

**ANALYTIC CONTINUATION AND HIGH ENERGY ESTIMATES
FOR THE RESOLVENT OF THE LAPLACIAN ON FORMS ON
ASYMPTOTICALLY HYPERBOLIC SPACES**

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ABSTRACT. We prove the analytic continuation of the resolvent of the Laplacian on asymptotically hyperbolic spaces on differential forms, including high energy estimates in strips. This is achieved by placing the spectral family of the Laplacian within the framework developed, and applied to scalar problems, by the author recently, roughly by extending the problem across the boundary of the compactification of the asymptotically hyperbolic space in a suitable manner. The main novelty is that the non-scalar nature of the operator is dealt with by relating it to a problem on an asymptotically Minkowski space to motivate the choice of the extension across the conformal boundary.

1. INTRODUCTION

Suppose that (X, g) is an n -dimensional asymptotically hyperbolic space with an even metric in the sense of Guillarmou [12]. That is, g is Riemannian on X , X has a compactification \bar{X} with boundary defining function x , and there is a neighborhood $U = [0, \epsilon)_x \times \partial X$ of ∂X on which g is of the warped product form $\frac{dx^2+h}{x^2}$, with $h = h(x, \cdot)$ a smooth family of symmetric 2-cotensors on ∂X whose Taylor series at $x = 0$ is even, and $h(0, \cdot)$ is positive definite. We refer to [12] for a more geometric version, and to Graham and Lee [11, Section 5] for how to put an arbitrary asymptotically hyperbolic metric, i.e. one for which x^2g is Riemannian on \bar{X} and $|dx|_{x^2g} = 1$ at $x = 0$, into a warped product form. We write \bar{X}_{even} for \bar{X} equipped with the even smooth structure, i.e. using coordinate charts $[0, \epsilon^2)_\mu \times O$, O a coordinate chart in ∂X , in the product decomposition above, where $\mu = x^2$. (So a C^∞ function on \bar{X} is in $C^\infty(\bar{X}_{\text{even}})$ if and only if its Taylor series has only even terms at $x = 0$.)

Let Δ_k denote the Laplacian on k -forms on the complete Riemannian manifold (X, g) . Thus, Δ_k with domain $C_c^\infty(X; \Lambda^k X)$ is essentially self-adjoint, and is indeed non-negative, so in particular $(\Delta_k - \lambda)^{-1}$ exists for $\lambda \in \mathbb{C} \setminus [0, \infty)$. We show that

Theorem 1.1. *The operators*

$$\delta d(\Delta_k - \sigma^2 - (n - 2k - 1)^2/4)^{-1}, \quad d\delta(\Delta_k - \sigma^2 - (n - 2k + 1)^2/4)^{-1}$$

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have a meromorphic continuation from $\text{Im } \sigma \gg 1$ to \mathbb{C} with finite rank poles and with non-trapping, resp. mildly trapping, high energy estimates in strips $|\text{Im } \sigma| < C$ if g is a non-trapping, resp. mildly trapping, metric.

Here recall that g non-trapping means that all geodesics approach ∂X as the time parameter goes to $\pm\infty$, while mildly trapping, defined in [20, Section 2], is an analytic assumption on a model problem near the trapping (roughly polynomial bounds for the model resolvent) and the nearby bicharacteristic flow; we recall this briefly at the end of Section 4. Non-trapping high-energy estimates mean that for all $C_0 > 0$ and s with $s + 3/2 > C_0$ there is $C > 0$ and $R > 0$ such that

$$(1.1) \quad \begin{aligned} & \|\delta d(\Delta_k - \sigma^2 - (n - 2k - 1)^2/4)^{-1}\|_{\mathcal{L}(\mathcal{Y}_{\delta d}^{s+1}, \mathcal{X}_{\delta d}^s)} \leq C|\sigma|, \\ & \|d\delta(\Delta_k - \sigma^2 - (n - 2k + 1)^2/4)^{-1}\|_{\mathcal{L}(\mathcal{Y}_{d\delta}^{s+1}, \mathcal{X}_{d\delta}^s)} \leq C|\sigma|, \\ & |\text{Im } \sigma| < C_0, \quad |\text{Re } \sigma| > R, \end{aligned}$$

where the norms are on suitable (high-energy) Sobolev spaces, namely

$$\begin{aligned} \mathcal{X}_{\delta d}^s &= x^{-i\sigma + (n-2k-1)/2} H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}}), \\ \mathcal{Y}_{\delta d}^{s+1} &= x^{-i\sigma + (n-2k-1)/2 + 2} H_{|\sigma|^{-1}}^{s+1}(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}}), \\ \mathcal{X}_{d\delta}^s &= \{u \in \mathcal{C}^{-\infty}(X) : x^{i\sigma - (n-2k-3)/2} u \in H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}}), \\ & \quad x^{i\sigma - (n-2k-3)/2 - 2} d\mu \wedge u \in H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}})\} \\ & \subset x^{-i\sigma + (n-2k-3)/2} H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}}), \\ \mathcal{Y}_{d\delta}^{s+1} &= \{f \in \mathcal{C}^{-\infty}(X) : x^{i\sigma - (n-2k-3)/2 - 2} f \in H_{|\sigma|^{-1}}^{s+1}(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}}), \\ & \quad x^{i\sigma - (n-2k-3)/2 - 4} d\mu \wedge f \in H_{|\sigma|^{-1}}^{s+1}(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}})\} \\ & \supset x^{-i\sigma + (n-2k-3)/2 + 4} H_{|\sigma|^{-1}}^{s+1}(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}}). \end{aligned}$$

Here the power of $|\sigma|$ on the right hand side of (1.1) is 1 rather than -1 due to the presence of $d\delta$ and δd on the left hand side which are $|\sigma|^2$ times second order semiclassical differential operators, as recalled below. Mildly trapping estimates mean that $|\sigma|$ is replaced by $|\sigma|^{\varkappa+1}$ for a $\varkappa > 0$ arising from the polynomial models on the trapped model. Notice that as $\text{Re } \sigma$ is assumed sufficiently large, the thresholds $(n - 2k \pm 1)^2/4$ are irrelevant in these estimates. Recall also briefly that on a compact manifold, possibly with boundary, the semiclassical Sobolev spaces are L^2 -based Sobolev spaces in which each derivative is weighted with $|\sigma|^{-1}$, $|\sigma| \geq 1$. In particular $|\sigma|^{-2}\delta d$, $|\sigma|^{-2}d\delta$ are second order semiclassical operators.

Denoting the Hodge star operator on X by $*$, and adding a subscript to the form spaces to denote the form degree, it is straightforward to check that $*$: $\mathcal{X}_{\delta d, k}^s \rightarrow \mathcal{X}_{d\delta, n-k}^s$ and $*$: $\mathcal{Y}_{\delta d, k}^{s+1} \rightarrow \mathcal{Y}_{d\delta, n-k}^{s+1}$ are isomorphisms, so the estimates corresponding to coexact and exact forms indeed match up. Note that under the mapping $k \mapsto n - k$, the threshold $(n - 2k + 1)^2/4$ becomes $(n - 2k - 1)^2/4$.

We also mention that when one only wants to estimate the operators in Theorem 1.1 away from ∂X , one can use semiclassical elliptic regularity to make the differential order of the domain and target spaces equal. There is a real loss at ∂X in terms of standard $\overline{X}_{\text{even}}$ -derivatives since the operator which plays a crucial role in our analysis, on an extended space \tilde{X} , ceases to be elliptic there.

Noting that

$$\lambda(\Delta_k - \lambda)^{-1} = -\text{Id} + \delta d(\Delta_k - \lambda)^{-1} + d\delta(\Delta_k - \lambda)^{-1},$$

and noting that strips

$$|\text{Im} \sqrt{\lambda - (n - 2k \pm 1)^2/4}| < C$$

are comparable (i.e. are contained within each other up to changing C by an arbitrarily small amount) as $|\text{Re} \sqrt{\lambda - (n - 2k \pm 1)^2/4}| \rightarrow +\infty$, we deduce that

Corollary 1.2. *Let Σ be the Riemann surface of the functions*

$$\lambda \mapsto \sqrt{\lambda - (n - 2k - 1)^2/4}, \quad \lambda \mapsto \sqrt{\lambda - (n - 2k + 1)^2/4};$$

thus $\lambda \mapsto \lambda$ defined on $\mathbb{C} \setminus [0, \infty)$ extends to a holomorphic function ϖ on Σ (cf. [2, p. 722]).

The operator family

$$\lambda \mapsto (\Delta_k - \lambda)^{-1},$$

has a meromorphic continuation from $\mathbb{C} \setminus [0, \infty)$ to the Riemann surface Σ with finite rank poles apart from a possible infinite rank pole at the zeros of ϖ (thus including $\lambda = 0$), and with non-trapping, resp. mildly trapping, high energy estimates in strips

$$|\text{Im} \sqrt{\lambda - (n - 2k - 1)^2/4}| < C$$

if g is a non-trapping, resp. mildly trapping, metric.

An analogous theorem on functions, without high energy estimates, is due to Mazzeo and Melrose [18] and Guillarmou [12], using the 0-calculus of Mazzeo and Melrose. A different proof, with high energy estimates, was provided by the author in [20] and [21]. Also, an analogous theorem (without high energy estimates) for the Dirac operator on a conformally compact spin manifold using the 0-calculus was proved by Guillarmou, Moroianu and Park [13]. The L^2 -Hodge theory was described by Mazzeo in [17], again using the 0-calculus. In the context of actual hyperbolic manifolds, i.e. quotients of real hyperbolic space (as well as complex and quaternionic hyperbolic spaces) the resolvent was constructed by Carron and Pedon [2] using explicit formulae for exact hyperbolic space; this followed the much earlier results of Donnelly [5] identifying the hyperbolic Laplacian up to unitary equivalence. In the more general asymptotically hyperbolic setting Kantor [16] has obtained an analytic continuation (without high energy estimates) except in middle degree using the 0-calculus and Pedon's explicit results, in part based on some notes provided by Guillarmou on the model case.

This theorem is proved by ‘conjugating’, or more precisely appropriately modifying, the Laplacian on differential forms to an operator which has a continuation across the boundary, as was done in the scalar setting by the author in [20] and [21]. However, here we emphasize an ‘ambient space’ point of view, which, while by no means necessary, is very enlightening; it uses a one higher dimensional (Minkowski type) Lorentzian manifold to perform this continuation across the boundary. Ambient metric constructions in conformal geometry (relating the ‘bulk’ and the asymptotically hyperbolic boundary) were introduced by Fefferman and Graham [7], see [10] for a recent treatment, but there Ricci flatness was an important consideration, while here this plays no role, instead merely the \mathbb{R}^+ -equivariance is relevant. We also refer to the recent monograph by Fefferman and Graham [8] for a more thorough treatment, including what they call ‘pre-ambient metrics’ (without a Ricci

condition). There is also the very recent work of Gover, Latini and Waldron [9] using the tractor calculus to analyze the geometric connection between asymptotically hyperbolic and ambient frameworks on differential forms.

The operator obtained in this extension process is an operator P_σ acting on two copies of the form bundle. In order to explain how this arises, and to motivate the subsequent constructions, we start by considering the d'Alembertian $\square_{\tilde{g}}$ on Minkowski space $(\mathbb{R}^{n+1}, \tilde{g})$ and the hyperbolic Laplacian in the next section, in which case the extension will be to an operator on \mathbb{S}^n acting on two copies of the form bundle, and then finally extend the results to general X in the Section 3. *The connection between an operator on the extended space acting on two copies of the form bundle and the spectral family of the asymptotically hyperbolic Laplacian is the key point of the paper.* Once this connection is established, we simply need to recall the general analytic framework for such non-elliptic problems, which we do in Section 4, to complete the proof of the theorem. This final section is merely a summary of the relevant parts of [21] and [20] since no new analytic tools are required; the set-up in these papers was such that it included non-scalar operators with scalar principal symbols, which the Laplacian on forms possesses. If the reader does not want to take the analytic ingredients as a ‘black box’, it is recommended that [21] be read before this paper.

There are no infinite rank poles on functions or top forms; from the perspective of the present paper this is so since one can work with a line bundle, i.e. by restriction of the form degree one of the two copies in the sum discussed above becomes trivial. One should be able to perform a more detailed analysis at the zeros of ϖ to rule out the infinite rank poles in certain other degrees; they are well-known to occur in middle degree even on hyperbolic space, see [5]. We briefly point out an approach to this more detailed analysis at the end of Section 2.

While we use P_σ and complex absorption to analyze the asymptotically hyperbolic resolvent, in fact when combined with analysis of the Klein-Gordon operator on asymptotically de Sitter spaces, the complex absorption can be dropped and the argument is fully reversible. In particular, on functions, this reversibility holds in the sense that the poles of the resolvent of the Laplacian on X correspond (understood in pairs, at σ and at $-\sigma$, as a dual problem also enters), apart from some integer coincidences, to poles of P_σ^{-1} . This, including the connection of the Poisson operators and scattering matrices will be discussed in a companion paper [22]. A concrete application without complex absorption is the analysis of [1] in asymptotically Minkowski spaces (again, on functions).

We reiterate the structure of this paper: in Section 2 we discuss how the analysis of the spectral family of the Laplacian on differential forms on hyperbolic space \mathbb{H}^n (and de Sitter space dS^n) can be understood via analyzing a family of operators (no longer a spectral family!) on the sphere \mathbb{S}^n acting on *two copies* of the form bundle ΛS^n . We explain the motivation for the introduction of this operator via discussing the wave operator on differential forms on Minkowski space \mathbb{R}^{n+1} ; this is a (pre-)ambient metric perspective referred to above. Next, in Section 3 we perform an analogous construction for general even asymptotically hyperbolic spaces. Finally, in Section 4 we recall the analytic set-up from [20] and [21] to complete the proof of our main theorem.

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2. MINKOWSKI SPACE, HYPERBOLIC SPACE AND DE SITTER SPACE

In this section we connect the analysis on the form bundles on Minkowski, hyperbolic and de Sitter spaces. Here we underemphasize de Sitter space, but in fact the analysis of the wave operator on forms on it is completely parallel to our treatment of hyperbolic space, as we point this out occasionally in what follows. This connection has a direct extension, with simple modifications, to the general asymptotically hyperbolic/de Sitter setting, thus while the present section is a model case, it is the heart of the paper.

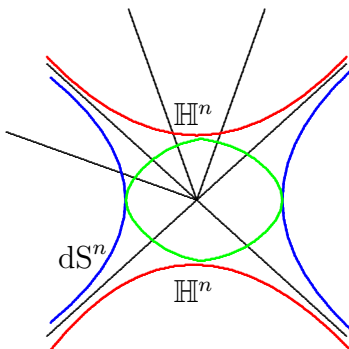


FIGURE 1. Minkowski space \mathbb{R}^{n+1} with the sphere \mathbb{S}^n being a smooth transversal to the \mathbb{R}^+ -action on $\mathbb{R}^{n+1} \setminus \{0\}$ by dilations (with some orbits shown). The hyperboloids and the corresponding regions, via the \mathbb{R}^+ action, on \mathbb{S}^n , namely the interior and exterior of the light cone, are also indicated.

The starting point of analysis is the manifold \mathbb{R}^{n+1} , or rather $\mathbb{R}^{n+1} \setminus o$, which is equipped with an \mathbb{R}^+ -action given by dilations: $(\lambda, z) \mapsto \lambda z$. A transversal to this action is, as a differentiable manifold, \mathbb{S}^n , which may be considered as the unit sphere with respect to the Euclidean metric, though the metric properties are not important here (since we are interested in the Minkowski metric after all). Thus, writing (z_1, \dots, z_{n+1}) as the coordinates, let

$$dz_1^2 + \dots + dz_n^2 + dz_{n+1}^2,$$

be the *Euclidean* metric, and let ρ be the Euclidean distance function on \mathbb{R}^{n+1} from the origin, namely

$$\rho = (z_1^2 + \dots + z_n^2 + z_{n+1}^2)^{1/2}.$$

Then \mathbb{S}^n is the 1-level set of ρ . One can identify $\mathbb{R}^{n+1} \setminus \{0\}$ via the Euclidean polar coordinate map with $\mathbb{R}_\rho^+ \times \mathbb{S}^n$, namely the map is $\mathbb{R}_\rho^+ \times \mathbb{S}^n \ni (\rho, \omega) \mapsto \rho\omega \in \mathbb{R}^{n+1} \setminus \{0\}$.

The Minkowski metric is given by

$$\tilde{g} = dz_{n+1}^2 - (dz_1^2 + \dots + dz_n^2),$$

and we also consider the Minkowski distance function r . Thus, away from the light cone, where $z_{n+1}^2 = z_1^2 + \dots + z_n^2$, let

$$r = |z_{n+1}^2 - (z_1^2 + \dots + z_n^2)|^{1/2}.$$

We are interested in $\square_{\tilde{g}}$ on differential forms. To analyze this, we conjugate $\rho^2 \square_{\tilde{g}}$ by the Mellin transform \mathcal{M}_ρ on $\mathbb{R}_\rho^+ \times \mathbb{S}^n$, identified with $\mathbb{R}^{n+1} \setminus \{0\}$ as above. To be precise, we identify the form bundle on $\mathbb{R}^{n+1} \setminus \{0\}$ with the pullback of $\Lambda \mathbb{S}^n \oplus \Lambda \mathbb{S}^n$ by decomposing a differential form into tangential and normal parts *relative to the Euclidean metric* i.e. writing forms as conormal forms plus orthogonal to these forms, which we think of as tangential forms. Notice that $T_\rho \mathbb{R}^+ \oplus T_\omega \mathbb{S}^n$ is an orthogonal decomposition of $T_{\rho\omega} \mathbb{R}^{n+1}$ relative to the Euclidean metric. Thus, a k -form on $\mathbb{R}^+ \times \mathbb{S}^n$ is written as

$$u = u_T + \frac{d\rho}{\rho} \wedge u_N,$$

where u_T and u_N are respectively k and $k-1$ forms on \mathbb{S}^n , depending on ρ ; we used $\frac{d\rho}{\rho}$ instead of $d\rho$ due to homogeneity reasons. The so-obtained operator,

$$P_{0,\tilde{\sigma}} = \mathcal{M}_\rho \rho^2 \square_{\tilde{g}} \mathcal{M}_\rho^{-1} \in \text{Diff}^2(\mathbb{S}^n; \Lambda \mathbb{S}^n \oplus \Lambda \mathbb{S}^n),$$

with $\tilde{\sigma}$ the Mellin dual parameter, fits into the framework of [20] and [21]. As an aside, we remark that it will be convenient to shift the Mellin parameter, or equivalently conjugate $\square_{\tilde{g}}$ by a power of ρ ; we shall do so later in (2.4), and this is the reason for adding the cumbersome subscript 0 to $P_{0,\tilde{\sigma}}$ presently. We explain the fit in more detail in the general asymptotically hyperbolic setting in Section 3, but we briefly indicate why this happens in terms of the scalar problem using special properties of the Minkowski metric here. Thus, the reason for the aforementioned fit into the framework is simple: in the case of the scalar d'Alembertian on Minkowski space this was shown in [20]; the d'Alembertian on forms is scalar on Minkowski space with respect to the decomposition of the form bundle relative to any basis of \mathbb{R}^{n+1} , identified with $T_z \mathbb{R}^{n+1}$ for all $z \in \mathbb{R}^{n+1}$, so with respect to this decomposition of the bundle (identifying the form bundle as a trivial bundle over \mathbb{S}^n), the (component-wise) Mellin transform fits into the framework as claimed. Now, the transition to the tangent plus normal form bundle decomposition amounts to a conjugation by a bundle endomorphism (we perform a similar one below) on \mathbb{S}^n ; such a conjugation preserves all the properties required for the analysis, except causing a form-degree dependent shift in the subprincipal term due to the different homogeneities of the forms (degree k on k -forms relative to the above trivialization, vs. degree 0 relative to the tangential plus normal decomposition).

While so far we explained why the Minkowski wave operator on forms can be analyzed by means of [20] and [21], we still need to connect this to asymptotically hyperbolic and de Sitter spaces. But in the region in \mathbb{S}^n corresponding to the interior of the future light cone, which can be identified with the hyperboloid

$$\mathbb{H}^n : z_{n+1}^2 - (z_1^2 + \dots + z_n^2) = 1, \quad z_{n+1} > 0,$$

via the \mathbb{R}^+ -quotient, one can also consider the Mellin transform of $r^2 \square_{\tilde{g}}$ with respect to the decomposition $\mathbb{R}_r^+ \times \mathbb{H}^n$, and the corresponding tangential-normal decomposition of the form bundle *relative to the Minkowski metric*, to get

$$\tilde{P}_{\tilde{\sigma}} = \mathcal{M}_r r^2 \square_{\tilde{g}} \mathcal{M}_r^{-1} \in \text{Diff}^2(\mathbb{H}^n; \Lambda \mathbb{H}^n \oplus \Lambda \mathbb{H}^n).$$

Now, $\tilde{P}_{\tilde{\sigma}}$ is not well-behaved at the boundary of the future light cone, but it is closely related to $P_{\tilde{\sigma}}$. If we use coordinates

$$\omega_j = \frac{z_j}{z_{n+1}}, \quad j = 1, \dots, n,$$

on the sphere away from the equator $z_{n+1} = 0$ (note that ω_j is not the j th component of ω with \mathbb{S}^n considered as a subset of \mathbb{R}^{n+1} !), then, with $|\cdot|$ the Euclidean norm on \mathbb{R}^n ,

$$r = F(\omega)\rho, \quad F(\omega) = \sqrt{\frac{1 - |\omega|^2}{1 + |\omega|^2}}.$$

Note that

$$\mu = F^2$$

is a smooth function on \mathbb{S}^n near (its intersection with) the light cone which vanishes non-degenerately at the light cone. On the other hand, the Poincaré ball model $\overline{\mathbb{H}^n}$ of \mathbb{H}^n arises by regarding it as a graph over \mathbb{R}^n in $\mathbb{R}^n \times \mathbb{R}$, and compactifying \mathbb{R}^n radially (or geodesically) to a ball, with boundary defining function, say, $(z_1^2 + \dots + z_n^2)^{-1/2}$, or, ρ^{-1} – these two differ by a smooth positive multiple on $\overline{\mathbb{H}^n}$. As $r = 1$ on \mathbb{H}^n , this means that F is a valid boundary defining function in the Poincaré model, in contrast with the natural F^2 defining function of the light cone. In particular, with y_j , $j = 1, \dots, n-1$, denoting local coordinates on \mathbb{S}^{n-1} , identified with $\partial\overline{\mathbb{H}^n}$, hence the light cone within \mathbb{S}^{n-1} , differential forms on \mathbb{S}^n have the form

$$c_I dy^I + c_J dF^2 \wedge dy^J,$$

with c_I and c_J smooth. We remark that pulling back the Minkowski metric to \mathbb{H}^n , which by definition yields the hyperbolic metric, a straightforward calculation yields that that

$$(2.1) \quad g = \frac{(dF)^2}{F^2(1-F^2)} + \frac{1-F^2}{2F^2} h(y, dy),$$

with h the round metric on the sphere \mathbb{S}^{n-1} ; this satisfies $F^2 g$ a smooth metric up to the boundary, $F = 0$ (with a polar coordinate singularity at $F = 1$; F and y are not valid coordinates there, though F is still \mathcal{C}^∞ near $F = 1$, and the metric is still \mathcal{C}^∞ there as well, as can be seen by using valid coordinates), with the coefficients even functions of F . The metric g can be put in the normal form $g = \frac{dx^2 + h}{x^2}$ by letting $x = \frac{F}{1 + \sqrt{1 - F^2}}$, which is an equivalent boundary defining function, but this is not necessary here.

We remark at this point that de Sitter space can be approached in a completely parallel manner. Recall that de Sitter space is the hyperboloid

$$dS^n : z_{n+1}^2 - (z_1^2 + \dots + z_n^2) = -1$$

in Minkowski space with the pulled back metric. Now, the region in \mathbb{S}^n corresponding to the ‘equatorial belt’, i.e. the exterior of the future and past light cones, can be identified with this hyperboloid via the \mathbb{R}^+ -quotient. In this region one can also consider the Mellin transform of $r^2 \square_{\tilde{g}}$ with respect to the decomposition $\mathbb{R}_r^+ \times dS^n$, and the corresponding tangential-normal decomposition of the form bundle relative to the Minkowski metric, to get

$$\tilde{P}_{dS, \tilde{\sigma}} = \mathcal{M}_r r^2 \square_{\tilde{g}} \mathcal{M}_r^{-1} \in \text{Diff}^2(dS^n; \text{AdS}^n \oplus \text{AdS}^n).$$

Returning to \mathbb{H}^n and rewriting a form in tangential-normal decomposition with respect to the Minkowski metric as such with respect to the Euclidean metric, where \mathbb{H}^n is identified as an open subset of \mathbb{S}^n , one has

$$v_T + \frac{dr}{r} \wedge v_N = \left(v_T + \frac{dF}{F} \wedge v_N \right) + \frac{d\rho}{\rho} \wedge v_N = u_T + \frac{d\rho}{\rho} \wedge u_N,$$

with

$$\begin{bmatrix} v_T \\ v_N \end{bmatrix} = J \begin{bmatrix} u_T \\ u_N \end{bmatrix}, \quad J = \begin{bmatrix} \text{Id} & \frac{dF}{F} \wedge \\ 0 & \text{Id} \end{bmatrix}, \quad J^{-1} = \begin{bmatrix} \text{Id} & -\frac{dF}{F} \wedge \\ 0 & \text{Id} \end{bmatrix}.$$

Since for f taking values in a bundle over \mathbb{S}^n

$$\mathcal{M}_\rho f(\tilde{\sigma}, \omega) = \int_0^\infty \rho^{-i\tilde{\sigma}} f \frac{d\rho}{\rho},$$

with a similar formula for \mathcal{M}_r , we have, if we identify \mathbb{H}^n with an open subset of \mathbb{S}^n (the interior of the future light cone), and correspondingly identify the form bundles, on $\mathcal{C}^\infty(\mathbb{H}^n)$,

$$(2.2) \quad \mathcal{M}_\rho \rho^2 \square_{\tilde{g}} \mathcal{M}_\rho^{-1}(\tilde{\sigma}) = J^{-1} F^{i\tilde{\sigma}-2} \mathcal{M}_r r^2 \square_{\tilde{g}} \mathcal{M}_r^{-1} F^{-i\tilde{\sigma}} J.$$

We next compute $\mathcal{M}_r r^2 \square_{\tilde{g}} \mathcal{M}_r^{-1}$; this is feasible since $\mathbb{R}^+ \times \mathbb{H}^n$ is an orthogonal decomposition relative to \tilde{g} . Concretely, the Minkowski metric is

$$\tilde{g} = dr^2 - r^2 g,$$

where g is the hyperbolic metric, since by definition the hyperbolic metric is the negative (with our sign convention) of the restriction of the Minkowski metric to the hyperboloid \mathbb{H}^n . This form of \tilde{g} is the analogue of the Euclidean metric being of the form $d\rho^2 + \rho^2 k$, k being the round metric on the sphere, which again corresponds to the sphere, $\rho = 1$, being given the pull-back of the Euclidean metric.

The metric \tilde{g} is thus a (non-Riemannian) conic metric, whose Laplacian was computed by Cheeger [3, Equation (3.8)]. This is best done relative to a tangential-normal decomposition of the form bundle of \mathbb{R}^{n+1} relative to \mathbb{H}^n and the Minkowski metric, i.e. writing forms as conormal forms plus orthogonal to these forms, which we again think of as tangential forms. Concretely, following Cheeger's decomposition, a k -form on $\mathbb{R}^+ \times \mathbb{H}^n$ is written as

$$v = \tilde{v}_T + dr \wedge \tilde{v}_N,$$

where v_T and v_N are respectively k and $k-1$ forms on \mathbb{H}^n . Then, in this decomposition, writing $v = (v_T, v_N)$, writing $X = \mathbb{H}^n$,

$$\square_{\tilde{g}} = \begin{bmatrix} -r^{-2} \Delta_X - r^{2k-n} \partial_r r^{n-2k} \partial_r & -2r^{-1} d_X \\ 2r^{-3} \delta_X & -r^{-2} \Delta_X - \partial_r r^{2(k-1)-n} \partial_r r^{n-2(k-1)} \end{bmatrix},$$

similarly to Cheeger's case with some sign changes due to the Lorentzian signature of \tilde{g} . Rewriting in a form that is more useful for homogeneity reasons,

$$v = v_T + \frac{dr}{r} \wedge v_N, \quad v_T = \tilde{v}_T, \quad v_N = r \tilde{v}_N,$$

$$r^2 \square_{\tilde{g}} = \begin{bmatrix} -\Delta_X - r^{2k-n+2} \partial_r r^{n-2k} \partial_r & -2d_X \\ 2\delta_X & -\Delta_X - r^3 \partial_r r^{2(k-1)-n} \partial_r r^{n-2(k-1)-1} \end{bmatrix}.$$

Thus, as

$$\begin{aligned}
r^{2k-n+2}\partial_r r^{n-2k}\partial_r &= (r\partial_r)^2 + (n-2k-1)r\partial_r \\
&= \left(r\partial_r + \frac{n-2k-1}{2}\right)^2 - \left(\frac{n-2k-1}{2}\right)^2, \\
r^3\partial_r r^{2(k-1)-n}\partial_r r^{n-2(k-1)-1} &= (r\partial_r - 2)(r\partial_r + (n-2k+1)) \\
&= \left(r\partial_r + \frac{n-2k-1}{2}\right)^2 - \left(\frac{n-2k+3}{2}\right)^2,
\end{aligned}$$

in this basis we have

$$\begin{aligned}
(2.3) \quad \text{Diff}^2(X; \Lambda X \oplus \Lambda X) \ni \mathcal{M}_r r^2 \square_{\tilde{g}} \mathcal{M}_r^{-1} \\
= \begin{bmatrix} -\Delta_X + \left(\tilde{\sigma} - \iota \frac{n-2k-1}{2}\right)^2 + \left(\frac{n-2k-1}{2}\right)^2 & -2d_X \\ 2\delta_X & -\Delta_X + \left(\tilde{\sigma} - \iota \frac{n-2k-1}{2}\right)^2 + \left(\frac{n-2k+3}{2}\right)^2 \end{bmatrix}.
\end{aligned}$$

In view of this formula, it is convenient to introduce

$$\sigma = \tilde{\sigma} - \iota \frac{n-2k-1}{2}$$

to simplify some expressions; so shifting the Mellin parameter amounts to conjugation of $\square_{\tilde{g}}$ by $r^{-(n-2k-1)/2}$, i.e. considering

$$(2.4) \quad \mathcal{M}_r r^2 r^{(n-2k-1)/2} \square_{\tilde{g}} r^{-(n-2k-1)/2} \mathcal{M}_r^{-1}(\sigma) = \mathcal{M}_r r^2 \square_{\tilde{g}} \mathcal{M}_r^{-1}(\tilde{\sigma}), \quad \tilde{\sigma} = \sigma + \iota \frac{n-2k-1}{2}.$$

Thus,

$$\begin{aligned}
(2.5) \quad \mathcal{M}_r r^2 r^{(n-2k-1)/2} \square_{\tilde{g}} r^{-(n-2k-1)/2} \mathcal{M}_r^{-1} \\
= \begin{bmatrix} -\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2}\right)^2 & -2d_X \\ 2\delta_X & -\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2}\right)^2 \end{bmatrix}.
\end{aligned}$$

Combining with (2.2) we deduce the following lemma:

Lemma 2.1. *Let*

$$P_\sigma = \mathcal{M}_\rho \rho^2 \rho^{(n-2k-1)/2} \square_{\tilde{g}} \rho^{-(n-2k-1)/2} \mathcal{M}_\rho^{-1}.$$

Then

$$\begin{aligned}
\begin{bmatrix} -\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2}\right)^2 & -2d_X \\ 2\delta_X & -\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2}\right)^2 \end{bmatrix} \\
= J F^{-\iota\sigma + (n-2k-1)/2 + 2} P_\sigma F^{\iota\sigma - (n-2k-1)/2} J^{-1}.
\end{aligned}$$

While (2.5) is not a diagonal matrix, the off-diagonal terms have a special structure. In particular, for v_T coclosed and v_N closed we have that

$$\begin{aligned}
\mathcal{M}_r r^2 r^{(n-2k-1)/2} \square_{\tilde{g}} r^{-(n-2k-1)/2} \mathcal{M}_r^{-1} \begin{bmatrix} v_T \\ v_N \end{bmatrix} \\
= \begin{bmatrix} -\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2}\right)^2 & 0 \\ 0 & -\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2}\right)^2 \end{bmatrix} \begin{bmatrix} v_T \\ v_N \end{bmatrix},
\end{aligned}$$

so

$$\begin{aligned}
& \mathcal{M}_r r^2 r^{(n-2k-1)/2} \square_{\bar{g}} r^{-(n-2k-1)/2} \mathcal{M}_r^{-1} \begin{bmatrix} \delta_X d_X & 0 \\ 0 & d_X \delta_X \end{bmatrix} \\
&= \begin{bmatrix} \delta_X d_X & 0 \\ 0 & d_X \delta_X \end{bmatrix} \mathcal{M}_r r^2 r^{(n-2k-1)/2} \square_{\bar{g}} r^{-(n-2k-1)/2} \mathcal{M}_r^{-1} \\
&= \begin{bmatrix} \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right) \delta_X d_X & 0 \\ 0 & \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \right) d_X \delta_X \end{bmatrix}.
\end{aligned}$$

Correspondingly, let $\iota_{X,T,k}$, resp. $\iota_{X,N,k-1}$ denote the inclusion maps $\Lambda^k X \rightarrow \Lambda^k X \oplus \Lambda^{k-1} X$, resp. $\Lambda^{k-1} X \rightarrow \Lambda^k X \oplus \Lambda^{k-1} X$ as 0 in the other summand, and $\pi_{X,T,k}$, resp. $\pi_{X,N,k-1}$ be the projection maps. Then, using (2.2),

$$\begin{aligned}
& \delta_X d_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right) \\
&= \delta_X d_X \pi_{X,T,k} \mathcal{M}_r r^2 r^{(n-2k-1)/2} \square_{\bar{g}} r^{-(n-2k-1)/2} \mathcal{M}_r^{-1} \iota_{X,T,k}, \\
& d_X \delta_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \right) \\
&= d_X \delta_X \pi_{X,N,k-1} \mathcal{M}_r r^2 r^{(n-2k-1)/2} \square_{\bar{g}} r^{-(n-2k-1)/2} \mathcal{M}_r^{-1} \iota_{X,N,k-1}.
\end{aligned}$$

We then have, via regarding $X = \mathbb{H}^n$ as a subset of $\tilde{X} = \mathbb{S}^n$ and using the corresponding identification of the form bundles

$$\begin{aligned}
& \delta_X d_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right) \\
&= \delta_X d_X \pi_{X,T,k} J F^{-i\sigma + (n-2k-1)/2 + 2} P_\sigma F^{i\sigma - (n-2k-1)/2} J^{-1} \iota_{X,T,k}, \\
& d_X \delta_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \right) \\
&= d_X \delta_X \pi_{X,N,k-1} J F^{-i\sigma + (n-2k-1)/2 + 2} P_\sigma F^{i\sigma - (n-2k-1)/2} J^{-1} \iota_{X,N,k-1}.
\end{aligned}$$

Correspondingly, for an appropriately chosen inverse P_σ^{-1} of the \mathcal{M}_ρ -conjugated operator we have for $\text{Im } \sigma \gg 1$, with r_X denoting restriction to X , e_X an extension map from X to \tilde{X} , i.e. $e_X : \mathcal{C}_c^\infty(X; \Lambda X \oplus \Lambda X) \rightarrow \mathcal{C}^\infty(\tilde{X}; \Lambda \tilde{X} \oplus \Lambda \tilde{X})$ with $r_X e_X = \text{Id}$ that

$$\begin{aligned}
(2.6) \quad & \delta_X d_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right)^{-1} = \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right)^{-1} \delta_X d_X \\
&= \delta_X d_X \pi_{X,T,k} J F^{-i\sigma + (n-2k-1)/2} r_X P_\sigma^{-1} e_X F^{i\sigma - (n-2k-1)/2 - 2} J^{-1} \iota_{X,T,k}
\end{aligned}$$

and

$$\begin{aligned}
(2.7) \quad & d_X \delta_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \right)^{-1} = \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \right)^{-1} d_X \delta_X \\
&= d_X \delta_X \pi_{X,N,k-1} J F^{-i\sigma + (n-2k-1)/2} r_X P_\sigma^{-1} e_X F^{i\sigma - (n-2k-1)/2 - 2} J^{-1} \iota_{X,N,k-1}.
\end{aligned}$$

It is useful to have a modification G_σ of P_σ^{-1} to ‘cut off’ the problem, namely to localize it to a neighborhood of the forward light cone. Concretely, G_σ is constructed using a complex absorption operator Q_σ , with Schwartz kernel supported in $(\tilde{X} \setminus \mathcal{U})^2$, as

$$G_\sigma = (P_\sigma - iQ_\sigma)^{-1}.$$

In fact, $P_\sigma - \iota Q_\sigma$ is Fredholm between appropriate spaces recalled in the next section, with a meromorphic inverse and with non-trapping high energy estimates under non-trapping assumptions on X . (Technically Q is defined only for a certain set of σ , or rather one needs to use different operators Q in different subsets of \mathbb{C} , but in strips, or even in somewhat larger conic sectors, which are our main interest, a single Q suffices. We refer the reader to [20, Section 4.7] for further details.) This modification is just as good as P_σ^{-1} for our purposes:

Lemma 2.2. *If G_σ is chosen so that it maps $\mathcal{C}_c^\infty(X; \Lambda X \oplus \Lambda X)$ to \mathcal{C}^∞ forms on \tilde{X} in a neighborhood \mathcal{U} of \bar{X} , and so that $r_X P_\sigma G_\sigma = \text{Id}$ where r_X denotes restriction to X , then for $\text{Im } \sigma \gg 1$, then (2.6), resp. (2.7), hold on $\mathcal{C}_c^\infty(X; \Lambda^k X)$, resp. $\mathcal{C}_c^\infty(X; \Lambda^{k-1} X)$, with G_σ replacing P_σ^{-1} .*

Proof. We consider (2.6); the treatment of (2.7) is completely analogous. Let $f \in \mathcal{C}_c^\infty(X; \Lambda^k X)$, $\text{Im } \sigma \gg 1$. We claim that

$$u = \delta_X d_X \pi_{X,T,k} J F^{-\iota\sigma + (n-2k-1)/2} r_X G_\sigma e_X F^{\iota\sigma - (n-2k-1)/2 - 2} J^{-1} \iota_{X,T,k} f \in \mathcal{C}^\infty(X; \Lambda^k X)$$

satisfies

$$(2.8) \quad \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right) u = \tilde{f}, \quad \tilde{f} = \delta_X d_X f.$$

Indeed, using $r_X P_\sigma = P_\sigma r_X$,

$$\begin{aligned} & \begin{bmatrix} -\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 & -2d_X \\ 2\delta_X & -\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \end{bmatrix} \\ & \times J F^{-\iota\sigma + (n-2k-1)/2} r_X G_\sigma e_X F^{\iota\sigma - (n-2k-1)/2 - 2} J^{-1} \iota_{X,T,k} f \\ & = J F^{-\iota\sigma + (n-2k-1)/2 + 2} P_\sigma r_X G_\sigma e_X F^{\iota\sigma - (n-2k-1)/2 - 2} J^{-1} \iota_{X,T,k} f = r_X \iota_{X,T,k} f = \begin{bmatrix} f \\ 0 \end{bmatrix}, \end{aligned}$$

so with

$$\tilde{u} = J F^{-\iota\sigma + (n-2k-1)/2} r_X G_\sigma e_X F^{\iota\sigma - (n-2k-1)/2 - 2} J^{-1} \iota_{X,T,k} f,$$

so $u = \delta_X d_X \pi_{X,T,k} \tilde{u}$, one has

$$\begin{aligned} \delta_X d_X f &= \delta_X d_X \pi_{X,T,k} \begin{bmatrix} f \\ 0 \end{bmatrix} \\ &= \delta_X d_X \pi_{X,T,k} \begin{bmatrix} -\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 & -2d_X \\ 2\delta_X & -\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \end{bmatrix} \tilde{u} \\ &= \pi_{X,T,k} \begin{bmatrix} \delta_X d_X & 0 \\ 0 & d_X \delta_X \end{bmatrix} \begin{bmatrix} -\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 & -2d_X \\ 2\delta_X & -\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \end{bmatrix} \tilde{u} \\ &= \pi_{X,T,k} \begin{bmatrix} \delta_X d_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right) & 0 \\ 0 & d_X \delta_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \right) \end{bmatrix} \tilde{u} \\ &= \delta_X d_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right) \pi_{X,T,k} \tilde{u} = \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right) u, \end{aligned}$$

and

$$\begin{aligned} u &\in \pi_{X,T,k} J F^{-\iota\sigma+(n-2k-1)/2} \mathcal{C}^\infty(\bar{X}; \Lambda^k \tilde{X} \oplus \Lambda^{k-1} \tilde{X}) \\ &\subset F^{-\iota\sigma+(n-2k-1)/2} \mathcal{C}^\infty(\bar{X}; \Lambda^k \tilde{X}) + F^{-\iota\sigma+(n-2k-1)/2} \frac{dF}{F} \wedge \mathcal{C}^\infty(\bar{X}; \Lambda^{k-1} \tilde{X}), \end{aligned}$$

and thus is in $L^2(X; \Lambda^k X)$. Therefore, as given $\tilde{f} \in L^2(X; \Lambda^k X)$ there is a unique L^2 form solving (2.8), we conclude that $\left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2}\right)^2\right)^{-1} \delta_X d_X f$ is indeed given by the right hand side of the second equality in (2.6). Since

$$\delta_X d_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2}\right)^2\right)^{-1} f$$

also solves (2.8) due to the fact that Δ_X and $\delta_X d_X$ commute as operators on $\mathcal{C}^\infty(X; \Lambda X)$, and as it is in L^2 (for $\text{Im } \sigma \gg 1$), the first equality in (2.6) also holds. \square

Then, as the right hand side of (2.6) is meromorphic on \mathbb{C} with finite rank poles and has appropriate high energy estimates under non-trapping assumptions, one obtains such an extension of the left hand side. A similar argument applies for (2.7), but we need to note this is acting on $k-1$ forms on X , and thus we need to replace k by $k+1$ throughout to obtain a formula for the k -form Laplacian, resulting in the shift in the statement of Theorem 1.1.

A bit of care is needed in order to derive the precise form of the mapping properties and the corresponding high energy estimates. Namely, as we recall in the next sections,

$$(P_\sigma - \iota Q_\sigma)^{-1} : H^{s-1}(\tilde{X}; \Lambda \tilde{X}) \rightarrow H^s(\tilde{X}; \Lambda \tilde{X}), \quad s > 1/2 - \text{Im } \sigma,$$

with the high energy estimate that for fixed s , σ satisfying $-s + 1/2 < \text{Im } \sigma$, $|\text{Re } \sigma| > R$, $R > 0$ sufficiently large,

$$\|(P_\sigma - \iota Q_\sigma)^{-1}\|_{\mathcal{L}(H_{|\sigma|^{-1}}^{s-1}(\tilde{X}; \Lambda \tilde{X}), H_{|\sigma|^{-1}}^s(\tilde{X}; \Lambda \tilde{X}))} \leq C |\sigma|^{-1},$$

where $H_{|\sigma|^{-1}}^s(\tilde{X})$ is the semiclassical Sobolev space in which derivatives come with a prefactor of $|\sigma|^{-1}$; see the introduction of [21] for more details. Now all the other operators in (2.6)-(2.7) are straightforward to estimate, being bundle maps or differential operators. However, these are singular maps: F vanishes at ∂X , and J^{-1} involves $\frac{dF}{F} \wedge$ when applied to normal forms, i.e. essentially $\frac{d\mu}{\mu} \wedge$. Thus, dropping the bundles from the notation momentarily,

$$\begin{aligned} &\|(P_\sigma - \iota Q_\sigma)^{-1} e_X F^{\iota\sigma-(n-2k-1)/2-2} J^{-1} \iota_{X,T,k} f\|_{H_{|\sigma|^{-1}}^s(\tilde{X})} \\ &\leq C \|F^{\iota\sigma-(n-2k-1)/2-2} f\|_{H_{|\sigma|^{-1}}^{s-1}(\bar{X}_{\text{even}})}, \\ &\|(P_\sigma - \iota Q_\sigma)^{-1} e_X F^{\iota\sigma-(n-2k-1)/2-2} J^{-1} \iota_{X,N,k-1} f\|_{H_{|\sigma|^{-1}}^s(\tilde{X})} \\ &\leq C (\|F^{\iota\sigma-(n-2k-1)/2-2} f\|_{H_{|\sigma|^{-1}}^{s-1}(\bar{X}_{\text{even}})} + \|F^{\iota\sigma-(n-2k-1)/2-4} d\mu \wedge f\|_{H_{|\sigma|^{-1}}^{s-1}(\bar{X}_{\text{even}})}) \\ &\leq \tilde{C} \|F^{\iota\sigma-(n-2k-1)/2-4} f\|_{H_{|\sigma|^{-1}}^{s-1}(\bar{X}_{\text{even}})}, \end{aligned}$$

where the loss for normal forms relative to tangential forms (in terms of a simple Sobolev space, given on the right hand side of the last inequality) comes from the singular factor in J giving rise to the $d\mu \wedge$ term, and where the spaces on \tilde{X} and

$\overline{X}_{\text{even}}$ are the sections of appropriate degree parts of $\Lambda\tilde{X} \oplus \Lambda\tilde{X}$, resp. $\Lambda\overline{X}_{\text{even}}$. Further,

$$\begin{aligned} & \|F^{\iota\sigma-(n-2k-1)/2+2}\pi_{X,T,k}JF^{-\iota\sigma+(n-2k-1)/2}r_Xv\|_{H_{|\sigma|-1}^s(\overline{X}_{\text{even}})} \\ & \leq C' \left(\|F^{\iota\sigma-(n-2k-1)/2+2}\pi_{X,T,k}JF^{-\iota\sigma+(n-2k-1)/2}r_Xv\|_{H_{|\sigma|-1}^s(\overline{X}_{\text{even}})} \right. \\ & \quad \left. + \|F^{\iota\sigma-(n-2k-1)/2}d\mu \wedge \pi_{X,T,k}JF^{-\iota\sigma+(n-2k-1)/2}r_Xv\|_{H_{|\sigma|-1}^s(\overline{X}_{\text{even}})} \right) \\ & \leq C'' \|v\|_{H_{|\sigma|-1}^s(\tilde{X})}, \\ & \|F^{\iota\sigma-(n-2k-1)/2}\pi_{X,N,k-1}JF^{-\iota\sigma+(n-2k-1)/2}r_Xv\|_{H_{|\sigma|-1}^s(\overline{X}_{\text{even}})} \leq C' \|v\|_{H_{|\sigma|-1}^s(\tilde{X})}, \end{aligned}$$

where the loss is now in tangential forms due to J . However, these losses are merely apparent, as we momentarily show using the special structure of $d_X\delta_X$ and δ_Xd_X .

Indeed, $\delta_Xd_X, d_X\delta_X$ are even differential operators, i.e. when regarded as an operator on $\overline{X}_{\text{even}}$, they satisfy $\delta_Xd_X, d_X\delta_X \in \text{Diff}^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}})$, and even $\delta_Xd_X, d_X\delta_X \in \text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}})$; this can be seen from a direct calculation, which we discuss below in the general conformally compact case in Lemma 3.1. (Recall that $\mathcal{V}_b(\overline{X}_{\text{even}})$ is the set of smooth vector fields tangent to the boundary; Diff_b is generated by these.) In fact, an even stronger statement also holds for certain parts of this operator, namely, with $d\mu \wedge$ denoting the operator of wedge product with $d\mu$,

$$(d\mu \wedge)d_X\delta_X, \delta_Xd_X(d\mu \wedge) \in \mu\text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}});$$

see Lemma 3.1. Correspondingly, for any $\alpha \in \mathbb{C}$, basically relying on

$$\mu^{\alpha/2}(\mu\partial_\mu)\mu^{-\alpha/2} = \mu\partial_\mu - \alpha/2,$$

one has

$$F^\alpha\delta_Xd_XF^{-\alpha} \in \text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}}) \subset \text{Diff}^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}}),$$

and

$$F^\alpha\delta_Xd_XF^{-\alpha}\left(\frac{dF}{F}\wedge\right) \in \text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}}) \subset \text{Diff}^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}}),$$

with analogous statements for $d_X\delta_X$. Thus,

$$\begin{aligned} & F^{\iota\sigma-(n-2k-1)/2}\delta_Xd_X\pi_{X,T,k}JF^{-\iota\sigma+(n-2k-1)/2} \in \text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}}), \\ & F^{\iota\sigma-(n-2k-1)/2}d_X\delta_X\pi_{X,N,k-1}JF^{-\iota\sigma+(n-2k-1)/2} \in \text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}}), \\ & F^{\iota\sigma-(n-2k-1)/2-2}(d\mu \wedge)d_X\delta_X\pi_{X,N,k-1}JF^{-\iota\sigma+(n-2k-1)/2} \in \text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda\overline{X}_{\text{even}}). \end{aligned}$$

Therefore, for $s \in \mathbb{R}$, the operators

$$\begin{aligned} & F^{\iota\sigma-(n-2k-1)/2}\langle|\sigma|\rangle^{-2}\delta_Xd_X\pi_{X,T,k}JF^{-\iota\sigma+(n-2k-1)/2}, \\ & F^{\iota\sigma-(n-2k-1)/2}\langle|\sigma|\rangle^{-2}d_X\delta_X\pi_{X,N,k-1}JF^{-\iota\sigma+(n-2k-1)/2}, \\ & F^{\iota\sigma-(n-2k-1)/2-2}\langle|\sigma|\rangle^{-2}(d\mu \wedge)d_X\delta_X\pi_{X,N,k-1}JF^{-\iota\sigma+(n-2k-1)/2} \end{aligned}$$

are uniformly bounded in $\mathcal{L}(H_{|\sigma|^{-1}}^{s+2}(\overline{X}_{\text{even}}; \Lambda \overline{X}_{\text{even}}), H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}}; \Lambda \overline{X}_{\text{even}}))$. In summary, using (2.6)-(2.7) for $\text{Im } \sigma > -s - 3/2$, $|\text{Re } \sigma|$ sufficiently large,

$$\begin{aligned} & \|F^{\nu\sigma - (n-2k-1)/2} \delta_X d_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right)^{-1} f\|_{H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}})} \\ & \leq C_0 |\sigma| \|F^{\nu\sigma - (n-2k-1)/2-2} f\|_{H_{|\sigma|^{-1}}^{s+1}(\overline{X}_{\text{even}})}, \\ & \|F^{\nu\sigma - (n-2k-1)/2} d_X \delta_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \right)^{-1} f\|_{H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}})} \\ & + \|F^{\nu\sigma - (n-2k-1)/2-2} (d\mu \wedge) d_X \delta_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2} \right)^2 \right)^{-1} f\|_{H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}})} \\ & \leq C_0 |\sigma| \|F^{\nu\sigma - (n-2k-1)/2-2} f\|_{H_{|\sigma|^{-1}}^{s+1}(\overline{X}_{\text{even}})} + \|F^{\nu\sigma - (n-2k-1)/2-4} d\mu \wedge f\|_{H_{|\sigma|^{-1}}^{s+1}(\overline{X}_{\text{even}})}. \end{aligned}$$

Since $\delta_X d_X + d_X \delta_X = \Delta_X$, combining (2.6)-(2.7) gives the meromorphic continuation of $\left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right)^{-1}$ itself, but with another branch arising from closed forms, i.e. the meromorphic continuation is not merely to the Riemann surface of the inverse function of $\lambda \mapsto \sqrt{\lambda - \left(\frac{n-2k-1}{2} \right)^2}$, rather the joint Riemann surface of this and $\lambda \mapsto \sqrt{\lambda - \left(\frac{n-2k+1}{2} \right)^2}$. Further, what one actually obtains is

$$\begin{aligned} & \Delta_X \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right)^{-1} \\ & = -\text{Id} + \left(\sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right) \left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right)^{-1}, \end{aligned}$$

and thus an infinite rank pole is allowed at points where the analytic continuation of $\lambda \mapsto \lambda$ vanishes (note that $\lambda = \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2$ is the spectral parameter in the above formula.) We write

$$\mathcal{R}_X(\sigma)$$

for the meromorphic continuation of $\left(-\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2} \right)^2 \right)^{-1}$.

We remark that with slightly more work the ‘cross terms’, i.e. $\delta_X d_X$ with normal forms and $d_X \delta_X$ with tangential forms can also be analyzed, and then the nature of the possible pole at zero can be described more precisely, but this is not our focus here. The basic point is that for an operator mapping between direct sums of Banach spaces, $\mathcal{X}_0 \oplus \mathcal{X}_1 \rightarrow \mathcal{Y}_0 \oplus \mathcal{Y}_1$, if $D : \mathcal{X}_1 \rightarrow \mathcal{Y}_1$ is invertible then the invertibility of $A - BD^{-1}C : \mathcal{X}_0 \rightarrow \mathcal{Y}_0$ and $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ are equivalent, with

$$(2.9) \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}.$$

There is an analogous formula if the role of the two components are interchanged. Since for top and bottom degree forms one of the two components is trivial (as $\Lambda^{-1}X$, resp. $\Lambda^{n+1}X$ are trivial), one can proceed inductively from the two extremes towards middle degrees. Thus, for 1-forms, for instance, one uses that one has obtained D_σ on 0-forms to conclude that, provided that the domains remain compatible, one has a meromorphic continuation for 1-forms with at most

a finite rank pole at 0 since, writing the right hand side of (2.9) as $\begin{bmatrix} E & F \\ G & H \end{bmatrix}$, $A^{-1} = E - F(D - CA^{-1}B)G$, and $CA^{-1}B$ (recall that B and C are $-2d_X$ and $2\delta_X$) can be computed using the information already obtained above, including at 0.

3. CONFORMALLY COMPACT SPACES

We now extend the results to general even conformally compact spaces. That is, if (X, g) is Riemannian and even asymptotically hyperbolic, there is a product decomposition near the boundary Y_y of \bar{X} such that

$$g = \frac{dx^2 + \tilde{h}(x, y, dy)}{x^2},$$

with \tilde{h} even in x , i.e. $\tilde{h} = h(x^2, y, dy)$, with h smooth. We write \bar{X}_{even} for \bar{X} with the new smooth structure in which $\mu = x^2$ is a boundary defining function. We consider h as a symmetric 2-cotensor on Y valued function on \bar{X}_{even} defined near Y . Before considering the appropriate extension of an operator related to the spectral family of Δ_X across ∂X , we first discuss $d_X\delta_X$ and $\delta_X d_X$ in some detail.

Lemma 3.1. *Suppose that (X, g) is equipped with an even asymptotically hyperbolic metric, with \bar{X}_{even} being the compactification equipped with the even smooth structure. Then*

$$\delta_X d_X, d_X \delta_X \in \text{Diff}_b^2(\bar{X}_{\text{even}}; \Lambda \bar{X}_{\text{even}}).$$

Further, with $d\mu \wedge$ denoting the operator of wedge product with $d\mu$,

$$(d\mu \wedge) d_X \delta_X, \delta_X d_X (d\mu \wedge) \in \mu \text{Diff}_b^2(\bar{X}_{\text{even}}; \Lambda \bar{X}_{\text{even}}).$$

Proof. With $\mu = x^2$, we use a conormal vs. tangential decomposition of forms near Y on \bar{X}_{even} , i.e. we write k -forms as linear combinations of

$$dy^\alpha, d\mu \wedge dy^\beta, |\alpha| = k, |\beta| = k - 1.$$

In this basis, d_X has the form

$$d_X = \begin{bmatrix} d_Y & 0 \\ \partial_\mu & -d_Y \end{bmatrix},$$

while g has the form $g = \frac{d\mu^2}{4\mu^2} + \frac{h}{\mu}$, so the dual metric is $G = 4\mu^2 \partial_\mu^2 + \mu H$, where H is the dual metric of h . Correspondingly,

$$|dg| = \frac{1}{2\mu^{(n+1)/2}} d\mu dh = \frac{\sqrt{\det h}}{2\mu^{(n+1)/2}} d\mu |dy|,$$

and on k -forms the dual metric is

$$G_k = \begin{bmatrix} \mu^k H_k & 0 \\ 0 & 4\mu^{k+1} H_{k-1} \end{bmatrix},$$

where H_k is the dual metric of h on boundary k -forms. We compute δ_X as $\delta_X = G_{k-1}^{-1} d_{\text{base}}^* G_k$, d_{base} being the adjoint of d_X where the Euclidean inner product is used in the fibers of T^*X via a local trivialization, but the metric density $|dg|$ is used to integrate, i.e. $d_{\text{base}}^* = (\det g)^{-1/2} \delta_{\mathbb{R}^n} (\det g)^{1/2}$. This gives

$$\delta_{X,k} = \begin{bmatrix} \mu \delta_Y & -4\mu^2 \partial_\mu + 2\mu(n - 2k - 1) + \mu^2 \gamma \\ 0 & -\mu \delta_Y \end{bmatrix},$$

$$\gamma = -4H_{k-1}^{-1} (\det h)^{-1/2} \partial_\mu (\det h)^{1/2} H_{k-1}.$$

This yields

$$\begin{aligned} d_{X,k-1}\delta_{X,k} &= \begin{bmatrix} \mu d_Y \delta_Y & d_Y(-4\mu^2 \partial_\mu + 2\mu(n-2k-1) + \mu^2 \gamma) \\ \partial_\mu \mu \delta_Y & \partial_\mu(-4\mu^2 \partial_\mu + 2\mu(n-2k-1) + \mu^2 \gamma) + \mu d_Y \delta_Y \end{bmatrix}, \\ \delta_{X,k+1}d_{X,k} &= \begin{bmatrix} \mu \delta_Y d_Y + (-4\mu^2 \partial_\mu + 2\mu(n-2k-3) + \mu^2 \gamma) \partial_\mu & -(-4\mu^2 \partial_\mu + 2\mu(n-2k-3) + \mu^2 \gamma) d_Y \\ -\mu \delta_Y \partial_\mu & \mu \delta_Y d_Y \end{bmatrix}, \end{aligned}$$

which are indeed in $\text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda \overline{X}_{\text{even}})$. Furthermore,

$$\begin{aligned} (d\mu \wedge) d_{X,k-1} \delta_{X,k} &= \begin{bmatrix} \mu d_Y \delta_Y & d_Y(-4\mu^2 \partial_\mu + 2\mu(n-2k-1) + \mu^2 \gamma) \\ 0 & 0 \end{bmatrix}, \\ \delta_{X,k+1} d_{X,k} (d\mu \wedge) &= \begin{bmatrix} 0 & -(-4\mu^2 \partial_\mu + 2\mu(n-2k-3) + \mu^2 \gamma) d_Y \\ 0 & \mu \delta_Y d_Y \end{bmatrix}, \end{aligned}$$

which are in $\mu \text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda \overline{X}_{\text{even}})$, completing the proof. \square

Note that this in particular implies that $\Delta_X \in \text{Diff}_b^2(\overline{X}_{\text{even}}; \Lambda \overline{X}_{\text{even}})$. In the scalar setting, more is true: after one conjugates the spectral family, $\Delta_X - \sigma^2 - (n-1)^2/4$, by the appropriate power of μ , one can factor out μ and still have a differential operator with smooth coefficients. The appropriate power is closely related to the asymptotic behavior of the Green's function at ∂X . The diverse behavior of the form Laplacian on different kinds of forms makes this a more difficult process in the form valued setting. For instance, notice that one can factor μ out of $d_{X,k-1}\delta_{X,k}$ on the right, while for $\delta_{X,k+1}d_{X,k}$ this can be done on the left – and this ignores additional issues from the spectral family!

We now describe two possible ways of proceeding, with the first being an analogue of [21] but working with an extended system (not merely extending a form bundle, but working with two copies); we pursue the second one of these in detail, which is based on the Minkowski space model.

The first method is as follows. One may regard (2.2) as a statement that the right hand side, valid in $\mathbb{H}^n = X$, extends to a differential operator on $\mathbb{S}^n = \tilde{X}$ of the appropriate type, with smooth coefficients, acting on two copies of the form bundle on the sphere, after $J^{-1}F^{i\tilde{\sigma}-2}$ is applied from the left, $F^{-i\tilde{\sigma}}J$ applied from the right, and the smooth structure is changed to the smooth structure corresponding to the boundary defining function F^2 . In view of (2.3), this is a statement about the spectral family of a slightly modified version of Δ_X , incorporated into a system. This transformation only depends on a choice of F , and for most purposes the only relevant feature of F is that it is a boundary defining function, well-behaved relative to the evenness statement. That is, if (X, g) is Riemannian and even, there is a product decomposition near the boundary Y_y of \overline{X} such that

$$g = \frac{dx^2 + \tilde{h}(x, y, dy)}{x^2},$$

with \tilde{h} even in x , i.e. $\tilde{h} = h(x^2, y, dy)$. Taking $F = x$, one modifies the system (3.1)

$$\tilde{P}_\sigma = \begin{bmatrix} -\Delta_X + \sigma^2 + \left(\frac{n-2k-1}{2}\right)^2 & -2d_X \\ 2\delta_X & -\Delta_X + \sigma^2 + \left(\frac{n-2k+3}{2}\right)^2 \end{bmatrix} \in \text{Diff}^2(X; \Lambda^k X \oplus \Lambda^{k-1} X)$$

to the operator

$$(3.2) \quad P_\sigma|_{X_{\text{even}}} = J^{-1} F^{i\sigma - (n-2k-1)/2 - 2} \tilde{P}_\sigma F^{-i\sigma + (n-2k-1)/2} J,$$

which one now checks is the restriction of an operator P_σ defined on an extension \tilde{X} of \bar{X}_{even} across Y , and satisfying the requirements of [20] and [21]. This was checked explicitly on functions in [21]. Note that at the level of the principal symbol, given by the dual metric *function* (times the identity operator), this means that $F^{-2}G$ extends smoothly to $T^*\tilde{X}$, which is automatic for an even asymptotically hyperbolic metric.

A different way of proceeding, which we pursue instead, is via extending the metric to an ambient metric, playing the role of the Minkowski metric, which is homogeneous of degree 2. Thus, one considers $M = \mathbb{R}_r^+ \times \tilde{X}$, as well as $\mathbb{R}_r^+ \times X$, with $r = F\rho$, $F = x$, although we note that with F defined above in the Minkowski setting, the hyperbolic metric has some higher order (in x) dx^2 terms in view of (2.1), which however do not affect properties of the extension. On $\mathbb{R}_r^+ \times X$ the analogue of the Minkowski metric is

$$\tilde{g} = dr^2 - r^2g = r^2 \left(\frac{dr^2}{r^2} - g \right) = \rho^2 \left(F^2 \left(\frac{d\rho}{\rho} + \frac{dF}{F} \right)^2 - F^2g \right).$$

Substituting the form of g and writing $F = x$, $F^2 = \mu$,

$$\tilde{g} = \rho^2 \left(\mu \frac{d\rho^2}{\rho^2} + \frac{1}{2} \left(\frac{d\rho}{\rho} \otimes d\mu + d\mu \otimes \frac{d\rho}{\rho} \right) - h(\mu, y, dy) \right).$$

But now the desired extension is immediate to a neighborhood of \bar{X}_{even} in \tilde{X} (which is all that is required for the analysis), by simply extending h smoothly to a neighborhood. This is easily checked to be Lorentzian (and as for this part forms are irrelevant, there is nothing to check beyond what was done in the scalar setting in [21]), with $d\mu$ time-like in $\mu < 0$, and now the Mellin transform gives rise to a smooth family of operators P_σ on \tilde{X} , related to \tilde{P}_σ via the same procedure as in the Minkowski setting. (In the scalar setting, this is a special case of metrics under study by Baskin, Wunsch and the author [1] as this paper was written, termed ‘Lorentzian scattering metrics’.) Since the requirements for the analysis involve the principal symbol for the Mellin transform (including in the high energy sense), which is the same as in the scalar setting (times the identity), namely the dual metric function on M , with σ being the Mellin-dual variable of $\rho\partial_\rho$, plus *some* bound on the subprincipal symbol at N^*Y as a bundle endomorphism (which is automatic by the compactness of Y), the results of [20] and [21] are now applicable. Note that the σ -dependence of the subprincipal symbol can be read off from the b-principal symbol of $\square_{\tilde{g}}$, so the issue is finding a σ -independent function (which, again, at most shifts by a constant what spaces should be used).

However, it is actually instructive to compute the subprincipal symbol at N^*Y . It turns out that this is a scalar bundle map on $\Lambda^k \tilde{X} \oplus \Lambda^{k-1} \tilde{X}$. First, the dual metric of \tilde{g} is

$$\tilde{G} = \frac{2}{\rho} (\partial_\mu \otimes \partial_\rho + \partial_\rho \otimes \partial_\mu) - \frac{4\mu}{\rho^2} \partial_\mu^2 - \rho^{-2} H,$$

with H the dual metric of h , and the metric density is

$$|d\tilde{g}| = \frac{\rho^n}{2} d\rho d\mu |dh| = \frac{\rho^n}{2} \sqrt{|\det h|} d\rho d\mu |dy|.$$

Next, writing k -forms on M as linear combinations of

$$dy^\alpha, d\mu \wedge dy^\beta, d\rho \wedge dy^\gamma, d\rho \wedge d\mu \wedge dy^\delta, |\alpha| = k, |\beta| = k - 1 = |\gamma|, |\delta| = k - 2,$$

one obtains that on k -forms

$$d = \begin{bmatrix} d_Y & 0 & 0 & 0 \\ \partial_\mu & -d_Y & 0 & 0 \\ \partial_\rho & 0 & -d_Y & 0 \\ 0 & \partial_\rho & -\partial_\mu & d_Y \end{bmatrix}$$

and thus, using the expression for d on $k-1$ -forms to compute its adjoint on k -forms,

$$(3.3) \quad \delta_k = \rho^{-2} \begin{bmatrix} \delta_Y & -4(\partial_\mu)_h^* + 2\rho(-\partial_\rho - \frac{n-2k+1}{\rho}) & 2\rho(\partial_\mu)_h^* & 0 \\ 0 & -\delta_Y & 0 & 2\rho(\partial_\mu)_h^* \\ 0 & 0 & -\delta_Y & 4\mu(\partial_\mu)_h^* - 2\rho(-\partial_\rho - \frac{n-2k+2}{\rho}) \\ 0 & 0 & 0 & \delta_Y \end{bmatrix},$$

where

$$(\partial_\mu)_h^* = -\frac{1}{\sqrt{\det h}} h \partial_\mu \sqrt{\det h} H.$$

This computation is analogous to the computation of $d_X \delta_X$ and $\delta_X d_X$ above, but is more complicated as one needs to work with a four-by-four system. It can again be done in steps, first computing the adjoint

$$d_{\text{base}}^* = \rho^{-n} (\det h)^{-1/2} \delta_{\mathbb{R}^n} (\det h)^{1/2} \rho^n$$

of d relative to the Euclidean inner product on the fibers of T^*M in local coordinates (ρ, μ, y) but with the actual metric density in the base, which is straightforward, and then computing $\delta = \tilde{G}^{-1} d_{\text{base}}^* \tilde{G}$, where \tilde{G} also stands for the dual metric on the form bundle, which is a block matrix of the form

$$\tilde{G}_k = \begin{bmatrix} \rho^{-2k} (-H)_k & 0 & 0 & 0 \\ 0 & -\frac{4\mu}{\rho^2} \rho^{-2(k-1)} (-H)_{k-1} & \frac{2}{\rho} \rho^{-2(k-1)} (-H)_{k-1} & 0 \\ 0 & \frac{2}{\rho} \rho^{-2(k-1)} (-H)_{k-1} & 0 & 0 \\ 0 & 0 & 0 & -\frac{4}{\rho^2} \rho^{-2(k-2)} (-H)_{k-2} \end{bmatrix}$$

on k -forms, where $(-H)_j$ is the inner product induced by $-H$ on j -forms on $Y = \partial X$. Note that the k -dependent powers of ρ arise from the degree of the form in the y -variables. Thus, $\delta_k = \tilde{G}_{k-1}^{-1} d_{\text{base}}^* \tilde{G}_k$ gives rise to (3.3).

Now one can compute $\Delta_k = d_{k-1} \delta_k + \delta_{k+1} d_k$ in a straightforward, if computationally slightly messy, manner. To state the result of the computation, it is convenient to rewrite k -forms on M as linear combinations of

$$dy^\alpha, d\mu \wedge dy^\beta, \frac{d\rho}{\rho} \wedge dy^\gamma, \frac{d\rho}{\rho} \wedge d\mu \wedge dy^\delta, |\alpha| = k, |\beta| = k - 1 = |\gamma|, |\delta| = k - 2.$$

Then one obtains that, with $\mathcal{V}_b(\tilde{X}; Y)$ denoting set of vector fields on \tilde{X} tangent to Y , $\text{Diff}_b^m(\tilde{X}; Y)$ denoting finite products up to m factors of these, and $\text{Diff}_b^m(\tilde{X}; Y; E)$ the corresponding operators acting on sections of a vector bundle E on \tilde{X} (with the action defined via trivialization as matrices of scalar operators),

$$\begin{aligned} \mathcal{M}_\rho \rho^2 \square_{\tilde{g}} \mathcal{M}_\rho^{-1} &= (4\partial_\mu \mu \partial_\mu - 4(\imath \tilde{\sigma} + (n - 2k - 1)/2) \partial_\mu) \otimes \text{Id} + \tilde{Q}, \\ \tilde{Q} &\in \text{Diff}_b^2(\tilde{X}; Y; \Lambda^k \tilde{X} \oplus \Lambda^{k-1} \tilde{X}), \end{aligned}$$

or

$$(3.4) \quad \begin{aligned} P_\sigma &= \mathcal{M}_\rho \rho^2 \rho^{(n-2k-1)/2} \square_{\tilde{g}} \rho^{-(n-2k-1)/2} \mathcal{M}_\rho^{-1} \\ &= (4\partial_\mu \mu \partial_\mu - 4i\sigma \partial_\mu) \otimes \text{Id} + Q, \quad Q \in \text{Diff}_b^2(\tilde{X}; Y; \Lambda^k \tilde{X} \oplus \Lambda^{k-1} \tilde{X}). \end{aligned}$$

This means that the spaces for Fredholm analysis, briefly recalled below from [21], are

$$\begin{aligned} P_\sigma &: \mathcal{X}^s \rightarrow \mathcal{Y}^{s-1}, \\ \mathcal{X}^s &= \{u \in H^s(\tilde{X}; \Lambda^k \tilde{X} \oplus \Lambda^{k-1} \tilde{X}) : P_\sigma u \in H^{s-1}(\tilde{X}; \Lambda^k \tilde{X} \oplus \Lambda^{k-1} \tilde{X})\}, \\ \mathcal{Y}^{s-1} &= H^{s-1}(\tilde{X}; \Lambda^k \tilde{X} \oplus \Lambda^{k-1} \tilde{X}), \quad s > -\text{Im } \sigma + 1/2, \end{aligned}$$

and elements of the distributional kernel of P_σ behave as $(\mu \pm i0)^{i\sigma}$; these are just outside the space \mathcal{X}^s when $s > -\text{Im } \sigma + 1/2$. As mentioned before, P_σ^{-1} , or rather $(P_\sigma - iQ_\sigma)^{-1}$, where Q_σ is the complex absorbing operator, is related to the resolvent family of Δ_X via the same procedure as in the Minkowski setting; no special properties of the Minkowski metric were used in the proof of (2.6)-(2.7).

4. ANALYSIS

We finally recall the analytic set-up from [20] and [21] to complete the picture. This brief discussion is not meant as a thorough introduction to the analytic methods of these papers; it merely recalls the key points, stating the analytically relevant properties that are satisfied. In this discussion P_σ is exactly the family of operators constructed in the previous section.

The key part of proving the theorem is obtaining estimates in a strip $|\text{Im } \sigma| < C$; working in this region means that even in the large parameter sense (i.e. when σ is large) the principal symbol of P_σ is a real scalar. More precisely, the general setup, satisfied by our operator P_σ , is that P_σ , of order m , has real scalar principal symbol even with σ as a large parameter (even if σ is complex but is in a strip, in the principal symbol sense it may be regarded real, and we often do so for convenience), i.e. the principal symbol is $p_\sigma \text{Id}$, with p_σ real valued. The classical principal symbol (without σ as a large parameter) is denoted by $p \text{Id}$, and p is assumed to be independent of σ . It is convenient to rescale the problem to a semiclassical one for large parameter issues, i.e. consider $P_{\hbar, z} = \hbar^2 P_{\hbar^{-1}z}$, with $|\text{Im } z| < Ch$, i.e. at the principal symbol level z is real; the semiclassical principal symbol is $p_{\hbar, z}$. We emphasize that p_σ and p are exactly the same as the symbols arising in [21], which is the reason all the hypotheses of the general setting are satisfied (when g is non-trapping, in the semiclassical sense as well) since they are shown in [21].

Next, we consider the characteristic set Σ of p ; in the general setting one assumes that this is a union of disjoint sets Σ_+, Σ_- , each of which is a union of connected components of Σ . Due to Hörmander's theorem [15], [6], one has real principal type propagation where \mathbf{H}_p is not *radial*, i.e. the Hamilton vector field \mathbf{H}_p is not a multiple of the radial vector field (the generator of dilations on the fibers of $T^*\tilde{X} \setminus o$). One assumes (though a more general setting is discussed in [20]; this is needed there since the conormal bundle of the event horizon in Kerr-de Sitter space is not radial, though it is an invariant Lagrangian submanifold) that the set of radial points of p , i.e. where \mathbf{H}_p is radial, is a union of conic Lagrangian submanifolds; in this case under a non-degeneracy assumption it is automatically a source or sink for the Hamilton flow within Σ_\pm .

At radial points, the basic theorem due to Melrose in asymptotically Euclidean scattering [19], proved in this generality by the author in [20], and refined by Haber and the author [14]. The result states that if one has a solution u of $P_\sigma u = f$, and u possesses a priori regularity beyond a threshold level at the radial set, then one has hyperbolic type estimates (loss of one derivative relative to elliptic estimates), i.e. u is $m - 1$ Sobolev orders more regular than f . Note that there is no need to assume that u has this $m - 1$ order improved regularity anywhere, unlike for real principal type propagation, where one can merely propagate such estimates. On the other hand, below a threshold level, one has the real principal type result in that without having to assume any regularity on u at the radial set, one can propagate regularity (up to, i.e. below, this threshold) from a punctured neighborhood of the radial set to the radial set, up to $m - 1$ order improved relative to f . Such results are local to each component of the radial set, and indeed can be localized even within the radial set, as shown in [14]. Here the threshold value is by no means mysterious; if P_σ is formally self-adjoint, it is $(m - 1)/2$. In general it is given by $(m - 1)/2$ plus the ratio of the imaginary (or skew-adjoint) part of the subprincipal symbol and the Hamilton vector field applied to the logarithm of a positive homogeneous degree one function evaluated at the Lagrangian when the subprincipal symbol is scalar (but possibly variable), if it is not scalar, one needs to take an operator bound of the skew-adjoint part of the subprincipal symbol as a self-adjoint operator. In the present case, the shift is $-\text{Im } \sigma$; as $m = 2$, this gives a threshold value of $1/2 - \text{Im } \sigma$.

Finally we introduce complex absorption. This is a pseudodifferential operator Q_σ , with real scalar principal symbol q , and one considers $P_\sigma - \iota Q_\sigma$. Here q and $q_{\bar{h},z}$ are supported away from the radial sets of p , and they are harmless in the elliptic set of p and $p_{\bar{h},z}$. In the real principal type region Q breaks down the symmetry of the propagation estimates (forward vs. backwards); for $q \geq 0$ one can propagate estimates forwards, for $q \leq 0$ backwards. Of course, adding Q changes the operator, so we want Q to be supported outside the region we care about (such as \bar{X}_{even} above).

Now, in order to have a Fredholm problem we need that all bicharacteristics of p in Σ_\pm are non-trapped, i.e. that they escape both in the forward and in the backward directions to locations where they can be controlled, i.e. either they enter $\{q \neq 0\}$ in finite time, or they tend to Λ_\pm . More concretely, if we label Σ_\pm so that Λ_+ is a source and Λ_- is a sink, which is the labelling of [21] (and the opposite of the labelling of [20]) then we require that each bicharacteristic in $\Sigma_+ \setminus \Lambda_+$ either tends to Λ_+ or enters $\{q > 0\}$ in finite time in the backward direction, and enters $\{q > 0\}$ in finite time in the forward direction, while each bicharacteristic in $\Sigma_- \setminus \Lambda_-$ either tends to Λ_- or enters $\{q < 0\}$ in finite time in the forward direction, and enters $\{q < 0\}$ in finite time in the backward direction. (Note that in $\{q > 0\}$ and $\{q < 0\}$ the requirements are automatically satisfied!) Thus, in high regularity spaces (with s bigger than a threshold) we can propagate estimates away from $\Lambda_+ \cup \Lambda_-$ (and towards the support of the complex absorption), while in the low regularity spaces we can proceed in the opposite direction.

Thus, if s is greater than the threshold value at Λ_+ and Λ_- , then one can propagate regularity and estimates from $\Lambda_+ \cup \Lambda_-$ to $\{q > 0\} \cup \{q < 0\}$. For the adjoint operator under these assumptions one has a similar result if one works with low regularity spaces, namely if one replaces s by $-s + (m - 1)$, which is exactly the relevant space for duality arguments; one then propagates the estimates *in the*

opposite direction. Concretely, one has estimates

$$\|u\|_{H^s} \leq C(\|(P_\sigma - \imath Q_\sigma)u\|_{H^{s-m+1}} + \|u\|_{H^{-N}})$$

and

$$\|u\|_{H^{-s+m-1}} \leq C(\|(P_\sigma^* + \imath Q_\sigma^*)u\|_{H^{-s}} + \|u\|_{H^{-N'}})$$

for appropriate N, N' with compact inclusion into the spaces on the right hand side, yielding that, with

$$\mathcal{Y}^s = H^s, \quad \mathcal{X}^s = \{u \in H^s : (P_\sigma - \imath Q_\sigma)u \in H^{s-m+1}\}$$

(note that the last statement in the definition of \mathcal{X}^s depends on the principal symbol of $P_\sigma - \imath Q_\sigma$ only, which is independent of σ),

$$P_\sigma : \mathcal{X}^s \rightarrow \mathcal{Y}^{s-m+1}, \quad P_\sigma^* : \mathcal{X}^{-s+m+1} \rightarrow \mathcal{Y}^{-s}$$

are Fredholm. Further, if $P_\sigma - \imath Q_\sigma$ depends holomorphically on σ (for σ in an open subset of \mathbb{C}), then $P_\sigma - \imath Q_\sigma$ is a holomorphic Fredholm family, while $P_\sigma^* + \imath Q_\sigma^*$ is antiholomorphic. Note also that if $P_\sigma - \imath Q_\sigma$ is invertible (or if simply $u \in \mathcal{X}^s$, $f \in \mathcal{Y}^{s-m+1}$, $P_\sigma u = f$), and $\text{WF}(f)$ is disjoint from Λ_\pm then $\text{WF}(P_\sigma^{-1}f)$ is also disjoint from this Lagrangian. Further, if f is \mathcal{C}^∞ , then $P_\sigma^{-1}f$ is also \mathcal{C}^∞ . For the adjoint, corresponding to propagation in the opposite direction, we have $\text{WF}((P_\sigma^*)^{-1}f) \subset \Lambda_+ \cup \Lambda_-$ when f is \mathcal{C}^∞ .

For the semiclassical problem, a natural assumption is *non-trapping*, i.e. all semiclassical bicharacteristics in Σ_\pm apart from those in the radial sets, in $\Sigma_{\hbar,\pm}$ are required to tend to $L_\mp \cup \{\pm q_{\hbar,z} > 0\}$ in the forward direction and $L_\pm \cup \{\pm q_{\hbar,z} > 0\}$ in the backward direction. Here L_\pm is the image of Λ_\pm in $S^*\tilde{X}$ under the quotient map, and one considers $S^*\tilde{X}$ as the boundary of the radial compactification of the fibers of $T^*\tilde{X}$. Under this assumption, one has non-trapping semiclassical estimates (analogues of hyperbolic estimates, i.e. with a loss of \hbar relative to elliptic estimates). This in particular proves that for small \hbar the operator is invertible (not just Fredholm), and thus the non-semiclassical Fredholm family has a meromorphic inverse with finite rank poles.

This completes the description of the analytic ingredients in the non-trapping setting, which are satisfied in view of p_σ and p being as in [21], and the subprincipal symbol being also the same at N^*Y as shown in (3.4), proving Theorem 1.1 and Corollary 1.2.

We refer to [20, Section 2, Definition 2.18] for *semiclassical mildly trapping* assumptions. These roughly state that there is a compact subset K of T^*X (the ‘trapped set’), a neighborhood O of K and a convex function F on T^*X which is ≥ 2 on K and 1 outside O , and if one adds a complex absorption \tilde{Q}_σ which vanishes near K but is elliptic outside O , then $(P_{\hbar,z} - \imath\tilde{Q}_{\hbar,z})^{-1}$ satisfies polynomial bounds, $Ch^{-\varkappa-1}$, in $\text{Im } z > -C_0\hbar$ (i.e. in a strip without the semiclassical rescaling), and such that the bicharacteristics of $p_{\hbar,z}$ are non-trapped once one regards O as non-trapped, i.e. entering O in finite time is regarded as good as entering $\{\pm q > 0\}$ in finite time. Due to the gluing construction of [4], semiclassical mildly trapping can be immediately be combined with the analysis developed for non-trapping P_σ , see [20, Theorem 2.19], roughly by placing complex absorption near K but inside O to obtain a non-trapping ‘exterior’ model, which can be glued with the ‘interior’ model $(P_{\hbar,z} - \imath\tilde{Q}_{\hbar,z})^{-1}$.

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