INVERSE PROBLEMS IN N-BODY SCATTERING

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Scattering theory is the analysis of the motion of several interacting particles. Inverse problems in scattering theory seek the answer to the question: can one determine the interactions between particles by a scattering experiment? That is, if one shoots a number of particles at each other in an accelarator and observes the outcome, can one find out how the particles interact? In this note we attempt to explain this problem and the existing results, and we also sketch the proof of the extension of our recent three-body result to the many-body case: this requires only minor modifications. We also indicate how to extend the inverse result of [30] to the many-body case.

Before stating many-body results, we recall the simpler two-body setting, where (after removal of the center of mass) one studies the Hamiltonian $\Delta + V$ on $\mathbb{R}^n$ for some n, where Δ is the positive Laplacian on $\mathbb{R}^n$, and V is a function on $\mathbb{R}^n$ that decays at infinity. Thus, V can be considered a perturbation of Δ, and hence a relatively simple analysis of scattering is possible. We recall that at high energies V can be considered a perturbation in an even stronger sense. Roughly, at high energies, V is not only relatively compact, but also relatively small (compared to Δ), so one can use the Born approximation to study inverse scattering. In particular, it is easy to recover V from the high energy ($\lambda \to +\infty$) asymptotics of the scattering matrix $\mathcal{S}(\lambda)$ – this is the object that captures the relationship between the initial ($t \to -\infty$) and final ($t \to +\infty$) state of the particles of energy λ, and which is described below. If one only knows the scattering matrix at a fixed energy λ, it is still possible to recover V, at least for compactly supported, or indeed exponentially decaying, potentials [26, 20, 19, 29]. For short-range potentials that are symbols, it is possible to recover the *asymptotics* of the potential at infinity using fixed energy information only, see [12]. For Schwartz potentials V (whose asymptotics is thus trivial) there are counterexamples to the fixed energy problem [8], namely there are 'transparent' potentials, but it would be interesting to know whether the S-matrix given in a finite interval determines the potential, since this would have direct applications in many-body scattering, see Theorem 1.

In fact, there are many inverse problems that one can study. If there are more than two particles, they may have a number of initial (or incoming) and final (or outgoing) configurations. For example, one can shoot N electrons at each other, let them interact, and observe the outcome. Since the electrons repel each other, we will have N (asymptotically) free particles as the outcome. This is N-cluster to N-cluster scattering: there were N clusters of particles (each consisting of a single electron) both initially and finally.

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In a physically much more relevant way, one may shoot an electron at a target which is a composite particle, for example an atom or ion. If the energy of the electron is high enough, the ion may break up e.g. a helium ion may break up into the nucleus and an electron. If the atom/ion breaks up into k parts, this is two-cluster to $k+1$-cluster scattering: initially there were two clusters, namely the electron and the ion, and finally there are $k+1$: the k parts of the atom/ion and the initial electron. For the helium ion, if the ion breaks up into the nucleus and an electron, $k=2$, so this is two-cluster to three-cluster scattering. Or, the atom/ion may not break up, though it may become excited, and eventually one will have an atom/ion and an electron, just as initially. This is two-cluster to two-cluster scattering. If in addition the ion is in the same final and initial states, the problem may be instead considered as a two-body problem, with the ion and the electron comprising the two (now composite) particles; indeed, this is the usual way one thinks of such an experiment physically, at least at low energies. A natural question is then whether this is an accurate description, i.e. whether the more complete three-particle analysis gives the same predictions as the simplified two-particle approach.

Another classification of inverse problems is by the energy range in which the scattering data are known. High energy problems have been extensively studied by Enss and Weder [3, 4], Novikov [21] and Wang [36, 35]. Namely, as mentioned above, at high energies the potential (the interaction) is small compared to the Laplacian (the kinetic energy), in an appropriate sense. Thus, one can use the Born approximation to study inverse problems and recover information about the interactions. In this note we discuss fixed energy problems, and problems where a component of the scattering matrix is known in a bounded interval of energies.

We first recall that the actual N-body Hamiltonian for d-dimensional particles takes the following form:

$$H = \sum_{i=1}^{N} \frac{\hbar^2}{2m_i}\Delta_{q_i} + \sum_{i<j} V_{ij}(q_i - q_j), \tag{1}$$

where $q_i \in \mathbb{R}^d$ is the position and m_i is the mass of particle i, while V_{ij} is the interaction between particles i and j, and $\hbar$ is Planck's constant. Note that H is an operator on (functions on) $\mathbb{R}^{Nd}$. The quantum N-body problem is then the analysis of the Schrödinger equation $i\hbar\partial_t u = Hu$, i.e. of the propagator e^{-iHt}, as $t \to \pm\infty$, or equivalently, by taking the t-Fourier transform, of tempered distributional solutions of $(H-\lambda)u = 0$.

It is convenient to generalize the framework, introducing some notation due to Agmon and Sigal; see [2] for a detailed discussion of the setup. We work on a vector space X_0 endowed with a translation invariant metric g; by introduction of an orthonormal basis we may of course identify it with $\mathbb{R}^n$, as we usually do below. Let $\Delta \geq 0$ be the positive Laplacian of g. We are also given a finite collection $\mathcal{X} = \{X_a : a \in I\}$ of linear subspaces X_a of $X_0 = \mathbb{R}^n$, called the collision planes. It is convenient to assume that $\mathcal{X}$ is closed under intersections, and includes $X_0 = \mathbb{R}^n$ and $X_1 = \{0\}$. We let $X^a = X_a^{\perp}$ be the orthocomplement of X_a in $\mathbb{R}^n$, so $\mathbb{R}^n = X_a \oplus X^a$. We write the corresponding coordinates as (x_a, x^a), and denote the projection to X^a by π^a. A many-body Hamiltonian is an operator of the form

$$H = \Delta + \sum_a (\pi^a)^* V_a,$$

where V_a is a real valued function on X^a in a certain class, for example V_a is a symbol on X^a of negative order: $V_a \in S^{-\rho}(X^a)$, $\rho > 0$. We drop the pull-back notation from now on and write $H = \Delta + \sum_a V_a$. Since here we are interested in spectral theory, and V_0 is a function on a point, hence a constant, it is convenient to assume that $V_0 = 0$: otherwise it would simply shift the spectral parameter.

In the standard example, (1), the collison planes are $X_{ij} = \{q_i = q_j\}$, as well as their intersections. Thus, at X_{ij} particles i and j are at the same place, hence the name 'collision plane'.

The main feature of many-body problems is that even if V_a decays at infinity on X^a, it does *not* decay at infinity in $\mathbb{R}^n$ since it is a constant along X_a as well as along its translates $X_a + \{\bar{x}^a\}$, $\bar{x}^a \in X^a$, so it does not decay if we go to infinity, say, along X_a.

In the two-body problem one actually has $H = \Delta_{q_1,q_2} + V_{12}(q_1 - q_2)$, i.e. V_{12} still does not decay at infinity, e.g. if one keeps $q_1 = q_2$ but lets $q_1 \to \infty$. However, one can easily remove the center of mass by performing a Fourier transform along X_{12}, and then one reduces the problem to the study of a Hamiltonian on X^{12} of the form $\Delta + V_{12}$ where V_{12} decays at infinity (we are working on X^{12}!). This reduced Hamiltonian can be considered as a perturbation of Δ, hence its analysis is rather simple.

The center of mass can also be removed in any actual many-body problem, but one still obtains a Hamiltonian with non-decaying potentials as before.

We also need to introduce subsystem Hamiltonians to describe bound states such as atoms and ions. We order the clusters by inclusion of the X^a, so

$$X^a \subset X^b \text{ iff } a \leq b.$$

For the collision planes the relation is thus reversed, so $X_a \subset X_b$ if and only if $a \geq b$. We say that a is a k-cluster if the maximum number of elements in a chain with minimal element a is k. Thus, if X_a is such that $X_b \subset X_a$ implies $b = a$ or $b = 1$, then a is a 2-cluster, while 1 is a 1-cluster. If 0 is an N-cluster, we call H an N-body Hamiltonian. It is easy to see that this agrees with the usual terminology for the physical N-body Hamiltonian (1), with center of mass removed. For example, in a three-body problem one gets that $X_a \cap X_b = \{0\}$ if $a \neq b$, $a, b \neq 0, 1$.

Corresponding to each cluster a we introduce the cluster Hamiltonian H^a as an operator on $L^2(X^a)$ given by

$$H^a = \Delta_{X^a} + \sum_{b \leq a} V_b = \Delta_{X^a} + V^a, \tag{2}$$

Δ_{X^a} being the Laplacian of the induced metric on X^a. Thus, if H is a N-body Hamiltonian and a is a k-cluster, then H^a is a $(N+1-k)$-body Hamiltonian. The L^2 eigenfunctions of H^a (also called bound states) play an important role in many-body scattering. We denote L^2 eigenvalues and normalized eigenfunctions of H^a by ϵ_α and ψ_α, so $(H^a - \epsilon_\alpha)\psi_\alpha = 0$. The pair (a, ψ_α) is called a channel. We remark that by a result of Froese and Herbst, [6], $\epsilon_\alpha \leq 0$ for all α (there are no positive eigenvalues). Moreover, $\mathrm{spec}_{pp}(H^a)$ is bounded below since H^a differs from Δ by a bounded operator. Note that $X^0 = \{0\}$, $H^0 = 0$, so the unique eigenvalue of H^0 is 0.

The eigenvalues of H^b can be used to define the set of thresholds of H^a. Namely, we let

$$\Lambda_a = \cup_{b<a} \operatorname{spec}_{pp}(H^b) \tag{3}$$

be the set of thresholds of H^a, and we also let

$$\Lambda'_a = \Lambda_a \cup \operatorname{spec}_{pp}(H^a) = \cup_{b\leq a} \operatorname{spec}_{pp}(H^b). \tag{4}$$

Thus, $0 \in \Lambda_a$ for $a \neq 0$ and $\Lambda_a \subset (-\infty, 0]$. It follows from the Mourre theory (see e.g. [7, 23]) that Λ_a is closed, countable, and $\operatorname{spec}_{pp}(H^a)$ can only accumulate at Λ_a. In addition, L^2 eigenfunctions of H^a with eigenvalues which are not thresholds are necessarily Schwartz functions on X^a (in fact, they decay exponentially, see [6]). We write

$$I_a = \sum_{b\not\leq a} V_b$$

for the intercluster interaction. Thus I_a is a function on $\mathbb{R}^n$.

The point of intruducing H^a and $H_a = \Delta_{X_a} + H^a$ is that near

$$X_{a,\mathrm{reg}} = X_a \setminus X_{a,\mathrm{sing}}, \text{ where } X_{a,\mathrm{sing}} = \cup_{b>a} X_b,$$

H_a is a good model for H since $I_a = H - H_a$ decays at infinity along $X_{a,\mathrm{reg}}$. (Here we are using that if $b \not\leq a$, then the collision plane $X_c = X_b \cap X_a$ satisfies $c > a$, i.e. $X_c \subsetneq X_a$.)

If we consider the two-cluster to two-cluster problem as a two-body problem by regarding the atom/ion as a single particle, we essentially work on X_a by projecting to a bound state ψ_α of H^a and using the effective interaction

$$V_{\alpha,\mathrm{eff}} = \int_{X^a} I_a |\psi_\alpha|^2 \, dx^a,$$

which is a function on X_a. For various reasons, including the results in high energy settings, it is clear that we can only hope to recover $V_{\alpha,\mathrm{eff}}$ (rather than I_a) by studying the corresponding (i.e. state α to itself) two-cluster to two-cluster scattering matrix.

We can now describe the S-matrices following [31]. We warn the reader that the normalization this gives is geometric as in [15], and is different from the standard wave-operator normalization. The difference is essentially given by pull-back by an antipodal map on the sphere. Thus, free particles propagate in straight lines, i.e. they leave in the opposite direction from where they came, so in the geometric formulation, in the absence of interactions, the outgoing data are supported in the image of the support of the incoming data under the antipodal map. An incoming wave of energy λ in the channel α is a function of the form

$$u_{\alpha,-} = e^{-i\sqrt{\lambda-\epsilon_\alpha}|x_a|}|x_a|^{-\frac{\dim X_a - 1}{2}} g_{\alpha,-}(\frac{x_a}{|x_a|})\psi_\alpha(x^a) + \text{faster decaying terms},$$

and an outgoing wave has the form

$$u_{\alpha,+} = e^{i\sqrt{\lambda-\epsilon_\alpha}|x_a|}|x_a|^{-\frac{\dim X_a - 1}{2}} g_{\alpha,+}(\frac{x_a}{|x_a|})\psi_\alpha(x^a) + \text{faster decaying terms},$$

i.e. the sign of the phase has changed. Here $g_{\alpha,\pm}$ may be taken e.g. L^2 functions on $\mathbb{S}_a$, the unit sphere in X_a, or ideally, at least one of them may be taken $\mathcal{C}^\infty$. One

is then interested in generalized eigenfunctions of H, i.e. tempered distributions u on $\mathbb{R}^n$ that solves $(H-\lambda)u = 0$. A typical example is of the form

$$u = u_{\alpha,-} - (H - (\lambda + i0))^{-1}((H-\lambda)u_{\alpha,-}); \tag{5}$$

here the faster decaying terms may even be dropped without affecting u and $g_{\alpha,-}$ can be specified to be any smooth function on $\mathbb{S}_a$. In general, even if the incoming data are in a single channel α, as in (5), the corresponding generalized eigenfunction u of H will have outgoing waves in all channels. The S-matrix $\mathcal{S}_{\alpha\beta}(\lambda)$ picks out the component in channel β by projection in a certain sense, see [31]. Thus, $\mathcal{S}_{\alpha\beta}(\lambda)$ maps functions on $\mathbb{S}_a$, the unit sphere in X_a, to functions on $\mathbb{S}_b$, by

$$\mathcal{S}_{\alpha\beta}(\lambda)g_{\alpha,-} = g_{\beta,+}$$

for u as in (5). For example, the free-to-free (i.e. N-cluster to N-cluster in N-body scattering) S-matrix $\mathcal{S}_{00}(\lambda)$ maps functions on $\mathbb{S}_0$, the unit sphere in $\mathbb{R}^n$, to functions on $\mathbb{S}_0$, more precisely $\mathcal{S}_{00}(\lambda) : L^2(\mathbb{S}_0) \to L^2(\mathbb{S}_0)$ is bounded. Below we write $\mathbb{S}_{a,\mathrm{reg}} = \mathbb{S}_a \cap X_{a,\mathrm{reg}}$.

The first many-body result we state, slightly informally, is an 'old' direct result of the second author [30] and Hassell [9], which in turn can be considered as the extension of a result of Melrose and Zworski [18] on asymptotically Euclidean spaces (i.e. in geometric two-body scattering). Its essential ingredient is the propagation of singularities, here meaning microlocal lack of decay at infinity, much as for real principal type PDEs in the usual microlocal setting, shown in the work of Hörmander [11, 10] and Kashiwara-Kawai [13] for $\mathcal{C}^\infty$ resp. analytic singularities in the absence of boundaries, by Melrose and Sjöstrand [16, 17] and Taylor [27] for smooth boundaries and $\mathcal{C}^\infty$ singularities, and for analytic singularities by Sjöstrand [24], and by Lebeau [14] for analytic singularities at corners. The relationship is that two-body scattering is a simpler version of the boundariless setting [15, 18], as shown by the Fourier transform, while 3-body scattering is analogous to the problems with smooth boundaries, and more than three bodies correspond to the corners.

Theorem 1. *Suppose that H is a three-body Hamiltonian and the V_a are Schwartz on X^a for all a. Then $\mathcal{S}_{00}(\lambda)$ is a finite sum of Fourier integral operators (FIOs) associated to the broken geodesic relation on $\mathbb{S}_0$ to distance π. Its canonical relation corresponds to the various collision patterns. The principal symbol of the term corresponding to a single collision at X_a is given by, and in turn determines, the 2-body S-matrix of H^a at energies $\lambda' \in (0, \lambda)$.*

This result presumably extends to short range symbolic potentials, using the same methods, though it is technically more complicated to write down the argument in that case, and it has not been done. However, it was shown in [32] and [33] that in N-body scattering, N arbitrary, the wave front relation of all S-matrices is given by the broken bicharacteristic relation. Thus, the *location* of the singularities of the Schwartz kernel of the S-matrix is known, but their *precise form,* e.g. that they are those of an FIO, is not known except in three-body scattering. (Also, in *some* parts of the wave front relation the singularities can be described precisely in the N-body setting, as we do below in Theorem 3.)

An immediate corollary, when combined with two-body results (e.g. analyticity of the S-matrix in λ' and the Born approximation) is the following inverse result.

Corollary 2. *In three-body scattering, if the V_a decay exponentially and* $\dim X_a \geq 2$ *for all a then $\mathcal{S}_{00}(\lambda)$ for a single value of λ determines all interactions.*

This result is analogous to the recovery of cracks in a material by directing sound waves at it and observing the singularities of the reflected waves, except the last step which uses two-body results to get the potentials from the two-body S-matrices – and apart from the fact that the crack recovery result does not seem to exist in the literature! Note that in 3-body scattering (and also in N-body scattering if the set of thresholds is discrete) there is only a finite number of possible collision patterns. Generalized broken bicharacteristics are curves in a compressed phase space described in [33], or in a more leasurely way in [34]. A subset of this family is the set of broken bicharacteristics, which are piecewise bicharacteristics of the metric function g, i.e. integral curves of its Hamilton vector field in $T^*X_0 = (X_0)_x \times (X_0^*)_\xi$, that satisfy in addition that at each break point at $X_{b,\mathrm{reg}} \times X_0$, the momenta ξ, ξ' of the two segments meeting there differ by an element of $(X^b)^*$, i.e. $\xi - \xi'$ vanishes on (or is conormal to) X_b. This means that in a collision at $X_{b,\mathrm{reg}}$, the energy and the component of the momentum tangential to X_b are conserved, similarly to geometric optics. A convenient way of making the broken bicharacteristics continuous is to compress the phase space, by identifying points (x,ξ) and (x,ξ') if $x \in X_{b,\mathrm{reg}}$ and $\xi - \xi' \in (X^b)^*$. Such a description is in fact necessary for generalized broken bicharacteristics, but the new analysis below involves only broken bicharacteristics, so we adopt the more naive piecewise continuous curve approach and work in $X_0 \times X_0^*$. (In 3-body scattering every generalized broken bicharacteristic is such a broken bicharacteristic, but this is not true in general.) Note that a bicharacteristic segment of g is a curve of the form

$$x(t) = \bar{x} + 2t\bar{\xi},\ \xi = \bar{\xi},\ t \in J,\ J \text{ an interval}, \tag{6}$$

i.e. is a straight line segment with constant momentum. In addition, $|\xi|^2$ is the kinetic energy, so in the free region, $X_{0,\mathrm{reg}}$, $|\xi|^2 = \lambda$, but for particles in a bound state of energy ϵ_α, we actually have $|\xi|^2 + \epsilon_\alpha = \lambda$.

In fact, in [33] the phase space is compactified, much as in [15], which here essentially means a quotient by the $\mathbb{R}^+$-action on $X_0 \times X_0^*$ that is simply the dilation in the X_0 component. Taking $\mathbb{S}_0 \times X_0^*$ as a transversal to this action, the bicharacteristic segments become curves on $\mathbb{S}_0 \times X_0^*$. *Below we freely use this identification without further commenting on it.* Then the projection of a broken bicharacteristic segment to $\mathbb{S}_0$ is a (reparameterized) geodesic, and in 3-body scattering the total length of these segments for a maximally extended broken bicharacteristic is π, see Figure 1. (One example, corresponding to an unbroken bicharacteristic, i.e. a straight line, is half of a great circle on $\mathbb{S}_0$, connecting antipodal points.)

Unfortunately, this was necessarily a very brief and incomplete discussion, and we refer the reader to [33, 34] for a much more detailed description of generalized broken bicharacteristics.

While the N-cluster to N-cluster S-matrix is more complicated in general than for $N = 3$, and in particular the FIO result is not known, it was proved in [33] that the wave front relation of the S-matrices is given by the generalized broken bicharacteristic relation. These generalized broken bicharacteristics can be quite complicated in the presence of bound states; not all of them are necessarily broken bicharacteristics. However, it turns out that for an open dense subset of the relation

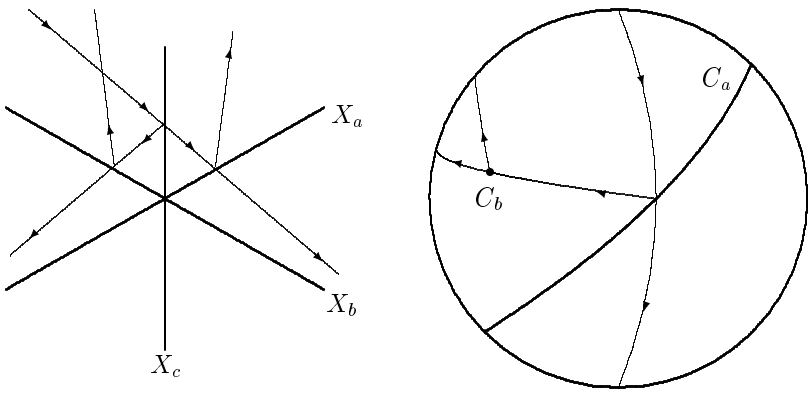

FIGURE 1. On the left, broken geodesics in $\mathbb{R}^n \setminus \{0\}$, $n = 2$, broken at the collision planes X_a, X_b and X_c. On the right, the projection of broken geodesics in $\mathbb{R}^n \setminus \{0\}$, $n = 3$, emanating from the north pole, to the unit sphere $\mathbb{S}_0$, better understood as the sphere at infinity. The C_a, C_b are the intersection of the collision planes X_a, X_b with $\mathbb{S}_0$; $\dim X_a = 2$, $\dim X_b = 1$.

corresponding to a single collision in an $(N-1)$-cluster b there are *no* generalized broken bicharacteristics relating these points except those with a single break at b.

Indeed, it is easy to write down this relation, Λ_b; this was done, for example, in [30, Section 6]. Recall that when discussing canonical relations, such as $\Lambda_b \subset T^*\mathbb{S}_0 \times T^*\mathbb{S}_0$, it is convenient to use the twisted symplectic form $\omega - \omega'$ where ω, resp. ω', is the standard symplectic form on the first, resp. second, copy of $T^*\mathbb{S}_0$. We identify points in $(y,\mu) \in T^*\mathbb{S}_0$ with pairs $(y,\mu) \in \mathbb{S}_0 \times X_0 = \mathbb{S}_0 \times \mathbb{R}^n$ such that $y \cdot \mu = 0$, and write $P_y^\perp$ for the orthogonal projection to the orthocomplement to $\operatorname{span}\{y\}$. Then the relation is the twisted conormal bundle of

$$\{(y,y') :\ y'_b = -y_b,\ y' \in \mathbb{S}_0 \setminus \mathbb{S}_b\} \tag{7}$$

with the zero section removed (twisting means that the sign of the dual variable of y' is changed); the restriction $y' \in \mathbb{S}_0 \setminus \mathbb{S}_b$ ensures that (7) is a smooth manifold. Its twisted conormal bundle is

$$\begin{aligned}\Lambda_b = \{(y,\mu,y',\mu') : y' \in \mathbb{S}_0 \setminus \mathbb{S}_b,\ y_b = -y'_b,\ \mu_b = -\mu'_b,\\ \mu^b = Cy^b,\ (\mu')^b = -C(y')^b,\ C = -\frac{\mu_b \cdot y_b}{|y_b|^2}\},\end{aligned} \tag{8}$$

i.e. is parameterized by (y', y^b, μ_b) with $y' \in \mathbb{S}_0 \setminus \mathbb{S}_b$, $y^b \in X^b$ satisfying $|y^b|^2 = 1 - |y'_b|^2$, and $\mu'_b \in X_b \setminus \{0\}$. Note that C is determined by $y \cdot \mu = 0 = y' \cdot \mu'$.

The (unbroken) geodesic of length π on $\mathbb{S}_0$ associated to (y',μ') is the arc given by $(\operatorname{span}\{y'\} + \{c\mu' :\ c > 0\}) \cap \mathbb{S}_0$; it is thus the great circle starting at y' in the direction μ'. Rather than using μ'_b as a parameter in (8), it is natural to use $z \in \mathbb{S}_b$ and $|\mu'| = |\mu|$, where z is the point at which the broken geodesic hits $\mathbb{S}_b$. Thus, $\mu' = cP^\perp_{y'} z$, $c > 0$, and an alternative description of Λ_b is

$$\begin{aligned}\Lambda_b = \{(y,\mu,y',\mu') : \exists z \in \mathbb{S}_b,\ c > 0,\ y' \in \mathbb{S}_0 \setminus \mathbb{S}_b,\ y_b = -y'_b,\\ \mu = -cP_y^\perp z,\ \mu' = cP^\perp_{y'} z\}.\end{aligned} \tag{9}$$

Thus, $y \in \mathbb{S}_{0,\mathrm{reg}}$, $z \in \mathbb{S}_b$, $y^b \in X^b$ with $|y^b|^2 = 1 - |y'_b|^2$, and $|\mu|$ parameterize Λ_b. (It is straightforward to express z in terms of (y', μ') as well, and show that the change of variables $(y', \mu') \to (y', z)$ is smooth.) The reader may find it helpful to study the parameterizations (11)-(12) below.

We also Λ_0 be the free bicharacteristic relation, which is thus the twisted conormal bundle of $y = -y'$.

Let

(10)

$$\Lambda'_b = \{(y, \mu, y', \mu') : \exists z \in \mathbb{S}_{b,\mathrm{reg}},\, c > 0,\ y_b = -y'_b,\ \mu = -cP_y^{\perp} z,\ \mu' = cP_{y'}^{\perp} z,\\ y \neq -y',\ y, y' \notin \mathrm{span}(z) \oplus X_a \text{ for any } a \neq 0, b\}.$$

Then we have the following new result, generalizing Theorem 1.

Theorem 3. *Suppose that H is an N-body Hamiltonian and the V_a are Schwartz on X^a for all a, $\dim X^a \geq 2$ for all a. With b an $(N-1)$-cluster, Λ_b, Λ'_b as above, Λ'_b is an open dense subset of Λ_b. Moreover, $\mathcal{S}_{00}(\lambda)$ is a Fourier integral operator microlocally at Λ'_b, and its principal symbol at $(y, \mu, y', \mu') \in \Lambda_b$ is determined by, and in turn determines, the Schwartz kernel of the S-matrix $\mathcal{S}^b(\lambda|y^b|)$ of H^b (which is a 2-body Hamiltonian) at $(\frac{y^b}{|y^b|}, \frac{(y')^b}{|y^b|})$. In particular, $\mathcal{S}_{00}(\lambda)$ determines $\mathcal{S}^b(\sigma)$ for $\sigma \in (0, \lambda]$, hence also V_b, provided that V_b decays exponentially.*

Proof. First, openness is clear since each additional condition in the definition (10) of Λ'_b (as compared to (9)) gives an open subset of Λ_b, and the intersection of these is open.

To see density, for $z \in \mathbb{S}_{b,\mathrm{reg}}$, note that $X_a \oplus \mathrm{span}(z)$ has codimension $\dim X^a - 1 \geq 1$ in X_0, so imposing the additional conditions $z \in \mathbb{S}_{b,\mathrm{reg}}$ and $y' \notin \mathrm{span}(z) \oplus X_a$ for any $a \neq 0, b$ certainly gives a dense subset of Λ_b.

Consider $y' \in \mathbb{S}_{0,\mathrm{reg}} \setminus \cup_{a \neq b,0}(X_a \oplus \mathrm{span}(z))$. Then a point in Λ_b is specified by giving $y \in \mathbb{S}_0$ satisfying $y_b = -y'_b$, and $|\mu|$. The projection of the corresponding bicharacteristic segments to the base space, $\mathbb{S}_0$, i.e. the corresponding geodesics connecting y' and z, resp. y and z, lie in $\mathrm{span}\{y', z\} \cap \mathbb{S}_0$ and $\mathrm{span}\{y, z\} \cap \mathbb{S}_0$. If these intersect X_a then $sy' + tz \in X_a$, resp. $sy + tz \in X_a$, for some $(s, t) \in \mathbb{R}^2$ so $y' \in X_a \oplus \mathrm{span}\{z\}$, resp. $y \in X_a \oplus \mathrm{span}\{z\}$, as $s \neq 0$. (For $s = 0$ gives $z \in X_a$, contradicting $z \in \mathbb{S}_{b,\mathrm{reg}}$.) Note that $y' \notin X_a \oplus \mathrm{span}\{z\}$ by assumption. It remains to show that the set of $y \in \mathbb{S}_{0,\mathrm{reg}}$ with $y_b = -y'_b$ and $y \notin X_a \oplus \mathrm{span}\{z\}$ is also dense inside

$$S = \{y \in \mathbb{S}_0 :\ y_b = -y'_b\}.$$

Thus, S is a sphere inside the affine space $\{-y'_b\} + X^b$, centered at $-y'_b$, of radius $\sqrt{1 - |y'_b|^2}$. The other conditions on y state that it does not lie on various affine subspaces W_a, namely $X_a \oplus \mathrm{span}\{z\}$, $a \neq 0, b$, and $W_b = X_b$, and in addition $y \neq -y'$. But the intersection of a sphere with an affine space W is a lower dimensional sphere, unless the whole sphere lies in W. Since $y = -y'$ does not lie on any of these affine subspaces W_a, $a \neq 0$, all of these intersections are lower dimensional spheres, so in fact the set of y satisfying $y \notin X_a \oplus \mathrm{span}\{z\}$ and $y \in \mathbb{S}_{0,\mathrm{reg}}$ is the complement of the union of compact codimension ≥ 1 submanifolds of S, and hence is dense in S. (A slightly different way of arguing is to say that if a point $\bar{y}$ lies in one of these affine spaces, then the arc between $\bar{y}$ and $-y'$ only meets this space inside $\mathrm{span}\{\bar{y}\}$, for otherwise $-y'$ would also lie in this affine space, giving the

density rather explicitly.) Since $y = -y'$ is a single point in S, and S has dimension $\dim X^b - 1 \geq 1$, the density follows even if we impose $y \neq -y'$.

The bicharacteristics (6) of the free Hamiltonian Δ, at energy λ, can be parameterized by $p = (y, \mu) \in T^*\mathbb{S}_0$ by

$$\gamma_p(t) = (x(t), \xi(t)), \ \ x(t) = -\frac{\mu}{\sqrt{\lambda}} + 2t\xi, \ \ \xi(t) = \sqrt{\lambda}\, y. \tag{11}$$

The analogous 'incoming' parameterization, $\gamma'_{p'}(t) = (x'(t), \xi'(t))$, $p' = (y', \mu')$, simply switches the sign of y and μ, i.e. it is

$$x(t) = \frac{\mu'}{\sqrt{\lambda}} + 2t\xi, \ \ \xi(t) = -\sqrt{\lambda}\, y', \tag{12}$$

so as $t \to +\infty$, $\frac{x(t)}{|x(t)|} \to y$ along γ_p, while as $t \to -\infty$, $\frac{x(t)}{|x(t)|} \to y'$ along $\gamma'_{p'}$. The normalization of these parameterizations may seem strange; they arise by considering the graph of $d(\sqrt{\lambda}x \cdot y)$, resp. $d(-\sqrt{\lambda}x \cdot y')$, (recall that $e^{\pm i\sqrt{\lambda}x\cdot y}$ is, up to constants, the Schwartz kernel of the free Poisson operator!), expressing x and ξ in terms of (y, μ) and an additional parameter t, and then changing the sign of the μ component, corresponding to the twisting.

Now suppose that $(p, p') \in \Lambda'_b$, $p = (y, \mu)$, $p' = (y', \mu')$. Then there is a unique generalized broken bicharacteristic 'connecting' p and p', i.e. with γ such that for large $|t|$, up to reparameterization, γ is given by γ_p and $\gamma'_{p'}$; say $\gamma|_{[T,+\infty)}$ coincides with γ_p and $\gamma|_{(-\infty,T']}$ coincides with $\gamma'_{p'}$, up to reparameterization. Moreover, a generalized broken bicharacteristic (being a bicharacteristic in $T^*X_{0,\mathrm{reg}}$) has a unique continuation until it hits $\mathbb{S}_c$ for some $c \neq 0$, so we may assume that $\gamma(T), \gamma(T') \in \mathbb{S}_{0,\mathrm{sing}}$. Note that γ_0, the broken bicharacteristic with a single break at $\mathbb{S}_b$ (with some parameterization), is such a bicharacteristic, so $\gamma(t) = \gamma_0(t - t_+)$, $t \geq T$, and $\gamma(t) = \gamma_0(t - t_-)$, $t \leq T'$. In particular, $\gamma_0(T' - t_-), \gamma_0(T - t_+) \in \mathbb{S}_{0,\mathrm{sing}}$. Since γ_0 has a unique such point, they are equal, so $T' - t_- = T - t_+$, and $q = \gamma(T') = \gamma(T)$. Since the radial momentum, $\frac{x\cdot\xi}{|x|}$, is strictly increasing along γ away from the radial points, in particular at q, we deduce that $T' = T$, hence $\gamma = \gamma_0$ as claimed. (Recall that radial points are those where the Hamilton vector field is radial, which in this case means that for $x \in X_{b,\mathrm{reg}}$, $\xi_b = cx$ for some $c \in \mathbb{R}$.)

To construct $\mathcal{S}_{00}(\lambda)$ microlocally, we construct the Poisson operators first. Microlocally a good approximation for these are the Poisson operators $\mathcal{P}^\sharp_{0,+}(\lambda)$ of H_b, which are easy to write down explicitly, see [30], since H_b is 'product-type': a 2-body Hamiltonian H^b with the 'center of mass' kinetic energy Δ_{X_b} added back. For suitable microlocal cutoffs T_+ (which are thus ps.d.o.'s)

$$\begin{aligned}
\mathcal{P}_{0,+}(\lambda)g &= \mathcal{P}^\sharp_{0,+}(\lambda)g - R(\lambda + i0)I_b\mathcal{P}^\sharp_{0,+}(\lambda)g,\\
\mathcal{S}_{00}(\lambda)g &= ((H - \lambda)T_+\mathcal{P}^\sharp_{0,-}(\lambda))^*\mathcal{P}_{0,+}(\lambda);
\end{aligned}$$

see [31] or [34, Equation (4.2)]. Now, the wave front relation of

$$((H - \lambda)T_+\mathcal{P}_{0,-}(\lambda))^*R(\lambda + i0)I_b\mathcal{P}^\sharp_{0,+}(\lambda)$$

is a subset of the generalized broken bicharacteristic relation, but it includes only those generalized broken bicharacteristics on which I_b is not Schwartz, i.e. which go through points over $\mathbb{S}_a$ with $a \neq 0, b$. (See the proof of Theorem 2.8 in [33, Section 10] for a similar argument.) In particular, it does not contain any points

in Λ'_b, so microlocally at Λ'_b, $\mathcal{S}_{00}(\lambda)$ is given by

$$((H-\lambda)T_+\mathcal{P}^\sharp_{0,-}(\lambda))^*\mathcal{P}^\sharp_{0,+}(\lambda) \tag{13}$$

But this is an FIO by exactly the same proof as in the three-body setting [30], proving the FIO statement.

The rest of the proof is exactly as in [30, Proposition 6.4], using the explicit form of (13). □

The other new result is in 2-cluster to 2-cluster scattering. To set it up, we first state a theorem of Skibsted [25].

Theorem. [25] *The two-cluster to two-cluster S-matrices have $\mathcal{C}^\infty$ Schwartz kernels, except the conormal singularity of $\mathcal{S}_{\alpha\alpha}(\lambda)$ corresponding to free motion, i.e. at the graph of the antipodal map on $\mathbb{S}_a$.*

Thus, principal symbol calculations do not help in this inverse problem. The new result is the following extension of [28] to the N-body setting.

Theorem 4. *Suppose that a is a 2-cluster, $\dim X_a \geq 2$, and V_b for $b \leq a$ is a symbol of negative order (i.e. may be long range). Suppose also that*

$$\inf\operatorname{spec} H^a < \inf \Lambda_a$$

(and hence $\inf\operatorname{spec} H^a$ is an L^2-eigenvalue of H^a), $I \subset (\epsilon_\alpha, \inf\Lambda_a)$ is a non-empty open set with $\sup I < \inf\Lambda_a$, and let

$$R = 2\sqrt{\sup I - \epsilon_\alpha}.$$

For any $\mu > \dim X_a$ there exists $\delta > 0$ such that the following holds.

Suppose that $\sup|(1+|x^b|)^\mu V_b(x^b)| < \delta$ for all b with $b \not\leq a$. Then $\mathcal{S}_{\alpha'\alpha''}(\lambda)$ given for all $\lambda \in I$ and for all bound states α', α'' of H^a with $\epsilon_{\alpha'}, \epsilon_{\alpha''} < \sup I$, determines the Fourier transform of the effective interaction $V_{\alpha,\mathrm{eff}}$ in the ball of radius R centered at 0.

Remark. Here δ depends on I only through $\sup I$; as long as $\sup I$ is bounded away from $\inf\Lambda_a$, δ may be chosen independently of I. This dependence on $\sup I$ was improperly omitted from the statement of Theorem 1.1 in [28].

The smallness assumption on the V_b, $b \not\leq a$ implies that $\inf\operatorname{spec} H^b$ is not much lower than $\inf\Lambda_a$, in particular it may be assumed to lie above $\sup I$.

This theorem says that if the unknown interactions are small then the effective interaction can be partly determined from the knowledge of all S-matrices with incoming and outgoing data in the cluster a in the relevant energy range. In fact, near-forward information suffices as in two-body scattering, where this was observed recently by Novikov [22], see [28]. Also, if one is willing to take small R and α is the ground state of H^a, it suffices to know $\mathcal{S}_{\alpha\alpha}(\lambda)$ to recover $\hat{V}_{\alpha,\mathrm{eff}}$. That is, one has

Theorem 5. *Suppose that a is a 2-cluster, $\dim X_a \geq 2$, and V_b for $b \leq a$ is a symbol of negative order (i.e. may be long range). Suppose also that*

$$\epsilon_\alpha = \inf\operatorname{spec} H^a < \inf\Lambda_a.$$

Let $\epsilon_{\alpha'}$ be the second eigenvalue of H^a, or the bottom of its essential spectrum if ϵ_α is the only eigenvalue. Suppose also that $I \subset (\epsilon_\alpha, \epsilon_{\alpha'})$ is a non-empty open set with $\sup I < \inf\Lambda_a$, and let

$$R = 2\sqrt{\sup I - \epsilon_\alpha}.$$

For any $\mu > \dim X_a$ there exists $\delta > 0$ such that the following holds.

Suppose that $\sup|(1+|x^b|)^\mu V_b(x^b)| < \delta$ for all b with $b \not\leq a$. Then $\mathcal{S}_{\alpha\alpha}(\lambda)$ given for all $\lambda \in I$ determines the Fourier transform of the effective interaction $V_{\alpha,\mathrm{eff}}$ in the ball of radius R centered at 0.

In case V_b decay exponentially on X^b for all $b \not\leq a$, then $V_{\alpha,\mathrm{eff}}$ decays exponentially on X_a, hence its Fourier transform is analytic, so $V_{\alpha,\mathrm{eff}}$ itself can be recovered from these S-matrices.

This result should extend to higher energies, i.e. $\sup I < \inf \Lambda_a$ is *not* expected to be essential. But it is hard to make R greater than $2\sqrt{\inf \Lambda_a - \epsilon_\alpha}$ even then. The reason is that our method relies on the construction of exponential solutions following Faddeev [5], Calderón [1], Sylvester and the first author [26] and Novikov and Khenkin [19], but in the many-body setting. One thus allows complex momenta $\rho \in \mathbb{C}(X_a)$, the complexification of X_a. We then seek solutions u of $(H-\lambda)u = 0$ of the form

$$u = u_\rho = e^{i\rho\cdot x_a}(\psi_\alpha(x^a) + v),\ \rho\cdot\rho = \lambda - \epsilon_\alpha,\ \rho \in \mathbb{C}(X_a), \tag{14}$$

where v is supposed to be 'small'. Note that $u_\rho^0 = e^{i\rho\cdot x_a}\psi_\alpha(x^a)$ satisfies $(\Delta + V^a - \lambda)u_\rho^0 = 0$ for ρ as in (14), so $(H-\lambda)u_\rho^0 = I_a\psi_\alpha$ decays at infinity since ψ_α decays in cones disjoint from X_a while I_a decays on X_a. (We are using here that a is a 2-cluster for otherwise I_a would only decay on $X_{a,\mathrm{reg}}$!) Writing

$$\rho = z\nu + \rho_\perp \text{ with } |\nu| = 1, \rho_\perp\cdot\nu = 0,\ \rho,\nu \text{ real},\ z \in \mathbb{C},$$

the atom/ion may break up for $|\rho_\perp| > \sqrt{\inf\Lambda_a - \epsilon_\alpha}$ even if $\lambda < \inf\Lambda_a$, i.e. where the conservation of energy does *not* allow this for *real* frequencies. On the other hand, one needs such large $\rho_\perp$ to recover $V_{\alpha,\mathrm{eff}}$ on larger balls.

We briefly indicate how the break-up can happen. One considers the conjugated Hamiltonian

$$P(\rho) = e^{-i\rho\cdot x_a}(H-\lambda)e^{i\rho\cdot x_a} = \Delta + 2\rho\cdot D_{x_a} + \rho\cdot\rho + V - \lambda$$

with $\rho \in \mathbb{C}(X_a)$ the complex frequency. The total energy of the system in state α is $\rho\cdot\rho + \epsilon_\alpha$ (kinetic+potential energy), and we are assuming that the total energy is λ, so

$$\rho\cdot\rho + \epsilon_\alpha = \lambda,\ \text{see (14)}, \tag{15}$$

so

$$P(\rho) = \Delta + 2\rho\cdot D_{x_a} + V - \epsilon_\alpha.$$

A good model for $P(\rho)$, under the smallness assumption on I_a, is

$$P_a(\rho) = \Delta + 2\rho\cdot D_{x_a} + V^a - \epsilon_\alpha. \tag{16}$$

Taking the Fourier transform in X_a makes the invertibility of $P_a(\rho)$ into a question on the behavior of the resolvent of $H^a = \Delta_{x^a} + V^a(x^a)$, uniformly across the spectrum. That is,

$$\mathcal{F}_{X_a}P_a(\rho)\mathcal{F}_{X_a}^{-1} = \Delta_{X^a} + V^a - (\epsilon_\alpha - |\xi_a|^2 - 2\rho\cdot\xi_a),$$

acting pointwise in ξ_a. Its right inverse is given by

$$\mathcal{F}_{X_a}G_a(\rho)\mathcal{F}_{X_a}^{-1} = R^a(\epsilon_\alpha - |\xi_a|^2 - 2\rho\cdot\xi_a), \tag{17}$$

with $R^a(\sigma) = (H^a - \sigma)^{-1}$, provided we show that this makes sense. Now, we need to keep

$$F_\rho(\xi_a) = \epsilon_\alpha - |\xi_a|^2 - 2\rho\cdot\xi_a \in \mathbb{C}\setminus[\inf\Lambda_a, +\infty), \tag{18}$$

for otherwise other channels also become open, i.e. the cluster may break up. In other words, on the range of F_ρ, we want to keep R^a invertible on the range of $\mathrm{Id} - e_a$, which would reflect that only the a-bound states form open channels. We thus need that either $\operatorname{Im} F_\rho(\xi_a) \neq 0$ or $\operatorname{Im} F_\rho(\xi_a) = 0$ but $\operatorname{Re} F_\rho(\xi_a) < \inf\Lambda_a$. Writing $\rho = z\nu + \rho_\perp$ as above,

$$\operatorname{Im} F_\rho(\xi_a) = -2(\operatorname{Im} z)\nu\cdot\xi_a,$$

so for $\operatorname{Im} z \neq 0$, i.e. for ρ non-real, $\operatorname{Im} F_\rho$ vanishes exactly on the hyperplane $\nu\cdot\xi_a = 0$. Moreover,

$$\operatorname{Re} F_\rho(\xi_a) = \epsilon_\alpha + \rho_\perp^2 - ((\xi_a)_\perp + \rho_\perp)^2 - (\nu\cdot\xi_a)^2 - 2(\operatorname{Re} z)(\nu\cdot\xi_a), \tag{19}$$

so if $\operatorname{Im} F_\rho(\xi_a) = 0$, the condition $\operatorname{Re} F_\rho(\xi_a) < \inf\Lambda_a$ amounts to

$$\epsilon_\alpha + \rho_\perp^2 - ((\xi_a)_\perp + \rho_\perp)^2 < \inf\Lambda_a.$$

This will *not* be satisfied for every ξ_a unless

$$|\rho_\perp| < \sqrt{\inf\Lambda_a - \epsilon_\alpha}. \tag{20}$$

The most unfortunate restriction of the theorem is the smallness assumption on the unknown interactions. While (20) ensures that the 2-cluster a may not break up into its subsystems, it does not rule out the existence of different channels, e.g. associated to other 2-clusters, at complex frequencies. The presence of such channels would significantly complicate the analysis. In particular, the crucial analyticity in z would not be clear at all.

We proceed to sketch the proof of Theorem 4, which is completely analogous to the three-body result shown in [28]. For

$$\lambda_1 \in [|\rho_\perp|^2 + \epsilon_\alpha, \inf\Lambda_a)\setminus\Lambda_a', \tag{21}$$

let $e_a(\lambda_1)$ be the orthogonal projection to the L^2 eigenfunctions of H^a with eigenvalue $\leq\lambda_1$, and let E_a be its extension to X_0 via tensoring by Id_{X_a}, so

$$E_a(\lambda_1) = \sum_{\alpha'} \mathrm{Id}_{X_a}\otimes(\psi_{\alpha'}\otimes\overline{\psi_{\alpha'}}). \tag{22}$$

Since eigenvalues of H^a can only accumulate at Λ_a, e_a is finite rank. We also let

$$\lambda_0 = \inf(\Lambda_a'\cap(\lambda_1, +\infty)) > \lambda_1. \tag{23}$$

The particular choice of λ_1, provided that it is sufficiently close to $\inf\Lambda_a$, does not play a major role in our arguments, so we usually simply write e_a for $e_a(\lambda_1)$, etc.

We restrict the region (20) slightly further and work in the region

$$\mathbb{C}(X_a)_\alpha^\circ = \{(z,\nu,\rho_\perp):\ \operatorname{Im} z\neq 0,\ |\rho_\perp|^2+\epsilon_\alpha \in (\epsilon_\alpha,\lambda_0)\setminus\Lambda_a'\}, \tag{24}$$

i.e. we also assume that $|\rho_\perp|^2 + \epsilon_\alpha$ is not an eigenvalue of H^a. Again, we do not indicate λ_0 explicitly in the notation.

Since the ranges of E_a and $\mathrm{Id} - E_a$ play a rather different role below, for $p\in\mathbb{R}$ we introduce weighted spaces that reflect this:

$$\begin{aligned}\mathcal{H}_p =& (L^2_p(X_a)\otimes\operatorname{Ran} e_a)\oplus(L^2(X_a)\otimes\operatorname{Ran}(\mathrm{Id}-e_a))\\ &\subset (L^2_p(X_a)\otimes\operatorname{Ran} e_a)\oplus L^2(X_0),\end{aligned} \tag{25}$$

with e_a considered as a bounded operator on $L^2(X^a)$. Thus, we allow weights on the range of E_a, but not on its orthocomplement. Since $\operatorname{Ran} e_a$ is finite dimensional, $\mathcal{H}_p$ is a Hilbert space with norms induced on the summands by the $L^2_p(X_a)$ and $L^2(X_0)$ norms respectively.

Returning to $P_a(\rho)$, we can define a right inverse $G_a(\rho)$ by (17) if we show that it makes sense, the only issues being the behavior of $R^a(\sigma)$ for real σ and bounds as $|\xi_a| \to \infty$. It is straightforward to analyze these, as in [28], and prove the following proposition.

Proposition 6. *Suppose that $(z, \nu, \rho_\perp) \in \mathbb{C}(X_a)^\circ_\alpha$. The operator*

$$G_a(\rho) = \mathcal{F}^{-1}_{X_a} R^a(\epsilon_\alpha - |.|^2 - 2\rho \cdot .)\mathcal{F}_{X_a}$$

is a bounded operator $\mathcal{H}_p \to \mathcal{H}_r$ for $p > 0$, $r < 0$, $r < p - 1$. It satisfies

$$\begin{aligned} P_a(\rho)G_a(\rho) &= \mathrm{Id} : \mathcal{H}_p \to \mathcal{H}_p, \\ G_a(\rho)P_a(\rho) &= \mathrm{Id} : \mathcal{H}_p \to \mathcal{H}_p. \end{aligned} \tag{26}$$

It is continuous in $\rho \in \mathbb{C}(X_a)^\circ_\alpha$ and analytic in $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, for $p > 0$, $r < 0$, $r < p - 1$, $\rho_\perp$ fixed, for any $C > 0$, $G_a(\rho)$ is uniformly bounded in $\mathcal{B}(\mathcal{H}_p, \mathcal{H}_r)$ in $|\operatorname{Im} z| \geq C|\operatorname{Re} z|$, and $\text{s-}\lim_{|z|\to\infty} G_a(\rho) = 0$ as an operator in $\mathcal{B}(\mathcal{H}_p, \mathcal{H}_r)$, provided that $|z| \to \infty$ in the region $|\operatorname{Im} z| \geq C|\operatorname{Re} z|$.

Sketch of proof. [See [28] for the completely analogous proof in the three-body setting.] Recall that We need to analyze F_ρ and $R^a(\sigma)$. First, by the spectral mapping theorem, for $\operatorname{Re}\sigma < \inf \operatorname{spec} H^a$,

$$\|R^a(\sigma)\|_{\mathcal{B}(L^2,L^2)} \leq (\inf \operatorname{spec} H^a - \operatorname{Re}\sigma)^{-1}.$$

Now, due to the $-|\xi_a|^2$ term in(18), for any fixed ρ there exists $C > 0$ such that

$$\operatorname{Re} F_\rho(\xi_a) < C - |\xi_a|^2/2. \tag{27}$$

Thus, for $\phi \in \mathcal{C}^\infty_c(X^*_a)$ identically 1 on a large enough ball, $G_a \mathcal{F}^{-1}_{X_a}(1 - \phi)\mathcal{F}_{X_a}$ is bounded on L^2. Hence, we only need to be concerned about what happens in a compact set in X^*_a.

For this we need two other bounds that hold by the selfadjointness of H^a and its spectral properties, namely

$$\begin{aligned} &\|(\mathrm{Id} - e_a)R^a(\sigma)\|_{\mathcal{B}(L^2,L^2)} \leq |\operatorname{Re}\sigma - \lambda_0|^{-1}, \ \operatorname{Re}\sigma < \lambda_0, \\ &\|R^a(\sigma)\|_{\mathcal{B}(L^2,L^2)} \leq |\operatorname{Im}\sigma|^{-1}, \ \operatorname{Im}\sigma \neq 0, \end{aligned} \tag{28}$$

λ_0 as in (23). It is easy to check using (19) that

$$\operatorname{Re} F_\rho(\xi_a) > \epsilon_\alpha + \rho^2_\perp \Rightarrow |\operatorname{Im} F_\rho(\xi_a)| > (\operatorname{Re} F_\rho(\xi_a) - (\epsilon_\alpha + \rho^2_\perp))\frac{|\operatorname{Im} z|}{|\operatorname{Re} z|} \tag{29}$$

In particular, $(z, \nu, \rho_\perp) \in \mathbb{C}(X_a)^\circ_\alpha$ implies that $\epsilon_\alpha + \rho^2_\perp < \inf \Lambda_a$, hence

$$\mathcal{F}^{-1}_{X_a} R^a(F_\rho(.))(\mathrm{Id} - E_a)\mathcal{F}_{X_a}$$

is well defined. Indeed, either $\operatorname{Re} F_\rho(\xi_a) > \epsilon_\alpha + \rho^2_\perp$, and then $\operatorname{Im} F_\rho(\xi_a) \neq 0$, so $R^a(F_\rho(\xi_a))$ is a bounded operator on L^2, or $\operatorname{Re} F_\rho(\xi_a) \leq \epsilon_\alpha + \rho^2_\perp < \inf \Lambda_a$, so $(\mathrm{Id} - e_a)R^a(F_\rho(\xi_a))$ is bounded on L^2.

On the range of $e_{a,\epsilon}$, the projection to the eigenspace with eigenvalue ϵ, $R^a(\sigma)$ is multiplication by $(\epsilon - \sigma)^{-1}$. This is a locally integrable function of σ near ϵ (in $\mathbb{C}$!), so the application of $R^a(F_\rho(.))$ to $\hat{u} = \mathcal{F}_{X_a} u$ is well defined, provided that

at every point ξ_a with $F_\rho(\xi_a) = \epsilon$, the differential of $F_\rho : X_a^* \to \mathbb{C}$ is surjective (with $\mathbb{C}$ considered as a 2-dimensional real manifold), i.e. $F_\rho(\xi_a) = \epsilon$ implies that $d\operatorname{Re} F_\rho(\xi_a)$ and $d\operatorname{Im} F_\rho(\xi_a)$ are linearly independent. But $d\operatorname{Im} F_\rho$ is nonzero, and is a multiple of $d(\nu\cdot\xi_a)$, so in view of (19), $d\operatorname{Im} F_\rho$ and $d\operatorname{Re} F_\rho$ are linearly independent if and only if $d(((\xi_a)_\perp + \rho_\perp)^2) \neq 0$, i.e. if and only if $(\xi_a)_\perp \neq -\rho_\perp$. But if $(\xi_a)_\perp = -\rho_\perp$ and $F_\rho(\xi_a)$ is real, then $\nu\cdot\xi_a = 0$, hence $F_\rho(\xi_a) = \epsilon_\alpha + \rho_\perp^2 \notin \Lambda_a'$ due to $(z,\nu,\rho_\perp) \in \mathbb{C}(X_a)_\alpha^\circ$. Now let

$$\begin{aligned} G_a(\rho) &= \mathcal{F}_{X_a}^{-1} R^a(\epsilon_\alpha - |.|^2 - 2\rho\cdot.)\mathcal{F}_{X_a} \\ &= \mathcal{F}_{X_a}^{-1} R^a(\epsilon_\alpha - |.|^2 - 2\rho\cdot.)(\operatorname{Id} - E_a)\mathcal{F}_{X_a} + \mathcal{F}_{X_a}^{-1} R^a(\epsilon_\alpha - |.|^2 - 2\rho\cdot.)E_a\mathcal{F}_{X_a}; \end{aligned}$$

both terms are well defined by the preceeding considerations when applied to functions in $\mathcal{S}(X_a; L^2(X^a))$. Indeed, application of $G_a(\rho)$ to the range of $E_{a,\epsilon}$ is the only issue, and there, with $u = v\otimes\psi_{\alpha'}$, $v \in L^2_p(X_a)$, $\hat v = \mathcal{F}_{X_a} v$,

$$(\mathcal{F}_{X_a} E_{a,\epsilon} G_a(\rho) v)(\xi_a, .) = (\epsilon - F_\rho(\xi_a))^{-1}\hat v \otimes \psi_{\alpha'},$$

so the mapping properties of $G_a(\rho)$ on $\operatorname{Ran} E_a$ are given by the two-body results of Weder [37, Theorem 1.1], see Remark 3.4 in [28] for its application in this context.

For the behavior of $G_a(\rho)$ as $|z| \to \infty$, we decompose G_a as above. On the range of E_a we use the analogous two-body result of Weder, while on the range of $\operatorname{Id} - E_a$ we use the uniform L^2 operator bounds on $R^a(\sigma)(\operatorname{Id} - e_a)$, valid uniformly away from $[\Lambda_a, +\infty)$, together with $|\operatorname{Im} F_\rho(\xi_a)| \to \infty$ for almost every ξ_a, namely the ξ_a such that $\xi_a\cdot\nu \neq 0$, so an application of the dominated convergence theorem yields the desired strong convergence. □

Since

$$P(\rho)G_a(\rho)v = (\operatorname{Id} + I_a G_a(\rho))v,\ v \in \mathcal{H}_p,\ p > 0,$$

we next investigate I_a on $\mathcal{H}_r$, again as in [28].

Lemma 7. *Suppose that $\mu > 0$, and $\langle x^b\rangle^\mu V_b \in L^\infty(X^b)$ for all $b \not\leq a$. The multiplication operator I_a is in $\mathcal{B}(\mathcal{H}_r, \mathcal{H}_p)$ provided that $p \leq r + \mu$, $p \leq \mu$, $r \geq -\mu$.*

Moreover, there exists $C > 0$ (independent of V_b) such that the norm of I_a as such an operator is bounded by $C\max_b \sup(\langle x^b\rangle^\mu |V_b|)$.

Sketch. See [28] *for details.* Consider the matrix decomposition of I_a corresponding to the direct sum in (25), and use the rapid decay of $\psi_{\alpha'}$ to analyze the terms involving E_a. The restrictions $p \leq r + \mu$, $p \leq \mu$, resp. $r \geq -\mu$ arise from the requirements of making $E_a I_a E_a$, $E_a I_a(\operatorname{Id} - E_a)$, resp. $(\operatorname{Id} - E_a) I_a E_a$ bounded. □

Thus, perturbation theory gives

Proposition 8. *Suppose that $\rho_\perp$ satisfies $|\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0)\setminus\Lambda_a'$, $\mu > \max(p, 1)$, $p > 0$, $r < 0$, $r < p - 1$. There exists $\delta > 0$ with the following property. Suppose that for all $b \not\leq a$, $\sup|\langle x^b\rangle^\mu V_b| < \delta$. Then the operator*

$$G(\rho) = G_a(\rho)(\operatorname{Id} + I_a G_a(\rho))^{-1} : \mathcal{H}_p \to \mathcal{H}_r \tag{30}$$

satisfies

$$\begin{aligned} P(\rho)G(\rho) &= \operatorname{Id} : \mathcal{H}_p \to \mathcal{H}_p, \\ G(\rho)P(\rho) &= \operatorname{Id} : \mathcal{H}_p \to \mathcal{H}_p. \end{aligned} \tag{31}$$

Moreover, $G(\rho)$ is a continuous function of ρ in $\mathbb{C}(X_a)_\alpha^\circ$, and an analytic function of $z \in \mathbb{C}\setminus\mathbb{R}$, and $\text{s-}\lim_{|z|\to\infty} G(\rho) = 0$ as a map $\mathcal{H}_p \to \mathcal{H}_r$ provided that $|z| \to \infty$ in $|\operatorname{Im} z| \geq C|\operatorname{Re} z|$, $C > 0$.

Since $I_a\psi_\alpha \in \mathcal{H}_p$ for some $p > 0$ if $\mu > \dim X_a/2$, we deduce the following corollary.

Corollary 9. *Suppose that* $\mu > \dim X_a/2$, $|\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a$. *Then*

$$u_\rho = u_\rho^0 - e^{i\rho\cdot x_a} G(\rho) I_a \psi_\alpha = e^{i\rho\cdot x_a}(\psi_\alpha - G(\rho) I_a \psi_\alpha) \tag{32}$$

satisfies $P(\rho)u_\rho = 0$, *and*

$$u_\rho^0 = e^{i\rho\cdot x_a}(\mathrm{Id} + G_a(\rho) I_a) e^{-i\rho\cdot x_a} u_\rho. \tag{33}$$

Moreover, $u_\rho - u_\rho^0 \to 0$ *in* $\mathcal{H}_r$, $r < 0$, $r < \mu - 1 - \frac{\dim X_a}{2}$, *as* $|z| \to \infty$ *in* $|\operatorname{Im} z| > C|\operatorname{Re} z|$, $C > 0$.

We now only need to connect u_ρ to the scattering matrix. This is done by letting ρ become real. Below we consider

$$\operatorname{Im} z \geq 0, \ \operatorname{Re} z \geq 0. \tag{34}$$

Proposition 10. *The operator* $G_a(\rho) : \mathcal{H}_p \to \mathcal{H}_r$ *extends continuously (from* $\mathbb{C}(X_a)^+_\alpha \cap \mathbb{C}(X_a)^\circ_\alpha$*) to* $\mathbb{C}(X_a)^+_\alpha$ *for* $p > 1/2$, $r < -1/2$, *and it satisfies* $P_a(\rho)G_a(\rho) = \mathrm{Id}$, $G_a(\rho)P_a(\rho) = \mathrm{Id}$, *on* $\mathcal{H}_p$, $p > 1/2$. *The limit* $G_a(\nu, \rho_\perp, z \pm i0)$ *satisfies*

$$\begin{aligned} &e^{i\rho\cdot x_a} G_a(\nu, \rho_\perp, z \pm i0) e^{-i\rho\cdot x_a} \\ &= R_a(\lambda + i0) + \sum_{\alpha'} \mathcal{F}_{X_a}^{-1} H(\nu\cdot(\xi_a - \operatorname{Re}\rho))[(|\xi_a|^2 - (\lambda - \epsilon_{\alpha'} - i0))^{-1} \\ &\qquad - (|\xi_a|^2 - (\lambda - \epsilon_{\alpha'} + i0))^{-1}]\mathcal{F}_{X_a} E_{a,\alpha'}, \end{aligned} \tag{35}$$

$\lambda = \rho^2 + \epsilon_\alpha$.

Sketch of proof. Write

$$G_a(\nu, \rho_\perp, z) = (\mathrm{Id} - E_a) G_a(\nu, \rho_\perp, z) + E_a G_a(\nu, \rho_\perp, z). \tag{36}$$

The first term is just $\mathcal{F}_{X_a}^{-1} R^a(F_\rho(.))(\mathrm{Id} - E_a)\mathcal{F}_{X_a}$, and $\operatorname{Im} F_\rho \to 0$ as $\operatorname{Im} z \to 0$. Moreover, in the region where the sign of $\operatorname{Im} z$ matters, i.e. where $\operatorname{Re} F_\rho \geq \Lambda_a$, $\operatorname{Im} F_\rho > 0$ by (29) which is valid with $\operatorname{Im} F_\rho$ in place of $|\operatorname{Im} F_\rho|$ due to (34), so the first term goes to $\mathcal{F}_{X_a}^{-1} R^a(|\xi_a|^2 + 2\rho\cdot\xi_a + \epsilon_\alpha + i0)(\mathrm{Id} - E_a)\mathcal{F}_{X_a}$ as $\operatorname{Im} z \to 0$. Conjugation by $e^{i\rho\cdot x_a}$ replaces ξ_a by $\xi_a + \rho$, hence yielding $R_a(\lambda + i0)(\mathrm{Id} - E_a)$, $\lambda = \rho\cdot\rho + \epsilon_\alpha$, in the limit.

The second term on the right hand side of (36) is dealt with exactly as in the two-body setting discussed by Weder. Namely, working on the range of $E_{a,\epsilon}$, $\mathcal{F}_{X_a} G_a(\rho)\mathcal{F}_{X_a}^{-1}$ is multiplication by $(\epsilon - F_\rho(\xi_a))^{-1}$. For each ξ_a with $\nu\cdot\xi_a \neq 0$, the limit as $\operatorname{Im} z \to 0$ is multiplication by the limit of $(\epsilon + |\xi_a|^2 + 2\rho\cdot\xi_a - \epsilon_\alpha)^{-1}$, which is $(\epsilon + |\xi_a|^2 + 2\rho\cdot\xi_a \pm i0 - \epsilon_\alpha)^{-1}$, with $+$ corresponding to $\nu\cdot\xi_a > 0$ and $-$ to $\nu\cdot\xi_a < 0$. Now a simple rewriting proves the proposition; see [28] for details. □

Perturbation theory now shows that $G(\rho)$ itself has a limit when ρ becomes real, provided that $\lambda = \rho^2 + \epsilon_\alpha < \lambda_0$. An immediate consequence is:

Corollary 11. *Suppose that* $|\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a$, $\mu > (\dim X_a + 1)/2$. *There exists* $\delta > 0$ *with the following property. Suppose that for all* $b \not\leq a$, $\sup |\langle x^b\rangle^\mu V_b| < \delta$. *Then* u_ρ *extends continuously to*

$$\{z : \operatorname{Im} z > 0\} \cup \{z \geq 0 : z^2 + \rho_\perp^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'\}, \tag{37}$$

with $u_\rho - u_\rho^0 \in \mathcal{H}_r$ *for all* $r < -1/2$, *and* u_ρ *is analytic in* z *in* $\operatorname{Im} z > 0$.

The last part of the proof is completely analogous to the two-body argument, expressing the S-matrix as a pairing, and considering an analogous pairing for the complex exponential solutions. For 2-clusters α, β, the knowledge of $\mathcal{S}_{\alpha\beta}(\lambda)$ is equivalent to that of a renormalization $\mathcal{S}^\sharp_{\alpha\beta}(\lambda)$ with free scattering removed, see [28, Section 2]. Rather than giving its detailed definition here, we simply give an expression for its Schwartz kernel that is derived in [28, Corollary 2.2]. For $\rho \in X_a$ (real!), we write

$$\begin{aligned} U_\rho &= (\mathrm{Id} - R(\lambda + i0)I_a)u^0_\rho,\ \lambda = \rho^2 + \epsilon_\alpha, \\ u^0_\rho &= u^0_{\alpha,\rho} = e^{i\rho\cdot x_a}\psi_\alpha(x^a). \end{aligned} \tag{38}$$

Then, for $\lambda \notin \Lambda'$, $\langle x^b\rangle^\mu V_b \in L^\infty(X^b)$, $\mu > \dim X_a$,

$$\mathcal{S}^\sharp_{\alpha\beta}(\lambda,\omega,\omega') = \int_{\mathbb{R}^n} I_b U_\omega(x)\overline{e^{ix_b\cdot\omega'}\psi_\beta(x^b)}\,dx = \int_{\mathbb{R}^n} I_b U_\omega(x)e^{-ix_b\cdot\omega'}\overline{\psi_\beta(x^b)}\,dx; \tag{39}$$

in particular, $\mathcal{S}^\sharp_{\alpha\beta}(\lambda)$ has a continuous kernel. The analogous pairing for the complex exponential solutions is

$$\begin{aligned} G_{\alpha\alpha'}(\rho,\overline{\rho}+\zeta) &= \int_{\mathbb{R}^n} I_a(x)u_\rho(x)\overline{u^0_{\alpha',\overline{\rho}+\zeta}(x)}\,dx \\ &= \int_{\mathbb{R}^n} I_a(x)u_\rho(x)\overline{\psi_{\alpha'}(x^a)}e^{-i(\rho+\zeta)\cdot x_a}\,dx. \end{aligned} \tag{40}$$

By Corollary 11, if $\langle x^b\rangle^\mu V_b \in L^\infty(X^b)$ for all b and for some $\mu > \dim X_a$, then the integral in (40) converges for all ρ for which u_ρ exists, and for $\zeta \in \mathbb{R}^n$, since then the real parts of the exponentials cancel, and $I_a\psi_\alpha\overline{\psi_{\alpha'}} \in L^1(X_0)$, and the same holds for $I_a\overline{\psi_{\alpha'}}G(\rho)(I_a\psi_\alpha)$. Other properties of (40) follow immediately from Corollary 11.

Proposition 12. *Suppose that $|\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a$, $\mu > \dim X_a$, and V_b as in Corollary 11. Then $G_{\alpha\alpha'}$ is an analytic function of z in $\mathbb{C}\setminus\mathbb{R}$, and extends to be continuous on* (37)*. In addition,*

$$\lim_{|z|\to\infty} G_{\alpha\alpha'}(\rho,\overline{\rho}+\zeta) = \int_{\mathbb{R}^n} I_a\psi_\alpha\overline{\psi_{\alpha'}}e^{-i\zeta\cdot x_a}\,dx, \tag{41}$$

provided that $|z|\to\infty$ in $|\operatorname{Im} z| > C|\operatorname{Re} z|$, $C > 0$.

For fixed ρ real and ζ satisfying

$$\lambda = \rho^2 + \epsilon_\alpha = (\rho+\zeta)^2 + \epsilon_{\alpha''},$$

i.e. the equality of incoming and outgoing energies, we can relate $G_{\alpha\alpha''}(\rho,\rho+\zeta)$, ρ real, to the S-matrices as follows. Under our assumptions,

$$e^{i\rho\cdot x_a}(\mathrm{Id} + G_a(\rho)I_a)e^{-i\rho\cdot x_a}u_\rho = u^0_\rho = (\mathrm{Id} + R_a(\lambda+i0)I_a)U_\rho. \tag{42}$$

Applying $(\mathrm{Id} + R_a(\lambda+i0)I_a)^{-1}$ to both sides of (42), and using (35), we deduce that

$$\begin{aligned} U_\rho &= u_\rho \\ &\quad + 2\pi i\sum_{\alpha'}(\mathrm{Id} + R_a(\lambda+i0)I_a)^{-1}\mathcal{F}^{-1}_{X_a}H(\nu\cdot(\xi_a - \operatorname{Re}\rho))\delta_{|\xi_a|^2-(\lambda-\epsilon)}\mathcal{F}_{X_a}E_{a,\alpha'}I_a u_\rho. \end{aligned} \tag{43}$$

Here we used

$$(\rho-(\lambda-\epsilon-i0))^{-1}-(\rho-(\lambda-\epsilon+i0))^{-1}=-2\pi i\delta_{\rho-(\lambda-\epsilon)}$$

with $\rho=|\xi_a|^2$; the δ distribution here is the pull-back of the standard delta distribution on $\mathbb{R}$ by the map $\xi_a\mapsto|\xi_a|^2-(\lambda-\epsilon)$. Integrating against $I_a e^{-i(\rho+\zeta)\cdot x_a}\overline{\psi_{\alpha''}(x^a)}$, and yields

$$\begin{aligned}\mathcal{S}^\sharp_{\alpha\alpha''}(\lambda,\rho,\rho+\zeta)=&G_{\alpha\alpha''}(\rho,\rho+\zeta)\\ &-\sum_{\alpha'}\frac{i}{2\sqrt{\lambda-\epsilon_{\alpha'}}}\int_{\mathbb{S}_a(\sqrt{\lambda-\epsilon_{\alpha'}})}\mathcal{S}^\sharp_{\alpha'\alpha''}(\lambda,\rho',\rho+\zeta)\\ &\qquad H(\nu\cdot(\rho'-\operatorname{Re}\rho))G_{\alpha\alpha'}(\rho,\rho')\,d\rho'.\end{aligned}\tag{44}$$

This is an integral equation for $G_{\alpha\alpha''}$ in terms of $\mathcal{S}^\sharp_{\alpha\alpha''}(\lambda)$, $\mathcal{S}^\sharp_{\alpha'\alpha''}(\lambda)$, $\lambda=\rho^2+\epsilon_\alpha$. It can be solved due to the smallness assumption on I_a, so we deduce the following result, with the same proof as in [28].

Proposition 13. *([28, Proposition 5.2]) Suppose that $\rho^2+\epsilon_\alpha=\lambda=(\rho+\zeta)^2+\epsilon_{\alpha''}\in(-\infty,0)\setminus\Lambda'$, $|\rho_\perp|^2+\epsilon_\alpha\in(\epsilon_\alpha,\lambda_0)\setminus\Lambda'_a$, $\mu>\dim X_a$. There exists $\delta>0$ with the following property.*

Suppose that for all $b\not\leq a$, $\sup|\langle x^b\rangle^\mu V_b|<\delta$. Then the pairings $G_{\alpha\alpha''}(\rho,\rho+\zeta)$ are determined by the operators $\mathcal{S}^\sharp_{\alpha'\alpha''}(\lambda)$ given for all α' and α''.

We can now prove Theorem 4. Fix a non-empty open interval $I\subset(\epsilon_\alpha,\inf\Lambda_a)$ with $\sup I<\inf\Lambda_a$, and let $R=2\sqrt{\sup I-\epsilon_\alpha}$. Let $\zeta\in X_a$ satisfy $|\zeta|<R$, and $\frac14|\zeta|^2+\epsilon_\alpha\notin\Lambda'_a$. The last condition excludes a discrete set of values of $|\zeta|^2$ in $[0,R)$. Note that $\frac14|\zeta|^2+\epsilon_\alpha<0$.

We now choose $\lambda_1<\inf\Lambda_a$ in (21) so that $\frac14|\zeta|^2+\epsilon_\alpha<\lambda_1$, and

$$I\cap(\frac14|\zeta|^2+\epsilon_\alpha,\lambda_1)\neq\emptyset.\tag{45}$$

Let $\rho_\perp=-\zeta/2$, so $\rho_\perp^2+\epsilon_\alpha\in(\epsilon_\alpha,\lambda_1)\setminus\Lambda'_a$. Let $\nu\in X_a$ be such that $|\nu|^2=1$, $\nu\cdot\zeta=0$. Then for $\rho=z\nu+\rho_\perp$, $z\in\mathbb{C}$, $(\rho+\zeta)^2-\rho^2=2\rho\cdot\zeta+\zeta^2=0$. Having fixed ζ, $\rho_\perp$, ν, consider the energy $\rho^2+\epsilon_\alpha$ when z becomes real. Since $\rho^2=z^2+|\rho_\perp|^2$, as z varies over $[0,\sqrt{\lambda_0-\rho_\perp^2-\epsilon_\alpha})$, the energy varies over $[|\rho_\perp|^2+\epsilon_\alpha,\lambda_0)$. This intersects the given interval I by (45) as $\lambda_1<\lambda_0$.

Let $z_0>0$ satisfy $\lambda=\rho_\perp^2+z_0^2+\epsilon_\alpha\in(\epsilon_\alpha,\lambda_0)\setminus\Lambda'$. By Proposition 13 the limit of the pairing $G_{\alpha\alpha}(\rho,\rho+\zeta)$ as $z\to z_0$ (in $\operatorname{Im}z>0$) is determined by the scattering matrices $\mathcal{S}^\sharp_{\alpha'\alpha''}(\rho_\perp^2+z_0^2+\epsilon_\alpha)$. Thus, there exists a non-empty open interval of these values z_0 at which $G_{\alpha\alpha}(\rho,\rho+\zeta)$ is determined by $\mathcal{S}^\sharp_{\alpha'\alpha''}(\lambda)$, $\lambda\in I$.

Since the limit on any open interval in the boundary of its domain determines an analytic function, we deduce that knowing the S-matrix $\mathcal{S}^\sharp_{\alpha'\alpha''}$ in the interval I determines $G_{\alpha\alpha}(\rho,\zeta)$ for all ζ with $|\zeta|<R$.

Now let $z\to\infty$ through imaginary z. By (41), $G_{\alpha\alpha}(\rho,\rho+\zeta)$ converges to

$$\int_{\mathbb{R}^n}I_a|\psi_\alpha|^2e^{-i\zeta\cdot x_a}\,dx=\int_{X_a}e^{-i\zeta\cdot x_a}\left(\int_{X^a}I_a|\psi_\alpha|^2\,dx^a\right)dx_a.\tag{46}$$

But this is the Fourier transform of $\int_{X^a}I_a|\psi_\alpha|^2\,dx^a$ in X_a, evaluated at ζ. Hence $\mathcal{S}_{\alpha'\alpha''}(\lambda)$, $\lambda\in I$, determines the Fourier transform in x_a of the 'effective interaction'

$$\int_{X^a}I_a|\psi_\alpha|^2\,dx^a\tag{47}$$

in a ball of radius $2\sqrt{\sup I - \epsilon_\alpha}$, except possibly on the spheres $\frac{1}{4}|\zeta|^2 + \epsilon_\alpha \in \Lambda_a'$. However, $\int_{X^a} I_a |\psi_\alpha|^2 \, dx^a \in L^1(X_a)$, hence the Fourier transform is continuous, hence it is determined on the whole ball. This completes the proof of Theorems 4 and 5.

Finally, we return to the electron-ion scattering experiment, comparing its treatment as a two-body problem and as a many-body problem. If one only uses a fixed energy λ, and the potentials are exponentially decaying, in the two-body model the S-matrix determines the interaction, which is $V_{\alpha,\text{eff}}$ in this model (here α is the ground state of the ion). No such result exists if we consider the ion as a composite particle, i.e. if the experiment is treated as a many-body problem, although our theorem shows that if the 2-cluster to 2-cluster S-matrix is known in an *interval* of energies and if the interactions are weak, then we can determine $V_{\alpha,\text{eff}}$. Indeed, in the two-body fixed energy inverse result one has to let $\rho_\perp \to \infty$. If we consider the scattering experiment as a many-body problem, $\rho_\perp \to \infty$ causes serious complications, namely the possibility of the break up of the ion. This seems to limit the use of two-body results for composite particles, such as ions.

References

[1] Alberto-P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pages 65–73. Soc. Brasil. Mat., Rio de Janeiro, 1980.

[2] J. Dereziński and C. Gérard. *Scattering theory of classical and quantum N-particle systems.* Springer, 1997.

[3] V. Enss and R. Weder. The geometrical approach to multidimensional inverse scattering. *J. Math. Phys.*, 36:3902–3921, 1995.

[4] Volker Enss and Ricardo Weder. Inverse two-cluster scattering. *Inverse Problems*, 12(4):409–418, 1996.

[5] L. D. Faddeev. The inverse problem in the quantum theory of scattering. II. In *Current problems in mathematics, Vol. 3 (Russian)*, pages 93–180, 259. (loose errata). Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow, 1974.

[6] R. G. Froese and I. Herbst. Exponential bounds and absence of positive eigenvalues of N-body Schrödinger operators. *Commun. Math. Phys.*, 87:429–447, 1982.

[7] R. G. Froese and I. Herbst. A new proof of the Mourre estimate. *Duke Math. J.*, 49:1075–1085, 1982.

[8] Piotr G. Grinevich and Roman G. Novikov. Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials. *Comm. Math. Phys.*, 174(2):409–446, 1995.

[9] A. Hassell. Distorted plane waves for the 3 body Schrödinger operator. *Geom. Funct. Anal.*, 10:1–50, 2000.

[10] L. Hörmander. *The analysis of linear partial differential operators,* vol. 1-4. Springer-Verlag, 1983.

[11] Lars Hörmander. On the existence and the regularity of solutions of linear pseudo-differential equations. *Enseignement Math. (2)*, 17:99–163, 1971.

[12] Mark S. Joshi and Antônio Sá Barreto. Recovering asymptotics of short range potentials. *Comm. Math. Phys.*, 193(1):197–208, 1998.

[13] Masaki Kashiwara and Takahiro Kawai. Micro-hyperbolic pseudo-differential operators. I. *J. Math. Soc. Japan*, 27(3):359–404, 1975.

[14] G. Lebeau. Propagation des ondes dans les variétés à coins. *Ann. Scient. Éc. Norm. Sup.*, 30:429–497, 1997.

[15] R. B. Melrose. *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces.* Marcel Dekker, 1994.

[16] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math*, 31:593–617, 1978.

[17] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math*, 35:129–168, 1982.
[18] R. B. Melrose and M. Zworski. Scattering metrics and geodesic flow at infinity. *Inventiones Mathematicae*, 124:389–436, 1996.
[19] R. G. Novikov and G. M. Khenkin. The $\overline{\partial}$-equation in the multidimensional inverse scattering problem. *Uspekhi Mat. Nauk*, 42(3(255)):93–152, 255, 1987.
[20] Roman G. Novikov. The inverse scattering problem at fixed energy for the three-dimensional Schrödinger equation with an exponentially decreasing potential. *Comm. Math. Phys.*, 161(3):569–595, 1994.
[21] Roman G. Novikov. On inverse scattering for the N-body Schrödinger equation. *J. Funct. Anal.*, 159(2):492–536, 1998.
[22] Roman G. Novikov. On determination of the fourier transform of a potential from the scattering amplitude. *Inverse Problems*, 17:1243–1251, 2001.
[23] P. Perry, I. M. Sigal, and B. Simon. Spectral analysis of N-body Schrödinger operators. *Ann. Math.*, 114:519–567, 1981.
[24] J. Sjöstrand. Propagation of analytic singularities for second order Dirichlet problems. *Comm. in PDEs*, 5:41–94, 1980.
[25] E. Skibsted. Smoothness of N-body scattering amplitudes. *Reviews in Math. Phys.*, 4:619–658, 1992.
[26] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math.*, 125:153–169, 1987.
[27] M. Taylor. Grazing rays and reflection of singularities of solutions to wave equations. *Comm. Pure Appl. Math.*, 29:1–38, 1976.
[28] Gunther Uhlmann and András Vasy. Low-energy inverse problems in three-body scattering. *Inverse Problems*, 18(3):719–736, 2002.
[29] Gunther Uhlmann and András Vasy. Fixed energy inverse problem for exponentially decaying potentials. *Methods and Applications of Analysis*, To appear.
[30] A. Vasy. Structure of the resolvent for three-body potentials. *Duke Math. J.*, 90:379–434, 1997.
[31] A. Vasy. Scattering matrices in many-body scattering. *Commun. Math. Phys.*, 200:105–124, 1999.
[32] A. Vasy. Propagation of singularities in many-body scattering. *Ann. Sci. École Norm. Sup. (4)*, 34:313–402, 2001.
[33] A. Vasy. Propagation of singularities in many-body scattering in the presence of bound states. *J. Func. Anal.*, 184:177–272, 2001.
[34] A. Vasy. *Geometry and analysis in many-body scattering*. Cambridge University Press, To appear.
[35] X. P. Wang. On the uniqueness of inverse scattering for N-body systems. *Inverse Problems*, 10:765–784, 1994.
[36] X. P. Wang. High energy asymptotics for N-body scattering matrices with arbitrary channels. *Ann. Inst. H. Poincaré Phys. Théor.*, 65(1):81–108, 1996.
[37] Ricardo Weder. Generalized limiting absorption method and multidimensional inverse scattering theory. *Math. Methods Appl. Sci.*, 14(7):509–524, 1991.

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