates for the norm $\|x^{1/2}\psi_F\|$, uniform both in F and in ψ. A regularization argument in F then gives the exponential decay of ψ.

The positivity of $(\psi_F, i[\operatorname{Re} P, \operatorname{Im} P]\psi_F)$, in the sense described above, is easy to see if we replace $i[\operatorname{Re} P, \operatorname{Im} P]$ by $i[H - \lambda - \alpha^2, -2\alpha x^2 D_x]$: this commutator is a standard one considered in many-body scattering, although the even more usual one would be $i[H - \lambda - \alpha^2, -2xD_x]$, whose local positivity in the spectrum of H is the Mourre estimate [15, 16, 4]. Indeed, the latter commutator is the one considered by Froese and Herbst in Euclidean many-body potential scattering, and we could adapt their argument (though we would need to deal with numerous error terms) to our setting. However, the argument presented here is more robust, especially in the high energy sense discussed below, in which their approach would not work in the generality considered here. There is one exception: for $\alpha = 0$, $\operatorname{Im} P$ degenerates, and in this case we need to 'rescale' the commutator argument, and consider $i[H - \lambda - \alpha^2, -2xD_x]$ directly.

We next want to let $\alpha \to \infty$. Since most of the related literature considers semiclassical problems, we let $h = \alpha^{-1}$, and replace P above by $P_h = h^2 P$, which is a semiclassical differential operator, $P_h \in \operatorname{Diff}^2_{\mathrm{Sc},h}(\bar{X}, \mathcal{C})$. Here

$$\operatorname{Diff}_{\mathrm{Sc},h}(\bar{X}, \mathcal{C}) = \mathcal{C}^\infty([\bar{X}; \mathcal{C}]) \otimes_{\mathcal{C}^\infty(\bar{X})} \operatorname{Diff}_{\mathrm{sc},h}(\bar{X}),$$

and $\operatorname{Diff}_{\mathrm{sc},h}(\bar{X})$ is the algebra of semiclassical scattering differential operators discussed, for example, in [21]. It is generated by $h\mathcal{V}_{\mathrm{sc}}(\bar{X})$ over $\mathcal{C}^\infty(\bar{X} \times [0,1)_h)$. In this semiclassical sense, the first and zeroth order terms in H do not play a role in P_h: their contribution is in $h \operatorname{Diff}^1_{\mathrm{Sc},h}(\bar{X}, \mathcal{C})$, hence their contribution to the commutator $i[\operatorname{Re} P_h, \operatorname{Im} P_h]$ is in $xh^2 \operatorname{Diff}_{\mathrm{Sc},h}(\bar{X}, \mathcal{C})$. Moreover, if g is close to g_0, then $i[\operatorname{Re} P_h, \operatorname{Im} P_h]$ is close to the corresponding commutator with P_h replaced by $h^2(e^F \Delta_{g_0} e^{-F} - \lambda)$. Since in the latter case the commutator is positive, modulo terms than can be absorbed in the two squares in (1.3), $i[\operatorname{Re} P_h, \operatorname{Im} P_h]$ is also positive for g near g_0. This gives an estimate as above, from which the vanishing of ψ near $x = 0$ follows easily.

We remark that the estimates we use are related to the usual proof of unique continuation at infinity on $\mathbb{R}^n$ (i.e. not in the many-body setting), see [8, Theorem 17.2.8], and to Hörmander's solvability condition for PDE's in terms of the real and imaginary parts of the principal symbol. Indeed, although in [8, Theorem 17.2.8] various changes of coordinates are used first, which change the nature of the PDE at infinity, ultimately the necessary estimates also arise from a commutator of the kind $i[\operatorname{Re} P, \operatorname{Im} P]$. However, even in that setting, the proof we present appears more natural from the point of view of scattering than the one presented there, which is motivated by unique continuation at points in $\mathbb{R}^n$. In particular, the reader who is interested in the setting of [8, Theorem 17.2.8] should be able to skip the proof of Theorem 2.3, which is rather simple (a Poisson bracket computation) in that case. We remark that related estimates, obtained by different techniques, form the backbone of the (two-body type) unique continuation results of Jerison and Kenig [13, 12].

The true flavor of our arguments is most clear in the proof of the unique continuation theorem, Theorem 4.1. The reason is that on the one hand there is no need for regularization of F, since we are assuming super-exponential decay, on the other hand the positivity of $i[\operatorname{Re} P_h, \operatorname{Im} P_h]$ is easy to see.

The structure of the paper is the following. In Section 2 we discuss various preliminaries, including the structure of the conjugated Hamiltonian and a Mourre-type global positive commutator estimate. In Section 3 we prove the exponential decay of non-threshold eigenfunctions. In Section 4, we prove the unique continuation theorem at infinity. Finally, for the sake of completeness, and since technically the usual statements of the Mourre estimate do not discuss the present setting, we include its proof in the Appendix. We emphasize that the presence of bundles such as E makes no difference in the discussion, hence they are ignored in order to keep the notation manageable; see Remarks 2.2 and 2.4 for further information.

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2. Preliminaries

We first remark that the Riemannian density of a metric g has the form

$$dg = \sqrt{\det(g_{ij})}\, dx\, dy = \tilde{g}\, \frac{dx\, dy}{x^{n+1}} \;\; n = \dim X,\;\; \tilde{g} \in \mathcal{C}^\infty([X;\mathcal{C}]). \tag{2.1}$$

By our conditions on the form of g, the Laplacian takes the following form

$$\Delta_g = (x^2 D_x)^2 + \sum_j b_j x^2 P_j + \sum_j x c_j Q_j + xR$$

with $b_j, c_j \in \mathcal{C}^\infty([\bar{X};\mathcal{C}])$, $P_j \in \operatorname{Diff}^2(\partial\bar{X})$, $Q_j \in \operatorname{Diff}^1(\partial\bar{X})$, $R \in \operatorname{Diff}^2_{\mathrm{Sc}}(\bar{X},\mathcal{C})$. Hence, $H = \Delta_g + V$ takes the form

$$H = (x^2 D_x)^2 + \sum_j b'_j x^2 P'_j + \sum_j x c'_j Q'_j + e + xR',$$

with $b'_j, c'_j, e \in \mathcal{C}^\infty([\bar{X};\mathcal{C}])$, $P'_j \in \operatorname{Diff}^2(\partial\bar{X})$, $Q'_j \in \operatorname{Diff}^1(\partial\bar{X})$, $R' \in \operatorname{Diff}^2_{\mathrm{Sc}}(\bar{X},\mathcal{C})$.

Below we consider the conjugated Hamiltonian $H(F) = e^F H e^{-F}$, where F is a symbol of order 1. The exponential weights will facilitate exponential decay estimates, and eventually the proof of unique continuation at infinity. Let $x_0 = \sup_{\bar{X}} x$. By altering x in a compact subset of X, we may assume that $x_0 < 1/2$; we do this for the convenience of notation below. We let $S^m([0,1)_x)$ is the space of all symbols F of order m on $[0,1)$, which satisfy $F \in \mathcal{C}^\infty((0,1))$, vanish on $(1/2,1)$, and for which $\sup |x^{m+k}\partial_x^k F| < \infty$ for all k. The topology of $S^m([0,1))$ is given by the seminorms $\sup |x^{m+k}\partial_x^k F|$. Also, the spaces $S^m(\bar{X})$, resp. $S^m([\bar{X};\mathcal{C}])$, of symbols is defined similarly, i.e. it is given by seminorms $\sup |x^m PF|$, $P \in \mathrm{Diff}_{\mathrm{b}}^k(\bar{X})$, resp. $P \in \mathrm{Diff}_{\mathrm{b}}^k([\bar{X};\mathcal{C}])$. In the following lemma $\mathrm{Diff}_{\mathrm{scc}}(\bar{X})$, as usual, stands for non-classical (non-polyhomogeneous) scattering differential operators (i.e. scattering differential operators with non-polyhomogeneous coefficients), corresponding to the lack of polyhomogeneity of F. In particular, $\mathrm{Diff}^0_{\mathrm{scc}}(\bar{X}) = S^0(\bar{X})$ (considered as multiplication operators). Similarly, $\mathrm{Diff}_{\mathrm{Scc}}(\bar{X},\mathcal{C}) = S^0([\bar{X};\mathcal{C}]) \otimes_{S^0(\bar{X})} \mathrm{Diff}_{\mathrm{scc}}(\bar{X})$ stands for the corresponding calculus of many-body differential operators.

Lemma 2.1. *Suppose $\lambda \in \mathbb{R}$, $H\psi = \lambda\psi$, $\psi \in L^2_{sc}(\bar{X})$. Suppose also that $\alpha \geq 0$, and for all β we have $x^{-\beta}\exp(\alpha/x)\psi \in L^2_{sc}(\bar{X})$. Then with $F \in S^1([0,1))$, $F \leq \alpha/x + \beta|\log x|$ for some β, $\operatorname{supp} F \subset [0,1/2)$, $\psi_F = e^F\psi = e^{F(x)}\psi$,*

$$P = P(F) = e^F(H-\lambda)e^{-F} = H(F) - \lambda,\ H(F) = H + e^F[H, e^{-F}],$$

we have $\psi_F \in \dot{\mathcal{C}}^\infty(\bar{X})$,

$$P(F)\psi_F = 0, \tag{2.2}$$

$$P(F) = H - 2(x^2D_xF)(x^2D_x) + (x^2D_xF)^2 - \lambda + xR_1, \quad R_1 \in \mathrm{Diff}^2_{Scc}(\bar{X},\mathcal{C}), \tag{2.3}$$

with

$$\operatorname{Re} P(F) = H + (x^2D_xF)^2 - \lambda + xR_2,\ \operatorname{Im} P(F) = 2(x^2\partial_xF)(x^2D_x) + xR_3, \tag{2.4}$$

$R_2, R_3 \in \mathrm{Diff}^2_{Scc}(\bar{X},\mathcal{C})$, R_j bounded as long as $x^2\partial_xF$ is bounded in $S^0([0,1))$, hence as long as F is bounded in $S^1([0,1))$. The coefficients of the xR_2, xR_3 are in fact polynomials with vanishing constant term, in $(x^2\partial_x)^{m+1}F$, $m \geq 0$.

In particular,

$$\begin{aligned} i[\operatorname{Re} P(F), \operatorname{Im} P(F)] =& i[H + (x^2D_xF)^2, 2(x^2\partial_xF)(x^2D_x)] \\ &+ \operatorname{Re} P(F) xR_4 + xR_5 \operatorname{Re} P(F) \\ &+ \operatorname{Im} P(F) xR_6 + xR_7 \operatorname{Im} P(F) + x^2R_8, \end{aligned} \tag{2.5}$$

where $R_j \in \mathrm{Diff}^2_{Scc}(\bar{X},\mathcal{C})$ are bounded as long as $x^2\partial_xF$ is bounded in $S^0([0,1))$.

Remark 2.2. All but the first term on the right hand side of (2.5) should be considered error terms, even though they are only of the same order (in terms of decay at $\partial\bar{X}$) as

$$i[H + (x^2D_xF)^2 - \lambda, 2(x^2\partial_xF)(x^2D_x)],$$

due to the lack of commutativity of $\mathrm{Diff}_{\mathrm{Scc}}(\bar{X},\mathcal{C})$ even to top order. The reason is that these terms contain factors of $\operatorname{Re} P(F)$ and $\operatorname{Im} P(F)$, and we will have good control over $\operatorname{Re} P(F)\psi_F$ and $\operatorname{Im} P(F)\psi_F$.

For similar reasons, the presence of bundles E would make no difference, since even if they are present, (2.4) is unaffected, hence (2.5) holds as well.

Proof. First note that

$$[x^2D_x, e^F] = (x^2D_xF)e^F, \qquad x^2D_xF \in S^0([0,1)), \tag{2.6}$$

so $e^F[H, e^{-F}] \in \mathrm{Diff}^1_{\mathrm{scc}}(\bar{X}, \mathcal{C})$. The dependence of the terms of $P(F)$ on F thus comes from x^2D_xF, and its commutators through commuting it through other vector fields (as in rewriting $(x^2D_x)(x^2D_xF)$ as $(x^2D_xF)(x^2D_x)$ plus a commutator term), hence through $(x^2D_x)^{m+1}F$, $m \geq 0$.

Now, (2.2), which a priori holds in a distributional sense, $\psi_F \in x^rL^2_{\mathrm{sc}}(\bar{X})$ for all r, and the ellipticity of $\sigma_{\mathrm{Sc},2}(H)$ show that $\psi_F \in \dot{\mathcal{C}}^\infty(\bar{X})$.

We use

$$\begin{aligned}\operatorname{Re} P(F) = \frac{1}{2}(P(F) + P(F)^*) &= H - \lambda + \frac{1}{2}(e^F[H, e^{-F}] - [H, e^{-F}]e^F)\\ &= H - \lambda + \frac{1}{2}[e^F, [H, e^{-F}]]\end{aligned}$$

to prove (2.4) (note that only the $(x^2D_x)^2$ terms in H gives a non-vanishing contribution to the double commutator). Finally, (2.5) follows since

$$Q = \operatorname{Re} P(F) - (H + (x^2D_xF)^2 - \lambda) \in x\,\mathrm{Diff}^2_{\mathrm{Sc}}(\bar{X}, \mathcal{C}),$$

so $[Q, \operatorname{Im} P(F)] = Q \operatorname{Im} P(F) - \operatorname{Im} P(F)Q$ is of the form of the R_4 and R_5 terms, and similarly for $Q' = \operatorname{Im} P(F) - 2(x^2\partial_xF)(x^2D_x)$. □

In light of (2.5), we need a positivity result for $i[x^2D_x, H]$. Such a result follows directly from a Poisson bracket computation if H is a geometric 2-body type operator. In general, it requires a positive commutator estimate that is closely related to, and can be readily deduced from, the well-known Mourre estimate [15, 16, 4], whose proof goes through in this generality. We will briefly sketch its proof in the appendix for the sake of completeness. So let

$$d(\lambda) = \inf\{\lambda - \lambda' : \lambda' \leq \lambda,\ \lambda' \in \Lambda\}$$

be the distance of λ to the next threshold below it. Let $\chi \in \mathcal{C}^\infty_c([0,1))$ be supported near 0, identically 1 on a smaller neighborhood of 0, and let

$$B = \frac{1}{2}(\chi(x)x^2D_x + (\chi(x)x^2D_x)^*)$$

be the symmetrization of the radial vector field. Now x^2D_x is formally self-adjoint with respect to the measure $\frac{dx\,dy}{x^2}$, and if C is formally self-adjoint with respect to a density dg' then its adjoint with respect to $\alpha\,dg'$, α smooth real-valued, is $\alpha^{-1}C\alpha = C + \alpha^{-1}[C, \alpha]$. Since xD_x is tangent to all elements of $\mathcal{C}$, $xD_x(x^{-n+1}\tilde{g}) \in x^{-n+1}\mathcal{C}^\infty([\bar{X}; \mathcal{C}])$. In particular,

$$ib = x^{-1}(B - \chi(x)x^2D_x) \in \mathcal{C}^\infty([\bar{X}; \mathcal{C}])$$

and b is real-valued. It is easy to check that in the particular case when $g = g_0$ then $b \in \mathcal{C}^\infty(\bar{X})$ and $b|_{\partial X} = \frac{n-1}{2}$, $n = \dim X$.

The first order differential operator B gives rise to:

Theorem 2.3. *Suppose $\lambda \in \mathbb{R} \setminus \Lambda$. For $\epsilon > 0$ there exists $\delta > 0$ such that for $\phi \in \mathcal{C}^\infty_c(\mathbb{R}; [0,1])$ with $\operatorname{supp} \phi \subset (\lambda - \delta, \lambda + \delta)$,*

$$\phi(H)i[B, H]\phi(H) \geq 2(d(\lambda) - \epsilon)\phi(H)x\phi(H) - 4\phi(H)BxB\phi(H) + K,$$

$K \in \Psi^{-\infty,2}_{Sc}(\bar{X}, \mathcal{C})$.

Remark 2.4. Again, the presence of bundles would make no difference, as is apparent from the proof given below and in the appendix.

This theorem could be proved directly, without the use of the global positive commutator estimate, (2.8), but for notational (and reference) reasons, it is easier to proceed via (2.8).

Proof. First, note that changing B by any term $B' = xB'_1 \in x\,\mathrm{Diff}^1_{\mathrm{Sc}}(\bar{X};\mathcal{C})$ changes the left hand side by

$$\phi(H)i[B',H]\phi(H) = \phi(H)iB'(H-\lambda)\phi(H) - \phi(H)i(H-\lambda)B'\phi(H). \tag{2.7}$$

Now, $\|\phi(H)(H-\lambda)\| \leq \delta'$ if $\operatorname{supp}\phi \subset (\lambda-\delta',\lambda+\delta')$. Thus, multiplying (2.7) from the left and right by $\tilde{\phi}(H)$, $\tilde{\phi} \in \mathcal{C}^\infty_c(\mathbb{R};[0,1])$ supported in $(\lambda-\delta',\lambda+\delta')$, $\tilde{\phi}(H)iB'(H-\lambda)\tilde{\phi}(H)$ has the form $\tilde{\phi}(H)xT\tilde{\phi}(H)$, and $\|T\| \leq \delta'\|B'\|$. Hence, after this multiplication, both terms on the right hand side of (2.7) can be absorbed into $2(d(\lambda)-\epsilon)\tilde{\phi}(H)x\tilde{\phi}(H)$ at the cost of increasing $\epsilon > 0$ (which was arbitrary to start with) by an arbitrarily small amount. Thus, for each $\epsilon > 0$ ther existence of a $\delta > 0$ such that the estimate of the theorem holds only depends on the indicial operators of B, even though $\phi(H)i[B',H]\phi(H)$ is the same order as our leading term $2(d(\lambda)-\epsilon)\phi(H)x\phi(H)$ (the many-body calculus is not commutative even to top order!): the key being that this commutator is small at the 'characteristic variety'.

With $x^2D_x = x(xD_x)$, $A = \frac{1}{2}(\chi(x)xD_x + (\chi(x)xD_x)^*)$,

$$\begin{aligned} i[x^2D_x,H] &= i[x,H](xD_x) + xi[xD_x,H] = i[x,H]x^{-1}(x^2D_x) + xi[xD_x,H] \\ &= -(x^2D_x)x(x^2D_x) + x^{1/2}i[xD_x,H]x^{1/2} + K',\ K' \in \Psi^{-\infty,2}_{\mathrm{Sc}}(\bar{X},\mathcal{C}). \end{aligned}$$

Multiplying through by $\phi(H)$ from both the left and the right, the standard Mourre estimate,

$$\phi(H)i[A,H]\phi(H)) \geq 2(d(\lambda)-\epsilon)\phi(H)^2 + K', \tag{2.8}$$

$K' \in \Psi^{-\infty,1}_{\mathrm{Sc}}(\bar{X},\mathcal{C})$, proves the Theorem, since $x^{1/2}\phi(H)c\phi(H)x^{1/2} = \phi(H)cx\phi(H) + K''$, $K'' \in \Psi^{-\infty,2}_{\mathrm{Sc}}(\bar{X},\mathcal{C})$ as $[\phi(H),x^{1/2}] \in \Psi^{-\infty,3/2}_{\mathrm{Sc}}(\bar{X},\mathcal{C})$. As indicated above, we briefly recall the proof of the Mourre estimate in the appendix, since technically our setting is not covered e.g. by the proof of Froese and Herbst [4], even though their proof goes through without any significant changes. □

An equivalent, and for us more useful, version of this theorem is the following result.

Corollary 2.5. *Suppose $\lambda \in \mathbb{R}\setminus\Lambda$. For any $\epsilon > 0$, there exist $R \in x\,\Psi^{2,0}_{Sc}(\bar{X},\mathcal{C})$, $K \in x^2\,\Psi^{0,0}_{Sc}(\bar{X},\mathcal{C})$, such that*

$$i[B,H] \geq 2(d(\lambda)-\epsilon)x - 4BxB + (H-\lambda)R + R^*(H-\lambda) + K. \tag{2.9}$$

Remark 2.6. This corollary essentially states that the commutator $i[B,H]$ is positive, modulo BxB, on the 'characteristic variety', i.e. where $H-\lambda$ vanishes. Since this is a non-commutative setting (even to leading order), the vanishing on the characteristic variety has to be written by allowing error terms $(H-\lambda)R + R^*(H-\lambda)$: $[R,H-\lambda]$ has the same order as R!

Also, the inequality (2.9) is understood as a quadratic form inequality on $\dot{\mathcal{C}}^\infty(\bar{X})$.

Proof. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ be identically 1 near λ, supported in $(\lambda-\delta,\lambda+\delta)$, δ as above. Then

$$\begin{aligned} i[B,H] =& i\phi(H)[B,H]\phi(H) + i(\mathrm{Id}-\phi(H))[B,H]\phi(H) \\ &+ i\phi(H)[B,H](\mathrm{Id}-\phi(H)) + i(\mathrm{Id}-\phi(H))[B,H](\mathrm{Id}-\phi(H)), \end{aligned}$$

with a similar expansion for x. Since $\mathrm{Id}-\phi(H) = (H-\lambda)\tilde\phi(H)$, $\tilde\phi(t) = (t-\lambda)^{-1}(1-\phi(t))$, so $\tilde\phi \in S^{-1}(\mathbb{R})$ is a classical symbol, $\tilde\phi(H) \in \Psi_{\mathrm{Sc}}^{-2,0}(\bar X, \mathcal{C})$. Thus,

$$(\mathrm{Id}-\phi(H))[B,H]\phi(H) = (H-\lambda)R',\ R' \in \Psi_{\mathrm{Sc}}^{-\infty,1}(\bar X,\mathcal{C}),$$

etc., proving the corollary. $\square$

Remark 2.7. This Corollary is in fact equivalent to the statement of the theorem. For assuming (2.9), multiplying by $\phi_0(H)$ from the left and right, $\phi_0 \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ identically 1 near λ, replaces the term $(H-\lambda)R$ by $\phi_0(H)(H-\lambda)R\phi_0(H)$, and $\phi_0(H)(H-\lambda)$ has small norm, $\leq \delta_0$, if $\operatorname{supp}\phi_0 \subset (\lambda-\delta_0,\lambda+\delta_0)$. Multiplying from the left and right by $\phi(H)$ then gives Theorem 2.3, since these small terms can then be absorbed as $2(d(\lambda)-\epsilon-C\delta_0)\phi(H)x\phi(H)$.

3. Exponential decay

Using the preceeding lemma and the global positive commutator estimate, Theorem 2.3, we can now prove the exponential decay of non-threshold eigenfunctions. For this part of the paper, we could adapt the proof of Froese and Herbst [3] in Euclidean potential scattering, as was done in [18] in the geometric potential scattering setting. However, to unify the paper, we focus on the approach that will play a crucial role in the proof of unique continuation at infinity. Nonetheless, the Froese-Herbst commutator will play a role when $\alpha = 0$ (in the notation of Lemma 2.1), where conjugated Hamiltonian is close to being self-adjoint (in fact, it is, if $F = 0$), so we will use xD_x for a commutator estimate in place of $\operatorname{Im} P$.

Theorem 3.1. *Let $\lambda \in \mathbb{R}\setminus\Lambda$, and suppose that $\psi \in L^2_{sc}(\bar X)$ satisfies $H\psi = \lambda\psi$. Then $e^{\alpha/x}\psi \in L^2_{sc}(\bar X)$ for all $\alpha \in \mathbb{R}$ such that $[\lambda,\lambda+\alpha^2]\cap\Lambda = \emptyset$.*

Proof. The proof is by contradiction. First note that $\psi \in \dot{\mathcal{C}}^\infty(\bar X)$ by a result of [20] which only makes use of positive commutator estimates whose proof is unchanged in this greater generality. Let

$$\alpha_1 = \sup\{\alpha \in [0,\infty) :\ \exp(\alpha/x)\psi \in L^2_{\mathrm{sc}}(\bar X)\},$$

and suppose that $[\lambda,\lambda+\alpha_1^2]\cap\Lambda = \emptyset$. If $\alpha_1 = 0$, then let $\alpha = 0$, otherwise suppose that $\alpha < \alpha_1$, and $\alpha+\gamma > \alpha_1$. We show that for sufficiently small γ (depending only on α_1) $\exp((\alpha+\gamma)/x)\psi \in L^2_{\mathrm{sc}}(\bar X)$, which contradicts our assumption on α_1 if α is close enough to α_1. In what follows we assume that $\gamma \in (0,1]$.

Below we use two positivity estimates, namely (2.5) and the Mourre-type estimate, Corollary 2.5, at energy $\lambda+\alpha_1^2$, with $B = \chi(x)x^2D_x + (\chi(x)x^2D_x)^*$. That is, since $\lambda+\alpha_1^2 \notin \Lambda$, there exists $c_0 > 0$, $R \in \Psi_{\mathrm{Sc}}^{0,0}(\bar X,\mathcal{C})$, $K \in \Psi_{\mathrm{Sc}}^{2,0}(\bar X,\mathcal{C})$, such that for $\tilde\psi \in L^2_{\mathrm{sc}}(\bar X)$,

$$\begin{aligned} &(\tilde\psi, i[B,H]\tilde\psi) \\ &\geq c_0\|x^{1/2}\tilde\psi\|^2 - 4\operatorname{Re}(\tilde\psi, x(x^2D_x)^2\tilde\psi) \\ &\qquad + \operatorname{Re}((H-\lambda-\alpha_1^2)\tilde\psi, xR\tilde\psi) + \operatorname{Re}(x\tilde\psi, Kx\tilde\psi). \end{aligned} \tag{3.1}$$

We apply this below with $\tilde{\psi} = \psi_F$.

We first note that we certainly have for all $\beta \in \mathbb{R}$, $\exp(\alpha/x)x^\beta \psi \in L^2_{\mathrm{sc}}(\bar{X})$, due to our choice of α. We apply the Lemma 2.1 with

$$F = F_\beta = \frac{\alpha}{x} + \beta \log(1 + \frac{\gamma}{\beta x}),$$

and let $\psi_\beta = e^F \psi$. (Since x is bounded on $\bar{X}$, we may consider F compactly supported in $[0, x_0]$, $x_0 = \sup_{\bar{X}} x < 1/2$, as arranged for convenience in the preceeding section.) The reason for this choice is that on the one hand $F(x) \to (\alpha + \gamma)/x$ as $\beta \to \infty$, so in the limit we will obtain an estimate on $e^{(\alpha+\gamma)/x}\psi$, and on the other hand $F(x) \leq \frac{\alpha}{x} + \beta |\log x|$, so e^{F_β} is bounded by $x^\beta e^{\alpha/x}$, for all values of β, i.e. e^{F_β} provides a 'regularization' (in terms of growth) of $e^{(\alpha+\gamma)/x}$, so that Lemma 2.1 can be applied.

Note that $F = F_\beta \in S^1([0, 1))$, and F_β is uniformly bounded in $S^1([0, 1))$ for $\beta \in [1, \infty)$, $\alpha \in [0, \alpha_1)$ (or $\alpha = \alpha_1$ if $\alpha_1 = 0$), $\gamma \in [0, 1]$. Indeed,

$$0 \leq -x^2 \partial_x F = \alpha + \gamma (1 + \frac{\gamma}{\beta x})^{-1} \leq \alpha + \gamma,$$

and in general $(x\partial_x)^m (1 + \frac{\gamma}{\beta x})^{-1} = (-r\partial_r)^m (1 + r)^{-1}$, $r = \frac{\gamma}{\beta x}$, so the uniform boundedness of F follows from $(1 + r)^{-1}$ being a symbol in the usual sense on $[0, \infty)$. In particular, all symbol norms of $-x^2\partial_x F - \alpha$ are $\mathcal{O}(\gamma)$. Below, when $\alpha = 0$, we will need to consider $(-x^2\partial_x F)^{-1}(x^2\partial_x)^m(-x^2\partial_x F)$, $m \geq 0$. By Leibniz' rule, this can be written as $\sum_{j \leq m} c_j x^m (-x^2\partial_x F)^{-1}(x\partial_x)^j(-x^2\partial_x F)$. In terms of r, $(-x^2\partial_x F)^{-1}(x\partial_x)^j(-x^2\partial_x F)$ takes the form $(1 + r)(-r\partial_r)^m(1 + r)^{-1}$, hence it is still bounded on $[0, \infty)$, so in fact

$$x^{-m}(-x^2\partial_x F)^{-1}(x^2\partial_x)^m(-x^2\partial_x F), \quad m \geq 0, \tag{3.2}$$

is uniformly bounded on $[0, \infty)$. In fact, (3.2) is uniformly bounded in $S^0([0, 1))$, since applying $x\partial_x$ to it gives rise to additional factors such as

$$(-x^2\partial_x F)^{-k}(x\partial_x)^k(-x^2\partial_x F),$$

which are also uniformly bounded on $[0, \infty)$ by the same argument.

We remark first that $P(F)\psi_F = 0$, so by elliptic regularity,

$$\|\psi_F\|_{x^p H^k_{\mathrm{sc}}(\bar{X})} \leq b_{1,k,p}\|x^p \psi_F\|,$$

with $b_{1,k,p}$ independent of F as long as α is bounded. *In general, below b_j denote positive constants that are independent of α, β, γ in these intervals, and R_j denote operators which are uniformly bounded in* $\mathrm{Diff}^2_{Scc}(\bar{X}, \mathcal{C})$*, or on occasion in* $\Psi^{m,0}_{Scc}(\bar{X}, \mathcal{C})$*, for some m.* (Note that by elliptic regularity, the differential order never matters.)

The proof is slightly different in the cases $\alpha > 0$ and $\alpha = 0$ since in the latter case the usually dominating term, $-2\alpha x^2 D_x$, of $\operatorname{Im} P$ vanishes.

So assume first that $\alpha > 0$. The key step in the proof of this theorem arises from considering, with $P = P_\beta = H(F) - \lambda$,

$$P^*P = (\operatorname{Re} P)^2 + (\operatorname{Im} P)^2 + i(\operatorname{Re} P \operatorname{Im} P - \operatorname{Im} P \operatorname{Re} P),$$

so

$$0 = (\psi_F, P^*P\psi_F) = \|\operatorname{Re} P\psi_F\|^2 + \|\operatorname{Im} P\psi_F\|^2 + (\psi_F, i[\operatorname{Re} P, \operatorname{Im} P]\psi_F). \tag{3.3}$$

The first two terms on the right hand side are non-negative, so the key issue is the positivity of the commutator. Note that

$$\begin{aligned} \operatorname{Re} P &= H-\alpha^2-\lambda+\gamma R_1+xR_2,\\ \operatorname{Im} P &= -2\alpha x^2 D_x+\gamma R_3+xR_4. \end{aligned} \tag{3.4}$$

Below we use $H-\alpha^2-\lambda$ for a positive commutator estimate, local in the spectrum of H, in place of $\operatorname{Re} P$, to make the choice of the spectral cutoff ϕ independent of β and γ. (Otherwise we would need a uniform analogue of Theorem 2.3 for $\operatorname{Re} P$.) Thus, by (2.5),

$$\begin{aligned} i[\operatorname{Re} P,\operatorname{Im} P]=&2\alpha i[x^2D_x,H]+x\gamma R_5\\ &+xR_6\operatorname{Re} P+\operatorname{Re} PxR_7+xR_8\operatorname{Im} P+\operatorname{Im} PxR_9+x^2R_{10}, \end{aligned}$$

Hence, from (3.3) and (3.1),

$$\begin{aligned} 0\geq&\|\operatorname{Re} P\psi_F\|^2+\|\operatorname{Im} P\psi_F\|^2+2\alpha c_0\|x^{1/2}\psi_F\|^2+\gamma(\psi_F,xR_{11}\psi_F)\\ &+(\psi_F,xR_{12}\operatorname{Re} P\psi_F)+(\psi_F,\operatorname{Re} PxR_{13}\psi_F)\\ &+(\psi_F,xR_{14}\operatorname{Im} P\psi_F)+(\psi_F,\operatorname{Im} PxR_{15}\psi_F)+(\psi_F,x^2R_{16}\psi_F). \end{aligned} \tag{3.5}$$

Moreover, terms such as $|(\psi_F,x^2R_{16}\psi_F)|$ can be estimated by $b_2\|x\psi_F\|^2$, while $\gamma|(\psi_F,xR_{11}\psi_F)|$ may be estimated by $\gamma b_3\|x^{1/2}\psi_F\|^2$, and

$$\begin{aligned} |(\psi_F,xR_{12}\operatorname{Re} P\psi_F)|&\leq b_4\|x\psi_F\|\|\operatorname{Re} P\psi_F\|\leq b_4(\epsilon^{-1}\|x\psi_F\|^2+\epsilon\|\operatorname{Re}\psi_F\|^2)\\ |(\psi_F,xR_{14}\operatorname{Im} P\psi_F)|&\leq b_5\|x\psi_F\|\|\operatorname{Im} P\psi_F\|\leq b_5(\epsilon^{-1}\|x\psi_F\|^2+\epsilon\|\operatorname{Im}\psi_F\|^2) \end{aligned}$$

with similar estimates for the other terms. Putting this together, (3.5) yields

$$\begin{aligned} 0\geq&(1-b_6\epsilon)\|\operatorname{Re} P\psi_F\|^2+(1-b_7\epsilon)\|\operatorname{Im} P\psi_F\|^2\\ &+(2\alpha c_0-\gamma b_8)\|x^{1/2}\psi_F\|^2-b_9(\epsilon)\|x\psi_F\|^2. \end{aligned} \tag{3.6}$$

For $\delta>0$, in $x\geq\delta$, $x|\psi_F|=xe^F|\psi|\leq b_{10}(\delta)|\psi|$, so

$$\begin{aligned} \|x\psi_F\|^2&=\|x\psi_F\|^2_{x\leq\delta}+\|x\psi_F\|^2_{x\geq\delta}\\ &\leq\delta\|x^{1/2}\psi_F\|^2_{x\leq\delta}+b_{10}(\delta)\|\psi\|^2_{x\geq\delta}\\ &\leq\delta\|x^{1/2}\psi_F\|^2+b_{10}(\delta)\|\psi\|^2. \end{aligned}$$

Thus, (3.6) yields that

$$\begin{aligned} 0\geq&(1-b_6\epsilon)\|\operatorname{Re} P\psi_F\|^2+(1-b_7\epsilon)\|\operatorname{Im} P\psi_F\|^2\\ &+(2\alpha c_0-\gamma b_8-b_9(\epsilon)\delta)\|x^{1/2}\psi_F\|^2-b_{10}(\delta)\|\psi\|^2. \end{aligned} \tag{3.7}$$

Hence, choosing $\epsilon>0$ sufficiently small so that $b_6\epsilon<1$, $b_7\epsilon<1$, then choosing $\gamma_0>0$ sufficiently small so that $b_{11}=2\alpha c_0-\gamma_0 b_8>0$, we deduce that for $\gamma<\gamma_0$,

$$b_{10}(\delta)\|\psi\|^2\geq(b_{11}-b_9\delta)\|x^{1/2}\psi_F\|^2. \tag{3.8}$$

But, for $\delta\in(0,\frac{b_{11}}{b_9})$, this shows that $\|x^{1/2}\psi_F\|^2$ is uniformly bounded as $\beta\to\infty$. Noting that F is an increasing function of β and ψ_F converges to $e^{(\alpha+\gamma)/x}\psi$ pointwise, we deduce from the monotone convergence theorem that

$$x^{1/2}e^{(\alpha+\gamma)/x}\psi\in L^2_{\mathrm{sc}}(\bar X),$$

so for $\gamma'<\gamma$, $e^{(\alpha+\gamma')/x}\psi\in L^2_{\mathrm{sc}}(\bar X)$.

In case $\alpha = 0$, (3.3) still yields that, as $i[\operatorname{Re} P, \operatorname{Im} P] = \gamma x R_{17}$, with R_{17} uniformly bounded, that $\|\operatorname{Re} P\psi_F\| \leq b_{12}\|x^{1/2}\psi_F\|$, $\|\operatorname{Im} P\psi_F\| \leq b_{12}\|x^{1/2}\psi_F\|$. In particular, the former implies that

$$\|(H-\lambda)\psi_F\| \leq \gamma b_{13}\|\psi_F\| + b_{14}\|x^{1/2}\psi_F\|, \tag{3.9}$$

while the latter yields that

$$\|(x^2\partial_x F)x^2 D_x\psi_F\| \leq b_{15}\|x^{1/2}\psi_F\|. \tag{3.10}$$

However, instead of the degenerating commutator $[\operatorname{Re} P, \operatorname{Im} P]$, we consider $P^*A - AP$, with A as in Theorem 2.3, which is the expression considered by Froese and Herbst in [4]. Since A is xD_x, modulo lower order terms, and $\operatorname{Im} P$ is $2(x^2\partial_x F)(x^2 D_x)$, modulo lower order terms, A can be considered a rescaling of $\operatorname{Im} P$, in that the degenerating factor $x^2\partial_x F$ is removed. Now,

$$\begin{aligned} i(AP - P^*A) &= i[A, \operatorname{Re} P] - (\operatorname{Im} P A + A \operatorname{Im} P) \\ &= i[A, H-\lambda] + xR_{18} - 4Ax(x^2\partial_x F)A + R_{19}xA + AxR_{20} + \gamma R_{21}. \end{aligned}$$

Now, by (3.2) and Lemma 2.1, $R_{20} = (x^2\partial_x F)R'_{20} + xR''_{20}$ with R'_{20}, R''_{20} also bounded, and similarly for R_{19}. Thus,

$$\begin{aligned} 0 &= (\psi_F, i(AP - P^*A)\psi_F) \\ &\geq (\psi_F, i[A,H]\psi_F) + 4\|x^{1/2}(-x^2\partial_x F)^{1/2}A\psi_F\|^2 \\ &\qquad - b_{16}\|x^{1/2}\psi_F\|^2 - b_{17}\|\psi_F\|\|(x^2\partial_x F)x^2 D_x\psi_F\| - b_{18}\gamma\|\psi_F\|^2. \end{aligned}$$

Using the Mourre estimate (2.8), with $\phi(H)$ dropped but $H - \lambda$ inserted, as in Corollary 2.5, we deduce that (with $c'_0 > 0$)

$$\begin{aligned} 0 \geq c'_0\|\psi_F\|^2 + 4\|x^{1/2}(-x^2\partial_x F)^{1/2}A\psi_F\|^2 - b_{21}\|(H-\lambda)\psi_F\|\|\psi_F\| - b_{16}\|x^{1/2}\psi_F\|^2 \\ - b_{17}\|\psi_F\|\|(x^2\partial_x F)x^2 D_x\psi_F\| - b_{18}\gamma\|\psi_F\|^2. \end{aligned}$$

Using (3.9)-(3.10) we deduce, as above, that

$$\begin{aligned} 0 &\geq c'_0\|\psi_F\|^2 - \gamma b_{21}\|\psi_F\|^2 - b_{22}\|\psi_F\|\|x^{1/2}\psi_F\| - b_{23}\|x^{1/2}\psi_F\|^2 \\ &\geq (c'_0 - \gamma b_{21} - \epsilon_1 b_{22})\|\psi_F\|^2 - (b_{22}\epsilon_1^{-1} + b_{23})\|x^{1/2}\psi_F\|^2 \\ &\geq (c'_0 - \gamma b_{21} - \epsilon_1 b_{22} - (b_{22}\epsilon_1^{-1} + b_{23})\delta)\|\psi_F\|^2 - b_{22}\epsilon_1^{-1}\|\psi_F\|^2. \end{aligned}$$

Again, we fix first $\epsilon_1 > 0$ so that $c'_0 - \epsilon_1 b_{22} > 0$, then $\gamma_0 > 0$ so that $c'_0 - \gamma_0 b_{21} - \epsilon_1 b_{22} > 0$, finally $\delta > 0$ so that $c'_0 - \gamma_0 b_{21} - \epsilon_1 b_{22} - (b_{22}\epsilon_1^{-1} + b_{23})\delta > 0$. Now letting $\beta \to \infty$ gives that $e^{\gamma/x}\psi \in L^2_{\mathrm{sc}}(\bar{X})$ for $\gamma < \gamma_0$, as above. □

Having proved the exponential decay of non-threshold eigenfunctions, we can also prove that the thresholds are isolated from above inductively, using an observation of Perry [17]. This relies on the following uniform estimate.

Proposition 3.2. *Suppose that $\lambda_0 \notin \Lambda$, and let I be a compact interval with $\sup I < \lambda_0$. Then there exists $C > 0$ with the following property. If $H\psi = \lambda\psi$, $\lambda \in I$ and if $e^{\alpha'/x}\psi \in L^2_{sc}(\bar{X})$ for some $\alpha' > \sqrt{\lambda_0 - \lambda}$ then $\|x^{1/2}e^{\alpha/x}\psi\| \leq C\|\psi\|$ for $\alpha = \sqrt{\lambda_0 - \lambda}$.*

Proof. The proof is very close to that of the preceeding theorem. First, we may use $F = \alpha/x$ directly, i.e. take $\gamma = 0$. Again, all constants are uniform in α and ψ, provided that α is bounded. Thus, (3.8) yields that

$$b_{10}(\delta)\|\psi\|^2 \geq (b_{11} - b_9\delta)\|x^{1/2}\psi_F\|^2. \tag{3.11}$$

So taking $\delta \in (0, \frac{b_{11}}{b_9})$ shows that

$$\|x^{1/2}e^{\alpha/x}\psi\|^2 \leq b'_{12}\|\psi\|^2,$$

which proves the proposition. $\square$

We introduce the following terminology. If H is a many-body Hamiltonian, we say that the thresholds $\lambda \in \Lambda$ of H are isolated from above if

$$\lambda \in \Lambda \Rightarrow \exists \lambda' > \lambda \text{ such that } (\lambda, \lambda') \cap (\Lambda \cup \operatorname{spec}_{pp}(H)) = \emptyset.$$

Theorem 3.3. *Let Λ be the set of thresholds of H, and suppose that $\lambda \in \Lambda$. Then λ is isolated from above in $\Lambda \cup \operatorname{spec}_{pp}(H)$, i.e. there exists $\lambda' > \lambda$ such that $(\lambda, \lambda') \cap (\Lambda \cup \operatorname{spec}_{pp}(H)) = \emptyset$.*

Proof. Note that the statement of the theorem is certainly true for H_0, since $\Lambda_0 \cup \operatorname{spec}_{pp}(H_0) = \{0\}$. We prove inductively that if in all proper subsystems H_b of H_a, the thresholds are isolated from above, then the same holds for H_a. Assuming the inductive hypothesis, and recalling that

$$\Lambda_a = \cup_{X^b \subsetneq X^a}(\Lambda_b \cup \operatorname{spec}_{pp}(H^b)),$$

with the union being finite, we deduce that for any $\lambda \in \Lambda_a$ there exists $\lambda' > \lambda$ such that $(\lambda, \lambda') \cap \Lambda_a = \emptyset$. So we only need to show that $\operatorname{spec}_{pp}(H) \cap (\lambda, \frac{\lambda+\lambda'}{2})$ is finite.

Suppose otherwise, and let $\psi^{(j)}$ be an orthonormal sequence of eigenfunctions with eigenvalue $\lambda_j \in (\lambda, \frac{\lambda+\lambda'}{2})$, $j \geq 1$. Since by the Mourre estimate eigenvalues may only accumulate at thresholds, $\lim_{j\to\infty} \lambda_j = \lambda$. In particular, dropping ψ_j for a finite number of j, we may assume that $\lambda_j < \frac{7\lambda+\lambda'}{8}$ for all j. Let $\lambda_0 = \frac{3\lambda+\lambda'}{4}$. By Theorem 3.1, for all j, $\psi_j \in e^{-\gamma/x}L^2_{\mathrm{sc}}(\bar{X})$ for $\gamma < \sqrt{\frac{\lambda'-\lambda}{2}}$. Note that this holds in particular for some $\gamma > \sqrt{\lambda_0 - \lambda} > \sqrt{\lambda_0 - \lambda_j}$ for all j. Hence, by Proposition 3.2, with $\alpha_j = \sqrt{\lambda_0 - \lambda_j} > \frac{1}{4}\sqrt{\lambda' - \lambda} = \alpha_0$, there exists $C > 0$ such that $\|x^{1/2}e^{\alpha_j/x}\psi_j\|_{H^2_{\mathrm{sc}}(\bar{X})} \leq C$ for all j, hence $\|e^{\alpha_0/x}\psi_j\|_{H^2_{\mathrm{sc}}(\bar{X})} \leq C'$ for all j. But the inclusion $e^{-\alpha_0/x}H^2_{\mathrm{sc}}(\bar{X}) \hookrightarrow L^2_{\mathrm{sc}}(\bar{X})$ is compact, so ψ_j has a subsequence that converges in $L^2_{\mathrm{sc}}(\bar{X})$, which contradicts the orthogonality of the ψ_j. This completes the inductive step, proving the theorem. $\square$

4. Absence of positive eigenvalues – high energy estimates

We next prove that faster than exponential decay of an eigenfunction of H implies that it vanishes. This was also the approach taken by Froese and Herbst. However, we use a different, more robust, approach to deal with our much larger error terms. The proof is based on conjugation by $\exp(\alpha/x)$ and letting $\alpha \to +\infty$. Correspondingly, we require positive commutator estimates at high energies. In such a setting first order terms are irrelevant, i.e. V does not play a significant role below. On the other hand, $\Delta_g - \Delta_{g_0}$ is not negligible in any sense. However, we show that if g and g_0 are close in a $\mathcal{C}^\infty$ sense (keeping in mind that we are assuming that g has a special structure), then the corresponding unique continuation theorem is still true. Our argument also shows the very close connection with Hörmander's solvability condition. Indeed, we work semiclassically (writing $h = \alpha^{-1}$), and the key fact we use is that the commutator of the real and imaginary parts of the conjugated Hamiltonian has the correct sign on its 'non-commutative characteristic variety'.

Theorem 4.1. *Let $\lambda \in \mathbb{R}$ and let d denote a metric giving the usual topology on $\mathcal{C}^\infty$ sections of ${}^{Sc}T^*[\bar{X};\mathcal{C}] \otimes {}^{Sc}T^*[\bar{X};\mathcal{C}]$. There exists $\epsilon > 0$ such that if $d(g, g_0) < \epsilon$ and $H\psi = \lambda\psi$, $\exp(\alpha/x)\psi \in L^2_{sc}(\bar{X})$ for all α, then $\psi = 0$.*

Proof. Let $F = F_\alpha = \phi(x)\frac{\alpha}{x}$ where $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ is supported near 0, identically 1 in a smaller neighborhood of 0, and let $\psi_F = e^F\psi$. Then with $h = \alpha^{-1}$, $H_h = h^2 H(F)$ and $P_h = H_h - h^2\lambda$ are elliptic semiclassical differential operators, elliptic in the usual sense (differentiability), and

$$P_h\psi_h = 0,\ \psi_h = \psi_F,$$

so by elliptic regularity,

$$\|\psi_h\|_{x^p H^k_{\mathrm{sc},h}(\bar{X})} \leq C_1 \|\psi_h\|_{x^p L^2_{\mathrm{sc}}(\bar{X})}, \tag{4.1}$$

C_1 independent of $h \in (0, 1]$ (but depends on k and p). In general, below the C_j denote constants independent of $h \in (0, 1]$ (and $\delta > 0$).

The key step in the proof of this theorem arises from considering

$$P_h^* P_h = (\operatorname{Re} P_h)^2 + (\operatorname{Im} P_h)^2 + i(\operatorname{Re} P_h \operatorname{Im} P_h - \operatorname{Im} P_h \operatorname{Re} P_h)$$

so

$$0 = (\psi_h, P_h^* P_h\psi_h) = \|\operatorname{Re} P_h\psi_h\|^2 + \|\operatorname{Im} P_h\psi_h\|^2 + (\psi_h, i[\operatorname{Re} P_h, \operatorname{Im} P_h]\psi_h). \tag{4.2}$$

The first two terms on the right hand side are non-negative, so the key issue is the positivity of the commutator. More precisely, we need that there exist operators R_j bounded in $\mathrm{Diff}^{2,0}_{\mathrm{Sc},h}(\bar{X}, \mathcal{C})$ such that

$$\begin{aligned}
&(\psi_h, i[\operatorname{Re} P_h, \operatorname{Im} P_h]\psi_h)\\
&\quad\geq (\psi_h, (xh + \operatorname{Re} P_h xhR_1 + xhR_2 \operatorname{Re} P_h\\
&\qquad + \operatorname{Im} P_h xhR_3 + xhR_4 \operatorname{Im} P_h + xh^2 R_5 + x^2 hR_6)\psi_h).
\end{aligned} \tag{4.3}$$

The important point is that replacing both $\operatorname{Re} P_h$ and $\operatorname{Im} P_h$ by zero, the commutator is estimated from below by a positive multiple of xh, plus terms $O(xh^2)$ and $O(x^2 h)$.

We first prove (4.3), and then show how to use it to prove the theorem. First, modulo terms that will give contributions that are in the error terms, $\operatorname{Re} P_h$ may be replaced by $h^2\Delta_g - 1$, while $\operatorname{Im} P_h$ may be replaced by $-2h(x^2 D_x)$. Now, by a principal symbol calculation (which also gives the 'trivial case' of Theorem 2.3),

$$i[h^2\Delta_{g_0} - 1, -2h(x^2 D_x)] = xh(4h^2\Delta_{g_0} - 4h^2(x^2 D_x)^2 + R_7),\ R_7 \in x\,\mathrm{Diff}^2_{\mathrm{sc},h}(\bar{X}).$$

The key point here is the microlocal positivity of the commutator where $h^2\Delta_{g_0} - 1$ and $-2h(x^2 D_x)$ both vanish. Now, taking the commutator with $hx^2 D_x$ is continuous from $\mathrm{Diff}^{2,0}_{\mathrm{Sc},h}(\bar{X}, \mathcal{C})$ to $xh\,\mathrm{Diff}^{2,0}_{\mathrm{Sc},h}(\bar{X}, \mathcal{C})$, so

$$i[h^2\Delta_g - 1, -2h(x^2 D_x)] = xh(4h^2\Delta_g + R_8 - 4h^2(x^2 D_x)^2 + R_7),$$

$R_7 \in x\,\mathrm{Diff}^2_{\mathrm{sc},h}(\bar{X})$, $R_8 \in \mathrm{Diff}^2_{\mathrm{sc},h}(\bar{X})$, and

$$\|R_8\|_{\mathcal{B}(x^{1/2}H^2_{\mathrm{sc},h}(\bar{X}), x^{1/2}L^2_{\mathrm{sc}}(\bar{X}))} \leq \rho(d(g, g_0)),$$

with ρ continuous, $\rho(0) = 0$. Since

$$\|x^{1/2}R_8\psi_h\| \leq \|R_8\|_{\mathcal{B}(x^{1/2}H^2_{\mathrm{sc},h}(\bar{X}), x^{1/2}L^2_{\mathrm{sc}}(\bar{X}))} \|\psi_h\|_{x^{1/2}H^2_{\mathrm{sc},h}(\bar{X})},$$

we deduce from (4.1) that

$$|(\psi_h, hxR_8\psi_h)| \leq h\|x^{1/2}\psi_h\|\|x^{1/2}R_8\psi_h\| \leq C_1 h\rho(d(g,g_0))\|x^{1/2}\psi_h\|^2$$

This proves (4.3) if $C_1\rho(d(g,g_0)) < 3$, hence if g is close to g_0.

We now show how to use (4.3) to show unique continuation at infinity. Let $x_0 = \sup_{\bar{X}} x$. We first remark that

$$|(\psi_h, xhR_2 \operatorname{Re} P_h\psi_h)| \leq C_2 h\|x\psi_h\|\|\operatorname{Re} P_h\psi_h\| \leq C_2 h\|x\psi_h\|^2 + C_2 h\|\operatorname{Re} P_h\psi_h\|^2,$$
$$|(\psi_h, xhR_4 \operatorname{Im} P_h\psi_h)| \leq C_3 h\|x\psi_h\|\|\operatorname{Im} P_h\psi_h\| \leq C_3 h\|x\psi_h\|^2 + C_3 h\|\operatorname{Im} P_h\psi_h\|^2,$$

with similar expressions for the R_1 and R_3 terms in (4.3). Next,

$$|(\psi_h, xh^2R_5\psi_h)| \leq C_4 h^2\|x^{1/2}\psi_h\|^2$$
$$|(\psi_h, x^2hR_6\psi_h)| \leq C_5 h\|x\psi_h\|^2.$$

For $\delta > 0$, in $x \geq \delta$, $|\psi_h| = e^{1/xh}|\psi| \leq e^{1/(\delta h)}|\psi|$, so

$$\begin{aligned}\|x\psi_h\|^2 &= \|x\psi_h\|^2_{x\leq\delta} + \|x\psi_h\|^2_{x\geq\delta}\\ &\leq \delta\|x^{1/2}\psi_h\|^2_{x\leq\delta} + x_0^2 e^{2/(\delta h)}\|\psi\|^2_{x\geq\delta}\\ &\leq \delta\|x^{1/2}\psi_h\|^2 + x_0^2 e^{2/(\delta h)}\|\psi\|^2.\end{aligned}$$

Thus,

$$\|(\psi_h, x^2hR_6\psi_h)| \leq C_5 h\delta\|x^{1/2}\psi_h\|^2 + C_5 x_0^2 h e^{2/(\delta h)}\|\psi\|^2.$$

Hence, we deduce from (4.2)-(4.3) that

$$\begin{aligned}0 \geq (1 - C_6 h)\|\operatorname{Re} P_h\psi_h\|^2 + (1 - C_7 h)\|\operatorname{Im} P_h\psi_h\|^2 + h(1 - C_8 h - C_9\delta)\|x^{1/2}\psi_h\|^2\\ - C_{10} h e^{2/(\delta h)}\|\psi\|^2.\end{aligned}$$

Hence, there exists $h_0 > 0$ such that for $h \in (0, h_0)$,

$$C_{10} h e^{2/(\delta h)}\|\psi\|^2 \geq h(\frac{1}{2} - C_9\delta)\|x^{1/2}\psi_h\|^2 \tag{4.4}$$

Now suppose that $\delta \in (0, \min(\frac{1}{4C_9}, \frac{1}{h_0}))$ and $\operatorname{supp}\psi \cap \{x \leq \frac{\delta}{4}\}$ is non-empty. Since $e^{2/xh} \geq e^{4/\delta h}$ for $x \leq \frac{\delta}{2}$, we deduce that

$$\|x^{1/2}\psi_h\|^2 \geq C_{11} e^{4/(\delta h)},\ C_{11} = \|x^{1/2}\psi\|^2_{x\geq\delta/2} > 0.$$

Thus, we conclude from (4.4) that

$$C_{10}\|\psi\|^2 \geq (\frac{1}{2} - C_9\delta) C_{11} e^{2/(\delta h)}.$$

But letting $h \to 0$, the right hand side goes to $+\infty$, providing a contradiction.

Thus, ψ vanishes for $x \leq \delta/4$, hence vanishes identically on $\bar{X}$ by the usual Carleman-type unique continuation theorem [8, Theorem 17.2.1]. □

Appendix A. Proof of the Mourre estimate, (2.8)

In this section we recall briefly how the Mourre estimate, (2.8), is proved, relying on a now standard iterative argument for the indicial operators that originated in this form in [4]; see also [1]. Namely, to prove (2.8), one only needs to show that for all b, the corresponding indicial operators satisfy the corresponding inequality, i.e. that

$$\phi(\hat{H}_b) i\widehat{[A,H]}_b \phi(\hat{H}_b) \geq 2(d(\lambda) - \epsilon)\phi(\hat{H}_b)^2. \tag{A.1}$$

(This means that the operators on the two sides, which are families of operators on X^b, depending on $(y_b,\xi_b)\in {}^{\mathrm{sc}}T^*_{C_b}\bar X$, satisfy the inequality for all $(y_b,\xi_b)\in {}^{\mathrm{sc}}T^*_{C_b}\bar X$.) It is convenient to assume that ϕ is identically 1 near λ; if (A.1) holds for such ϕ, it holds for any ϕ_0 with slightly smaller support, as follows by multiplication by $\phi_0(\hat H_b)$ from the left and right.

Note that for $b=0$ the estimate certainly holds: it comes from the Poisson bracket formula in the scattering calculus, or from a direct computation yielding $i\widehat{[A,H]}_0=2\Delta_{g_0}$. Hence, if the the localizing factor $\phi(\hat H_0)=\phi(|\xi|^2)$ is supported in $(\lambda-\delta,\lambda+\delta)$ and $\lambda>0$, then (A.1) holds even with $d(\lambda)-\epsilon$ replaced with $\lambda-\delta$. Note that $\lambda\geq d(\lambda)$, if $\lambda>0$, since 0 is a threshold of H. On the other hand, if $\lambda<0$, both sides of (A.1) vanish for ϕ supported near λ, so the inequality holds trivially.

In general, we may assume inductively that at all clusters c with $C_c\subsetneq C_b$, i.e. $X^b\subsetneq X^c$, (A.1) has been proved with ϕ replaced by a cutoff $\tilde\phi$ and ϵ replaced by ϵ', i.e. we may assume that for all $\epsilon'>0$ there exists $\delta'>0$ such that for all c with $C_c\subsetneq C_b$, and for all $\tilde\phi\in\mathcal{C}^\infty_c(\mathbb{R};[0,1])$ supported in $\lambda-\delta,\lambda+\delta)$,

$$\tilde\phi(\hat H_c)i\widehat{[A,H]}_c\tilde\phi(\hat H_c)\geq 2(d(\lambda)-\epsilon)\tilde\phi(\hat H_c)^2. \tag{A.2}$$

But these are exactly the indicial operators of $\tilde\phi(\hat H_b)i\widehat{[A,H]}_b\tilde\phi(\hat H_b)$, so, as discussed in [19, Proposition 8.2], (A.1) implies that

$$\tilde\phi(\hat H_b)i\widehat{[A,H]}_b\tilde\phi(\hat H_b)\geq 2(d(\lambda)-\epsilon')\tilde\phi(\hat H_b)^2+K_b,\ K_b\in\Psi^{-\infty,1}_{\mathrm{Sc}}(X^b,\mathcal{C}^b). \tag{A.3}$$

Recall that this implication relies on a square root construction in the many-body calculus, which is particularly simple in this case.

Now, we first multiply (A.3) through by $\phi(H)$ from both the left and the right. Recall that we use coordinates (w_b,w^b) on $X_b\oplus X^b$ and (ξ_b,ξ^b) are the dual coordinates. We remark that $\hat H_b=|\xi_b|^2+H^b$, so if $\lambda-|\xi_b|^2$ is not an eigenvalue of H^b, then as $\operatorname{supp}\phi\to\{\lambda\}$, $\phi(H^b+|\xi_b|^2)\to 0$ strongly, so as K_b is compact, $\phi(H^b+|\xi_b|^2)K_b\to 0$ in norm; in particular it can be made to have norm smaller than $\epsilon'-\epsilon>0$. After multiplication from both sides by $\phi_1(\hat H_b)$, with ϕ_1 having even smaller support, (A.1) follows (with ϕ_1 in place of ϕ), with the size of $\operatorname{supp}\phi_1$ a priori depending on ξ_b. However, $i\phi_1(\hat H_b)\widehat{[A,H]}_b\phi_1(\hat H_b)$ is continuous in ξ_b with values in bounded operators on $L^2(X^b)$, so if (A.1) holds at one value of ξ_b, then it holds nearby. Moreover, for large $|\xi_b|$ both sides vanish as $\hat H_b=H^b+|\xi_b|^2$, with H^b bounded below, so the estimate is in fact uniform if we slightly increase $\epsilon>0$.

In general, if E denotes the projection on the L^2 eigenspace of H^b at $\lambda-|\xi_b|^2$, the argument just sketched works if we can replace $\phi(\hat H_b)$ by $(\mathrm{Id}-E)\phi(\hat H_b)$, in particular, it suffices to show that

$$i\tilde\phi(\hat H_b)\widehat{[A,H]}_b\tilde\phi(\hat H_b)\geq 2(d(\lambda)-\epsilon')\tilde\phi(\hat H_b)^2+(\mathrm{Id}-E)K_b'(\mathrm{Id}-E), \tag{A.4}$$

K_b' compact on $L^2(X^b)$.

To show (A.4), we follow an argument due to B. Simon (as explained in the paper [4] of Froese and Herbst). The key point is to replace E by a finite rank orthogonal projection F, which will later ensure that an error term is finite rank, hence compact. Thus, by the compactness of K_b (from (A.3)), there is a finite rank orthogonal projection F with $\operatorname{Ran}F\subset\operatorname{Ran}E$ (so F commutes with $\hat H_b$) such that

$$\|(\mathrm{Id}-E)K_b(\mathrm{Id}-E)-(\mathrm{Id}-F)K_b(\mathrm{Id}-F)\|<\epsilon'. \tag{A.5}$$

Multiplying (A.3) through by $(\mathrm{Id}-F)\phi(\hat{H}_b)=\phi(\hat{H}_b)-F$ from left and right and using (A.5) gives

$$\begin{aligned} i(\phi(\hat{H}_b)-F)\widehat{[A,H]}_b(\phi(\hat{H}_b)-F)\\ \geq 2(d(\lambda)-2\epsilon')(\phi(\hat{H}_b)^2-F)+\phi(\hat{H}_b)(\mathrm{Id}-E)K_b(\mathrm{Id}-E)\phi(\hat{H}_b). \end{aligned}\tag{A.6}$$

Moving the terms involving F from the left hand side to the right hand side yields that

$$\begin{aligned} i\phi(\hat{H}_b)\widehat{[A,H]}_b\phi(\hat{H}_b) \geq {} & 2(d(\lambda)-2\epsilon')(\phi(\hat{H}_b)^2-F)\\ & + F\phi(\hat{H}_b)i\widehat{[A,H]}_b(\phi(\hat{H}_b)-F)\\ & + (\phi(\hat{H}_b)-F)i\widehat{[A,H]}_b\phi(\hat{H}_b)F+FTF\\ & + \phi(\hat{H}_b)(\mathrm{Id}-E)K_b(\mathrm{Id}-E)\phi(\hat{H}_b). \end{aligned}\tag{A.7}$$

where $T=Ei\widehat{[A,H]}_bE$. But

$$\begin{aligned} &F\phi(\hat{H}_b)i\widehat{[A,H]}_b(\phi(\hat{H}_b)-F)=FC+F\phi(\hat{H}_b)i\widehat{[A,H]}_b(E-F),\\ &C=F\phi(\hat{H}_b)i\widehat{[A,H]}_b\phi(\hat{H}_b)(\mathrm{Id}-E), \end{aligned}$$

and for any $\epsilon_1>0$,

$$FC+C^*F\geq -\epsilon_1F-\epsilon_1^{-1}C^*C,$$

so (A.7) yields that

$$\begin{aligned} i\phi(\hat{H}_b)\widehat{[A,H]}_b\phi(\hat{H}_b) \geq {} & 2(d(\lambda)-2\epsilon')\phi(\hat{H}_b)^2\\ & -2(d(\lambda)-2\epsilon'+\epsilon_1/2)F\\ & +FT(E-F)+(E-F)TF+FTF\\ & +\phi(\hat{H}_b)(\mathrm{Id}-E)(K_b+K_b'')(\mathrm{Id}-E)\phi(\hat{H}_b), \end{aligned}\tag{A.8}$$

with $K_b''=-\epsilon_1^{-1}\widehat{[A,H]}_bF\phi(\hat{H}_b)^2\widehat{[A,H]}_b$, which is compact due to the appearance of F. Now, $A=xD_x+A'=w_b\cdot D_{w_b}+w^b\cdot D_{w^b}+A'$, $A'\in x\,\mathrm{Diff}^1_{\mathrm{Sc}}(\bar{X},\mathcal{C})$, hence, similarly to (2.7),

$$\begin{aligned} T=Ei\widehat{[A,H]}_bE = {} & Ei\widehat{[w_bD_{w_b},H]}_bE+Ei\widehat{[w^bD_{w^b},H]}_bE\\ & +E\widehat{A'}_b(\hat{H}_b-\lambda)E-E(\hat{H}_b-\lambda)\widehat{A'}_bE=Ei\widehat{[w_bD_{w_b},H]}_bE. \end{aligned}$$

since by the virial theorem $iE[w^bD_{w^b},H^b]E=0$ (see the remark below). Thus,

$$T=iE\widehat{[w_bD_{w_b},\Delta_{X_b}]}E\geq 2|\xi_b|^2E=2(\lambda-\lambda')E\geq 2d(\lambda)E;$$

here $\lambda'=\lambda-|\xi_b|^2$ is the eigenvalue of H^b to which E projects, and $\lambda-\lambda'\geq d(\lambda)$ since λ' is a threshold of H by definition. Note that $d(\lambda)$ enters the estimate at this point (i.e. this is the constant we need to use, rather than λ, which is the corresponding constant in the free region). Thus,

$$-2(d(\lambda)-2\epsilon'+\epsilon_1/2)F+FT(E-F)+(E-F)TF+FTF\geq(4\epsilon'-\epsilon_1)F\geq 0$$

if we choose $\epsilon_1<4\epsilon'$. Thus,

$$\begin{aligned} i\phi(\hat{H}_b)\widehat{[A,H]}_b\phi(\hat{H}_b)\geq {} & 2(d(\lambda)-2\epsilon')\phi(\hat{H}_b)^2\\ & +\phi(\hat{H}_b)(\mathrm{Id}-E)(K_b+K_b'')(\mathrm{Id}-E)\phi(\hat{H}_b), \end{aligned}\tag{A.9}$$

which proves (A.4). Hence the proof of (2.8) is complete.

Remark A.1. The statement $iE[w^bD_{w^b}, H^b]E = 0$ is formally a consequence of

$$iE[w^bD_{w^b}, H^b]E = iEw^bD_{w^b}(H^b - \lambda - |\xi_b|^2)E - iE(H^b - \lambda - |\xi_b|^2)w^bD_{w^b}E = 0,$$

but this requires justification since $w^bD_{w^b}$ is not bounded on $L^2(X^b)$. In fact, by elliptic regularity (namely using $E = E\phi(\hat{H}_b)$), the only issue is the lack of decay of $w^bD_{w^b}$ at infinity, but the computation is justified by replacing $w^bD_{w^b}$ by $\chi_0(|w^b|/C)w^bD_{w^b}$, $\chi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ identicaly 1 near 0, and observing that

$$\phi(\hat{H}_b)[\chi_0(w^b/C), H]w^bD_{w^b}$$

is uniformly bounded and goes to 0 strongly as $C \to \infty$.

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