SCATTERING POLES FOR NEGATIVE POTENTIALS

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1. Introduction

Let $V \in L^\infty_c(\mathbb{R}^n)$ be a real-valued potential, $n \geq 3$ odd. The scattering matrix, $S_V(\lambda)$, corresponding to V extends to be a meromorphic function in $\mathbb{C}$. Our normalization is that $\mathcal{P}$, the physical half plane, is the open lower half plane, $\mathbb{C}^-$ (cf. [2]). Thus, $S_V(\lambda)$ only has a finite number of poles in $\mathcal{P}$ and they correspond to the bound states of the Hamiltonian $\Delta + V$ (Δ is the positive Laplacian). In [5] Zworski has shown that the number of scattering poles $n(r)$ in a disc of radius r is bounded by $C(r+1)^n$, and in [6] he has given similar lower bounds for $n(r)$ for certain radial potentials. The question of the existence of pure imaginary scattering poles was investigated by Lax and Phillips in [1] for both obstacle and potential scattering. In the case of obstacle scattering they showed that the number $N(s)$ of pure imaginary scattering poles of absolute value less than s satisfies $C_1 s^2 \leq N(s)$ for $n = 3$ for some constant $C_1 > 0$, and there is an analogous upper bound if the obstacle is star-shaped. They also proved that much of the machinery can be translated into the setting of potential scattering by non-negative potentials. This reduces the problem of obtaining lower bounds for $N(s)$ to finding the corresponding bounds when V is the characteristic function of a ball. In this paper we make the simple observation that their proof can be modified to accommodate non-positive potentials, and we prove that if V or $-V$ is bounded below by a positive multiple of the characteristic function of a ball, then for some $C, C' > 0$

$$N(s) \geq C s^{n-1} - C'. \tag{1.1}$$

2. Scattering poles for potentials with a fixed sign

Let $V \in L^\infty(B(R))$ be real-valued; $B(R) \subset \mathbb{R}^n$ the ball of radius R centered at 0, $n \geq 3$ odd. The (modified) resolvent $R_V(\lambda) = (\Delta + V - \lambda^2)^{-1}$ of the perturbed Laplacian ($\operatorname{Im} \lambda < 0$) extends to a meromorphic family of operators in the complex plane. More precisely, given $T \geq 0$

$$R_0(\lambda) : L^2(B(R)) \to e^{T|z|} L^2(\mathbb{R}^n)$$

is a holomorphic family of bounded operators in $\operatorname{Im} \lambda < T$;

$$V R_0(\lambda) : L^2(B(R)) \to L^2(B(R))$$

is compact, and by analytic Fredholm theory

$$R_V(\lambda) = R_0(\lambda)(\operatorname{Id} + V R_0(\lambda))^{-1}$$

is meromorphic in $\operatorname{Im}\lambda < T$ with finite rank poles. For $\lambda \in \mathbb{R}\setminus\{0\}$ $\Phi_V(\lambda,.,\omega)$ is the unique solution of

$$(\Delta + V - \lambda^2)\Phi_V(\lambda,.,\omega) = 0$$

which is of the form $\Phi_V(\lambda,z,\omega) = e^{i\lambda z\cdot\omega} + e^{-i\lambda|z|}|z|^{-(n-1)/2}\phi_V(\lambda,z,\omega)$ with

$$|\phi_V(\lambda,|z|\theta,\omega) - \tilde{\phi}_V(\lambda,\theta,\omega)| < C|z|^{-1}$$

for some $\tilde{\phi}_V$ and C. We call $\tilde{\phi}_V$ the boundary value of ϕ at infinity; this terminology arises naturally if we consider the radial compactification of $\mathbb{R}^n$ via the stereographic projection (see [2]). The scattering matrix $S_V(\lambda)$ is then the bounded operator on $L^2(\mathbb{S}^{n-1})$ of the form $\mathrm{Id} + S'_V(\lambda)$ where the kernel of $S'_V(\lambda)$ is

$$k(\lambda,-\theta,\omega) = (\lambda/(2\pi i))^{(n-1)/2}\tilde{\phi}_V(\lambda,-\theta,\omega).$$

For $\lambda \in \mathbb{R}\setminus\{0\}$ we have $S_V(-\lambda) = S_V(\lambda)^{-1}$. The asymptotic expansion of $R_0(\lambda)f(z)$ for $f \in L^\infty_c(\mathbb{R}^n)$ actually gives that

$$k(\lambda,\theta,\omega) = i2^{-1}\lambda^{n-2}(2\pi)^{-(n-1)}\mathcal{F}((\mathrm{Id} + VR_0(\lambda))^{-1}(V(.)e^{i\lambda\omega\cdot.}))(-\lambda\theta)$$

where $\mathcal{F}$ is the Fourier transform. This can be used to extend the $S'_V(\lambda)$ to a meromorphic family of compact operators on $L^2(\mathbb{S}^{n-1})$; the extension still satisfies $S_V(-\lambda) = S_V(\lambda)^{-1}$. The poles of $S_V(\lambda)$ are thus a subset of the poles of $(\mathrm{Id} + VR_0(\lambda))^{-1}$ (with multiplicities). In particular, there are no poles in some lower half plane. The kernel of the analytic continuation of the $S'_V(\lambda)$ along the negative imaginary axis becomes (with $\sigma > 0$)

$$\begin{aligned} k(-i\sigma,\theta,\omega) &= (-1)^{(n-3)/2}2^{-1}\sigma^{n-2}(2\pi)^{-n+1} \\ &\qquad \int_{\mathbb{R}^n} e^{\sigma\theta\cdot z}(\mathrm{Id} + VR_0(-i\sigma))^{-1}(V(.)e^{\sigma\omega\cdot.})(z)\,dz. \end{aligned} \tag{2.1}$$

Now we restrict our attention to potentials with a fixed sign. In this case eigenvalue estimates of the scattering matrix can be reduced to cases where the eigenvalues can be calculated explicitly. Thus, we can obtain lower bounds for the number of pure imaginary scattering poles for these potentials. We shall employ the technique developed by Lax and Phillips in [1]. There they prove a form of the following proposition. We repeat the proof for the convenience of the reader.

Proposition 2.1. *Suppose that $n \equiv 1 \bmod 4$, $V \in L^\infty_c(\mathbb{R}^n)$, $V \geq 0$. Then the operator $k(-i\sigma)$ with kernel $k(-i\sigma,\theta,\omega)$ is self-adjoint, negative and a decreasing function of V for $\sigma > 0$. If $V \leq 0$ then there exists $\sigma_0 > 0$ such that for $\sigma \geq \sigma_0$ $k(-i\sigma)$ is self-adjoint, positive and an increasing function of $-V$. If $n \equiv 3 \bmod 4$ then the previous claims hold if increasing and decreasing, and positive and negative are respectively interchanged.*

Proof. Multiplying each side of (2.1) by $\overline{a_1(\theta)}a_2(\omega)$ ($a_j \in L^2(\mathbb{S}^{n-1})$, $j = 1,2$) and integrating with respect to both spherical variables gives (after interchanging the order of integration) that

$$(a_1, k(-i\sigma)a_2)_{L^2(\mathbb{S}^{n-1})} = c_n(\sigma)(A_1, (\mathrm{Id} + VR_0(-i\sigma))^{-1}VA_2)_{L^2(B(R))}$$

where

$$A_j(z) = \int_{\mathbb{S}^{n-1}} e^{\sigma\theta\cdot z}a_j(\theta)\,d\theta,$$

$$c_n(\sigma) = (-1)^{(n-3)/2}2^{-1}\sigma^{n-2}(2\pi)^{-(n-1)}.$$

Hence, we only need the following lemma. $\square$

Lemma 2.2. *Suppose that $V \in L^\infty(B(R))$, $V \geq 0$, $R > 0$. Then*

$$(\mathrm{Id} + VR_0(-i\sigma))^{-1}V \in \mathcal{B}(L^2(B(R)))$$

is positive, self-adjoint, and an increasing function of V. If $V \leq 0$ then there exists $\sigma_0 > 0$ such that for $\sigma \geq \sigma_0$ $(\mathrm{Id} + VR_0(-i\sigma))^{-1}V$ is negative, self-adjoint, and a decreasing function of $-V$.

Proof. Suppose first that V is bounded below by a positive constant on $B(R)$; then

$$(\mathrm{Id} + VR_0(-i\sigma))^{-1}V = (V^{-1} + R_0(-i\sigma))^{-1}.$$

Since $\Delta + \sigma^2$ is self-adjoint and positive, so is $R_0(-i\sigma)$. This gives the self-adjointness of the lemma. Monotonicity also follows since inversion reverses the order of positive operators.

In the general case for nonnegative V (V not necessarily bounded below by a positive constant) consider the family of potentials $V_\epsilon = V + \epsilon$, $\epsilon > 0$. In the operator topology $V_\epsilon \to V$, $(\mathrm{Id} + V_\epsilon R_0(-i\sigma))^{-1} \to (\mathrm{Id} + VR_0(-i\sigma))^{-1}$ as $\epsilon \to 0$. This proves the lemma for $V \geq 0$.

Suppose now that V is bounded from above by a negative constant. If we choose σ_0 such that for $\sigma \geq \sigma_0$ $V^{-1} + R_0(-i\sigma)$ is negative (which is certainly possible since the norm of $R_0(-i\sigma)$ goes to 0 as $\sigma \to \infty$) then a proof similar to the one for positive V shows the lemma. Note that σ_0 can be chosen so that it only depends on $\inf |V^{-1}|$, i.e. on $\sup |V|$. Hence, the general case is similar to the non-negative one. $\square$

Let $\mathcal{C}$ denote the set of potentials $V \in L^\infty_c(\mathbb{R}^n)$ which are bounded below by a positive multiple of the characteristic function of a ball (and hence they are non-negative). If $V \in \mathcal{C}$ then $k(-i\sigma)$ is strictly positive or negative depending on n mod 4. In fact, this follows from Proposition 2.1 and the strict positivity (or negativity) for potentials which are a constant (nonzero) multiple of the characteristic function of a ball; this second fact follows from the explicit formula (3.6). But $S_V(-i\sigma) - \mathrm{Id}$ is $Wk(-i\sigma)$ where W is the symmetric involution $(Wa)(\theta) = a(-\theta)$. Therefore we can apply a slightly modified version of Theorem 4.4 of Lax and Phillips [1] to deduce that the eigenvalues of $S_V(-i\sigma) - \mathrm{Id}$ are real, and the positive ones are an increasing, while the negative ones are a decreasing, function of V. If $-V \in \mathcal{C}$ then we deduce the same result for $\sigma \geq \sigma_0$.

Let $N(s)$ denote the number of pure imaginary poles of $S_V(\lambda)$ of absolute value less than s, i.e. the number of poles $i\sigma$, $\sigma \in \mathbb{R}$, with $|\sigma| \leq s$. Using the estimates of the next section (which give the corresponding statement for potentials which are multiples of the $\chi(B(1))$) we can now show the main theorem.

Theorem 2.3. *Suppose $V \in \mathcal{C}$ or $-V \in \mathcal{C}$. Then there exist constants $C, C' > 0$ such that $N(s) \geq Cs^{n-1} - C'$.*

Proof. Corollary 3.3 together with a simple scaling argument (as in [6]) shows that if $V = V_0\chi(B(r))$, $r > 0$, $V_0 \in \mathbb{R} \setminus \{0\}$, then there exist $C, C' > 0$ such that $S_V(-i\sigma) - \mathrm{Id}$ has at least $C\sigma^{n-1} - C'$ eigenvalues which are less than -1. The monotone dependence of the scattering matrix on the potential shows that the same result holds for $\pm V \in \mathcal{C}$ (where, of course, C, C' depend on V). Fix $\sigma_0 > 0$. Since $S_V(-i\sigma) - \mathrm{Id}$ is compact, it has only a finite number of eigenvalues less than -1. But the eigenvalues depend continuously on σ, so $S_V(-i\sigma)$ has at least

$Cs^{n-1} - C' - C_0$ zeros between $-i\sigma_0$ and $-is$. Since $S_V(\lambda)^{-1} = S_V(-\lambda)$, this proves the theorem. $\square$

3. Estimates for $V = V_0\chi(B(1))$

We now obtain the eigenvalue estimates of $k(-i\sigma)$ if $V = V_0\chi(B(1))$ is a non-zero multiple of the characteristic function of the unit ball. The Bessel functions $(J_\nu,\ I_\nu,\ K_\nu)$ and the Gegenbauer polynomials, C_m^ν, will play a crucial role. Let $l = (n-2)/2$,

$$a_m = I_{m+l}(\sqrt{\sigma^2 + V_0}), \qquad b_m = (2\sigma/\pi)^{1/2} K_{m+l}(\sigma),$$

$$a'_m = \sqrt{\sigma^2 + V_0}\, I'_{m+l}(\sqrt{\sigma^2 + V_0}), \qquad b'_m = \sigma(2\sigma/\pi)^{1/2} K'_{m+l}(\sigma),$$

$$\alpha_m = 2^l\Gamma(l)(m+l)\sigma^{-l} I_{m+l}(\sigma), \qquad \beta_m = 2^l\Gamma(l)(m+l)\sigma^{-l+1} I'_{m+l}(\sigma).$$

Thus, (see [4, Section 11.5])

$$e^{\sigma t} = \sum_{m\geq 0} \alpha_m C_m^l(t), \qquad \sigma^{-l+1}\partial_\sigma(\sigma^l t e^{\sigma t}) = \sum_{m\geq 0} \beta_m C_m^l(t).$$

We can now calculate $\tilde{\phi}_V$.

Lemma 3.1. *For the potential $V = V_0\chi(B(1))$, $V_0 \in \mathbb{R}$,*

$$\tilde{\phi}_V(-i\sigma, \theta, \omega) = \sum_{m\geq 0} c_m(\sigma) C_m^l(\theta\cdot\omega) \tag{3.1}$$

where

$$c_m = c_m(\sigma) = \frac{-a'_m\alpha_m + a_m\beta_m}{-a_m b'_m + a'_m b_m}. \tag{3.2}$$

Proof. The Bessel functions have the following asymptotic expansion for large s and fixed $\nu \in \mathbb{R}$ (see [4]):

$$J_\nu(s) \sim (2/\pi s)^{1/2}(\cos(s - (\nu\pi)/2 - \pi/4) + \mathcal{O}(s^{-1})).$$

Using this, the spherical symmetry of the potential together with the defining properties of Φ_V we see that for $r > 1$ (where the potential vanishes), $\lambda > 0$, Φ_V can be expanded as

$$\begin{aligned}\Phi_V(-\lambda, r\theta, \omega) = e^{-i\lambda r\theta\cdot\omega} + e^{i\pi/4}(\pi\lambda/2)^{1/2} r^{-l} \sum_{m\geq 0} c_m [e^{i\pi(m+l)/2} J_{m+l}(\lambda r) \\ + e^{-i\pi(m+l)/2} J_{-m-l}(\lambda r)] C_m^l(\theta\cdot\omega).\end{aligned} \tag{3.3}$$

Here the constants were chosen to insure that for $\sigma > 0$

$$\tilde{\phi}_V(-i\sigma, \theta, \omega) = \sum_{m\geq 0} c_m C_m^l(\theta\cdot\omega)$$

holds. The analytic continuation of Φ_V in λ and the relationships between the various Bessel functions imply that for $\sigma > 0$

$$\Phi_V(-i\sigma, r\theta, \omega) = e^{\sigma r\theta\cdot\omega} + (2\sigma/\pi)^{1/2} r^{-l} \sum_{m\geq 0} c_m K_{m+l}(\sigma r) C_m^l(\theta\cdot\omega).$$

For $r < 1$ $\Phi_V(-i\sigma, z, \omega)$ is $\mathcal{C}^\infty$ in z, so we can expand it as

$$\Phi_V(-i\sigma, r\theta, \omega) = r^{-l} \sum_{m\geq 0} c'_m I_{m+l}(\sqrt{\sigma^2 + V_0}\, r) C_m^l(\theta\cdot\omega).$$

As $(\Delta + V_0\chi(B(1)) + \sigma^2)\Phi_V(-i\sigma, ., \omega) = 0$, Φ_V is $\mathcal{C}^1$ in $\mathbb{R}^n$, so matching Φ_V and $\partial_r(r^l\Phi_V)$ at $r = 1$ shows that

$$c'_m a_m = \alpha_m + c_m b_m,$$

$$c'_m a'_m = \beta_m + c_m b'_m.$$

Hence (3.2) follows. $\square$

We shall now estimate c_m. We shall use the following asymptotic expansions of Bessel functions of large order which were obtained by Olver [3]. Here $z > 0$, $k > 0$, and the expansions are uniform in $z \in [\epsilon, \infty)$ for any $\epsilon > 0$:

$$K_k(kz) \sim (\pi/2k)^{1/2} e^{-k\zeta}(1+z^2)^{-1/4}(1 - k^{-1}U_1 + \mathcal{O}(k^{-2})),$$

$$I_k(kz) \sim (2\pi k)^{-1/2} e^{k\zeta}(1+z^2)^{-1/4}(1 + k^{-1}U_1 + \mathcal{O}(k^{-2})),$$

$$K'_k(kz) \sim -(\pi/2k)^{1/2} e^{-k\zeta}(1+z^2)^{1/4} z^{-1}(1 - k^{-1}V_1 + \mathcal{O}(k^{-2})),$$

$$I'_k(kz) \sim (2\pi k)^{-1/2} e^{k\zeta}(1+z^2)^{1/4} z^{-1}(1 + k^{-1}V_1 + \mathcal{O}(k^{-2}))$$

where $\zeta = \zeta(z) = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}$, $u = (1+z^2)^{-1/2}$, $U_1 = \frac{u}{8} - \frac{5u^3}{24}$, $V_1 = -\frac{3u}{8} + \frac{7u^3}{24}$. Note that $\zeta = \zeta(z)$ has a unique zero, z_0, in $(0, \infty)$, and for $z > z_0$ we have $\zeta > 0$. Let $k = m + l$, $z = \sigma/k$. We then obtain the asymptotic behavior of c_{k-l}.

Proposition 3.2. *With $c_m = c_m(\sigma)$ as in (3.2) we have*

$$c_{k-l}(kz) \sim -\frac{2^{l-5/2}\Gamma(l) V_0 \pi^{1/2} e^{2k\zeta}}{k^{l+3/2} z^{l+1/2}(1+z^2)}(1 + \mathcal{O}(k^{-1}))$$

uniformly in $z \in [\epsilon, \infty)$ for all $\epsilon > 0$.

Proof. Let $z' = \sqrt{\sigma^2 + V_0}/k$, so

$$z' \sim z(1 + V_0/(2k^2z^2) + \mathcal{O}(k^{-4})).$$

The asymptotic expansions of the Bessel functions give at once that as $k \to \infty$

$$-a_{k-l}b'_{k-l} + a'_{k-l}b_{k-l} \sim (2kz/\pi)^{1/2}(1 + \mathcal{O}(k^{-1}))$$

uniformly for $z \in [\epsilon, \infty)$. The numerator A_m of (3.2) is only slightly more complicated. Since $\Phi_0(-i\sigma, z, \omega) = e^{\sigma z\cdot\omega}$, for the zero potential $c_m = 0$. Let $\tilde{a}_m$ and $\tilde{a}'_m$ denote a_m and a'_m respectively in this case. Then, for any $V_0 \in \mathbb{R}$ we have

$$A_m = -(a'_m - \tilde{a}'_m)\alpha_m + (a_m - \tilde{a}_m)\beta_m.$$

Now, for $s \in \mathbb{R}$ we have

$$(1 + (z')^2)^s \sim (1+z^2)^s(1 + V_0 s/(k^2(1+z^2)) + \mathcal{O}(k^{-4})),$$

$$\zeta' \sim \zeta + V_0\xi/(2k^2) + \mathcal{O}(k^{-4})$$

where $\zeta' = \zeta(z')$ and $\xi = z^{-2}(1+z^2)^{1/2}$. Moreover,

$$e^{k\zeta'} \sim e^{k\zeta}(1 + V_0\xi/(2k) + V_0^2\xi^2/(8k^2) + \mathcal{O}(k^{-3})).$$

Hence,

$$
\begin{aligned}
a_{k-l} - \tilde{a}_{k-l} \sim & \frac{e^{k\zeta}}{\sqrt{2\pi k}k(1+z^2)^{1/4}}\Big(\frac{V_0\xi}{2} \\
& + k^{-1}\left(\frac{V_0^2\xi^2}{8} - \frac{V_0}{4(1+z^2)} + \frac{U_1 V_0 \xi}{2}\right) + \mathcal{O}(k^{-2})\Big),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
a'_{k-l} - \tilde{a}'_{k-l} \sim & (1+z^2)^{1/4}\frac{e^{k\zeta}}{\sqrt{2\pi k}}\Big(\frac{V_0\xi}{2} \\
& + k^{-1}\left(\frac{V_0^2\xi^2}{8} + \frac{V_0}{4(1+z^2)} + \frac{V_1 V_0 \xi}{2}\right) + \mathcal{O}(k^{-2})\Big).
\end{aligned}
\tag{3.5}
$$

Substituting the expansions of I_k into α_{k-l} and β_{k-l} gives after some cancellations that

$$
A_{k-l} \sim -2^l \Gamma(l) \frac{V_0 e^{2k\zeta}}{4\pi k^{l+1} z^l (1+z^2)} (1 + \mathcal{O}(k^{-1})).
$$

Hence the proposition follows. ∎

Corollary 3.3. *For the potential $V = V_0 \chi(B(1))$, $V_0 \in \mathbb{R}\backslash\{0\}$ there exist constants C, $C' > 0$ such that $S_V(-i\sigma) - \mathrm{Id}$ has at least $C\sigma^{n-1} - C'$ eigenvalues which are less than -1.*

Proof. First of all, we check the action of $S_V(-i\sigma) - \mathrm{Id}$ on the spherical harmonics. Let Y^p be a spherical harmonic of degree p. The Gegenbauer polynomials are closely related to the zonal harmonics; namely

$$
Z_m^{n-1}(\omega, \theta) = \gamma_n (m+l) C_m^l(\omega \cdot \theta)
$$

with $\gamma_n = (n-4)(n-6)...1[(n-3)(n-5)...2]^{-1}$. Hence,

$$
\begin{aligned}
(S_V(-i\sigma) - \mathrm{Id})Y^p(\theta) = & (-\sigma/(2\pi))^{(n-1)/2} \\
& \sum_{m\geq 0} c_m \int_{\mathbb{S}^{n-1}} C_m^l(-\theta\cdot\omega) Y^p(\omega)\, d\omega \\
= & (-\sigma/(2\pi))^{(n-1)/2} \gamma_n^{-1} (p+l)^{-1} c_p Y^p(-\theta) \\
= & (-1)^p (-\sigma/(2\pi))^{(n-1)/2} \gamma_n^{-1} (p+l)^{-1} c_p Y^p(\theta).
\end{aligned}
\tag{3.6}
$$

The spherical harmonics of degree p are therefore eigenfunctions of $S_V(-i\sigma) - \mathrm{Id}$ with eigenvalue

$$
\mu_p(\sigma) = (-1)^p (-\sigma/2\pi)^{(n-1)/2} \gamma_n^{-1} (p+l)^{-1} c_p.
$$

The uniform result of Proposition 3.2 shows that for $z_1 > z_0$ there exist $N \in \mathbb{N}$, $C > 0$ such that if $p \geq N$, $z \geq z_1$ then

$$
(-1)^{p+(n-3)/2} \operatorname{sign}(V_0) \mu_p((p+l)z) \geq C e^{(p+l)\zeta(z)}.
$$

But the right hand side is certainly greater than 1 for $p \geq N'$, $z \geq z_1$ (for some $N' \geq N$). Thus,

$$
(-1)^{p+(n-3)/2} \operatorname{sign}(V_0) \mu_p(\sigma) > 1 \tag{3.7}
$$

if $N' \leq p \leq z_1^{-1}\sigma$, i.e. there exist constants $C_1, C_2 > 0$ such that (3.7) holds for at least $C_1\sigma - C_2$ values of p, and this also holds if we restrict p to be even or to be odd. If $(-1)^{(n-3)/2}\operatorname{sign}(V_0) = 1$, take p to be odd, otherwise even. Taking into account the dimension of the space of spherical harmonics of degree p (which is a polynomial of degree $n-2$ in p) we deduce the claim of the corollary. $\square$

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