LOW ENERGY INVERSE PROBLEMS IN THREE-BODY SCATTERING

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1. Introduction

Scattering theory is the analysis of the motion of several interacting particles. Inverse problems in scattering theory seek the answer to the question: can one determine the interactions between particles by a scattering experiment, so, for instance, does the scattering matrix, or a part of the scattering matrix, determine the interaction? The answer, and the difficulty, greatly depends on the part of the scattering data one wishes to use. Before explaining the various settings, we remark that the inverse problems are highly non-linear since the scattering data do not depend linearly on the interactions. Hence, one of the usual methods in the field is to transfer the problem to an asymptotically linear one, which is then easier to analyze.

The simplest setting is the study of high energy asymptotics of the scattering matrix, for then the potentials behave like small perturbations of the Laplacian. In addition, the leading term in the asymptotics depends linearly on them. This problem was studied, under various assumptions, by Enss and Weder [3, 4], Novikov [13] and Wang [20].

Here we are interested in finite energy problems, i.e. where the scattering data are known either only at a fixed energy, or in a fixed bounded interval of energies. Thus, the problems are not immediately equivalent to a linear perturbation problem. The flavor of the problem greatly depends on the part of the scattering matrix one wishes to use.

In some situations a principal symbol calculation for an S-matrix allows one to use the 2-body inverse results. An example of this is free-to-free scattering: as shown in [17], in three-body scattering the singularities of the free-to-free S-matrix at energy $\lambda > 0$ determine the S-matrices in all proper subsystems at all energies in $(0, \lambda)$, which then determine the pair interactions by two-body results. More precisely, the principal symbol of the part of the free-to-free S-matrix corresponding to a single collision is essentially given by the subsystem S-matrix at the energies in $(0, \lambda)$. Slightly more involved arguments using the results of [19] are expected to work in the many-body setting to show that the free-to-free S-matrix determines all pair interactions.

However, one may wish to study inverse problems where the parts of the S-matrix that are known do not have any singularities, so the previous method cannot be

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applied. An example is two-cluster to any other cluster scattering. Indeed, Skibsted [15] has shown that the corresponding S-matrices have smooth kernels apart from the diagonal singularity of the 2-cluster to same 2-cluster S-matrix. The latter agrees with the diagonal singularity of the kernel of the identity operator if the potentials are Schwartz, hence is of no help for the inverse problem; this is also the case with the two-body problem with Schwartz potentials. In certain ways, these are the most realistic problems, for in a scattering experiment one typically shoots a particle at a nucleus, atom, molecule, or other such composite 'cluster', which may break up as a result of the collision. One then measures the outcome of the experiment – this is exactly the information contained in the two-cluster to other-cluster S-matrices.

We study two-cluster to same two-cluster scattering for three-body Hamiltonians with real-valued potentials under the assumption that all unknown interactions are short-range and small. We show that the S-matrices $S_{\alpha\alpha'}(\lambda)$ in an energy interval I_λ below the break-up energy determine the Fourier transform of the effective interaction in a ball, whose radius is determined by the energy of the bound state under consideration. More precisely we prove the following theorem, whose statement uses some notation that we describe in detail in the next section.

Theorem 1.1. *Suppose that α is a channel in a 2-cluster a, $\dim X_a \geq 2$, and $\mu > \dim X_a$, and $V_a \in S^{-s}(X^a)$ for some $s > 0$. There exists a constant $\delta > 0$ such that the following statement holds.*

Suppose that $\sup |\langle w^b\rangle^\mu V_b| < \delta$ for all $b \neq a$. Suppose also that $I \subset (\epsilon_\alpha, 0)$ is a non-empty open interval, and let $R = 2\sqrt{\sup I - \epsilon_\alpha}$. Then $S_{\alpha'\alpha''}(\lambda)$, $\lambda \in I$, given for all bound states α', α'' with energy $\epsilon_{\alpha'}, \epsilon_{\alpha''} < 0$, determines the Fourier transform $\hat{V}_\alpha$ of the effective interaction,

$$V_\alpha = \int_{X^a} I_a |\psi_\alpha|^2 \, dw^a, \tag{1.1}$$

in the ball $B_0(R)$ of radius R centered at the origin in X_a.

If we only want to determine $\hat{V}_\alpha$ in a smaller ball, we need even less information. There is a variety of statements one can make using different information; we only make the following one.

Theorem 1.2. *Suppose that a is a 2-cluster, $\dim X_a \geq 2$, $V_a \in S^{-s}(X^a)$ for some $s > 0$, α is the ground state of H^a with $\epsilon_\alpha < 0$. Let $\epsilon' > \epsilon_\alpha$ be the next eigenvalue of H^a, or 0 if this does not exist. Let $\mu > \dim X_a$. There exists a constant $\delta > 0$ such that the following statement holds.*

Suppose that $\sup |\langle w^b\rangle^\mu V_b| < \delta$, for all $b \neq a$. Suppose also that $I \subset (\epsilon_\alpha, 0)$ is a non-empty open interval, and let $R = 2\sqrt{\min(\sup I, \epsilon') - \epsilon_\alpha}$. Then $S_{\alpha\alpha}(\lambda)$, $\lambda \in I$, determines the Fourier transform $\hat{V}_\alpha$ of the effective interaction,

$$V_\alpha = \int_{X^a} I_a |\psi_\alpha|^2 \, dw^a, \tag{1.2}$$

in the ball $B_0(R)$ of radius R centered at the origin in X_a.

Since we are working below the break-up energy, heuristically one expects that the composite particle may be regarded as a single particle, and two-body methods may be applied. This turns out to be false, at least when taken literally. Indeed, many two-body methods, one of which we describe below, rely on allowing large complex momenta for the particles, which in turn permits the break-up of a cluster.

Hence, one of the themes of this paper is the extent to which composite particles may be regarded as a single unit below break-up energies for the purposes of inverse problems in scattering theory.

Our strategy is similar to how one approaches low energy inverse problems in two-body scattering, which we now briefly recall. Then the kernel of the (relative) S-matrix is also smooth for Schwartz potentials, once the kernel of Id is subtracted, and is conormal to the diagonal for symbolic potentials. Faddeev [6, 5, 7] started the study of exponential solutions, i.e. solutions of $(H-\lambda)u=0$ of the form $u=u_\rho=e^{i\rho\cdot w}(1+v_\rho(w))$, v_ρ 'small', ρ not necessarily real, and $\rho\cdot\rho=\lambda$. Even if λ is fixed, by allowing ρ to be complex, one can take $\rho\to\infty$, so that $v_\rho\to 0$ in an appropriate sense. Provided that one can relate the pairing

$$\int u_\rho V e^{-i\rho'\cdot w}\,dw, \tag{1.3}$$

taking the limit $\rho\to\infty$ (and $\rho'\to\infty$) becomes an analogue of the high energy limit, with the leading term linear in the potential. In other words, the high energy asymptotics is replaced by high complex momentum asymptotics, as pioneered by Calderón, see [1, 16, 11].

While the S-matrix is not analytic in the energy λ unless other assumptions are made, for a very large class of potentials (including Schwartz potentials) u_ρ is meromorphic (indeed, analytic if V is small) in the complex one-dimensional space (i.e. line) spanned by $\operatorname{Im}\rho$, provided this line is fixed. In other words, u_ρ is analytic in z (in $\operatorname{Im} z\neq 0$), where we write $\rho=z\nu+\rho_\perp$, $\nu,\rho_\perp$ real, $\nu\cdot\rho_\perp=0$. Moreover, from $\operatorname{Im} z>0$, u_ρ extends continuously to $\operatorname{Im} z=0$ if V is small, and to a large subset of the real line in z otherwise. This can be exploited in problems where the S-matrix is known in an interval, as in the work of Novikov [12], Weder [22] and Isozaki [9]. Indeed, one shows first that the S-matrix in an energy interval determines the pairing (1.3) in a corresponding interval in z, then uses that the boundary values of a meromorphic function determine the function, finally lets $z\to\infty$ and uses the high momentum limit to determine the Fourier transform of V.

In the three-body setting, there are similar exponential solutions corresponding to a bound state ψ_α of a subsystem H^a, so $(H^a-\epsilon_\alpha)\psi_\alpha=0$. Namely, one considers solutions of $(H-\lambda)u_\rho=0$ of the form $u_\rho=e^{i\rho\cdot w_a}(\psi_\alpha(w^a)+v_\rho(w))$ where ρ is in the complexification $\mathbb{C}(X_a)$ of X_a, $\rho\cdot\rho=\lambda-\epsilon_\alpha$, $\lambda\in\mathbb{C}$. In fact, this construction works in great generality, though the structure of u_ρ changes with ρ. In this paper we keep $|\rho_\perp|<\sqrt{-\epsilon_\alpha}$, in which case u_ρ can be constructed by perturbation theory. In particular, it is easy to see that u_ρ depends analytically on z. Here perturbation theory is understood loosely, for even if the unknown interactions V_b are small, they are *not* a compact perturbation of $H_a=\Delta+V_a$, for they do *not* decay at infinity. In particular, if V_b becomes large, the structure of u_ρ changes drastically, and its analyticity in z is far from clear. Even for small V_b, if we take $\rho_\perp$ large, the cluster will be allowed to break up, creating a major difficulty for fixed energy inverse problems.

On the other hand, the connection to the S-matrices is less immediate than in the two-body setting. In general, one expects that all parts of the S-matrix need to be known at a certain energy to determine the pairing $\int u_\rho I_a\overline{\psi_\alpha(w^a)}e^{-i\rho'\cdot w_a}\,dw$. This can be seen explicitly from the statement of our main theorem, when it is applicable: $S_{\alpha'\alpha''+}(\lambda)$ play a role in the statement for all α', α''. In fact, analogously to an observation of Novikov [14], by reducing $|\rho_\perp|$ further (than $|\rho_\perp| < \sqrt{-\epsilon_\alpha}$ mentioned above), some of the two-cluster to two-cluster S-matrices can be eliminated, as was done in the second theorem. However, the restriction on $\rho_\perp$ implies restrictions on the frequencies at which $\hat{V}_\alpha$ can be recovered.

The structure of this paper is the following. After recalling the usual many-body notation, we construct the exponential eigenfunctions, and we study their limit as ρ becomes real. We use this to relate the corresponding pairing to the S-matrix. Finally, we apply this to the study of the inverse problem by taking $\rho \to \infty$, and prove Theorems 1.1 and 1.2.

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2. Notation and preliminaries

Below the notation is that of [19], which is to say it is the standard many-body notation as in [2]. First, $X = X_0 = \mathbb{R}^n$ is the total configuration space, equipped with the standard Euclidean metric g. The collision planes, X_a, $a \in I$, I finite, are linear subspaces of X_0, and X^a is the orthocomplement of X_a in X_0. We assume that $\{X_a :\ a \in I\}$ is closed under intersections, includes X_0 and $X_1 = \{0\}$. We write $w = (w_a, w^a)$ for coordinates on $X = X_a \oplus X^a$, and identify X_a^* with X_a via the metric g.

We write $\mathbb{S}_a = C_a$ for the unit sphere in X_a (with respect to the metric inherited from X_0). Geometrically it is better to consider C_a as 'the sphere at infinity', but for the sake of simplicity (and to conform with the usual many-body conventions) we adopt the unit sphere point of view. We also let $C_{a,\mathrm{sing}} = \cap_{C_b \subsetneq C_a} C_b$ be the singular, $C_{a,\mathrm{reg}} = C_a \setminus C_{a,\mathrm{sing}}$ the regular part of C_a. Recall also that a two-cluster a, denoted by $\#a = 2$, is a (non-trivial) cluster such that $C_b \subset C_a$ implies $b = a$ (or $b = 1$ or $a = 1$ provided that $C_1 = \emptyset$ is included in the collection of C_c's). Thus, two-clusters are the most singular clusters, and in particular if a is a 2-cluster, then $C_{a,\mathrm{sing}} = \emptyset$. The collection of collision plans corresponds to a three-body geometry if every cluster, except 0 and 1, is a two-cluster.

Concerning the analytic aspects, we write $L^2_p(X_a)$ for the weighted L^2 space $L^2(X_a, \langle w_a \rangle^{2p}\, dw_a)$ on X_a. We also write $H^s_p(X_a) = H^{s,p}(X_a)$ for the Sobolev space corresponding to this weight.

We let H^a be the subsystem Hamiltonian on X^a, i.e.

$$H^a = \Delta_{X^a} + \sum_{X^b \subset X^a} V_b,$$

and I_a is the intercluster interaction $I_a = V - \sum_{X^b \subset X^a} V_b$. The unreduced subsystem Hamiltonian acts on functions on the whole space $\mathbb{R}^n$; it is

$$H_a = \Delta_{X_a} + H^a.$$

In addition, R^a, resp. R_a, denote the resolvent of the reduced, resp. unreduced, Hamiltonian of the subsystem a, i.e. $R^a(\sigma) = (H^a - \sigma)^{-1}$, $R_a(\sigma) = (H_a - \sigma)^{-1}$ for $\sigma \notin \mathbb{R}$. We write Λ for the set of thresholds of H, which is defined inductively over the proper subsystems by

$$\Lambda_a = \cup_{X_b \supsetneq X_a} \Lambda'_b, \quad \Lambda'_b = \Lambda_b \cup \mathrm{spec}_{pp}(H^b),$$

and we usually denote the spectral parameter by λ. In particular, for a three-body Hamiltonian H, if a is a 2-cluster then $\Lambda_a = \{0\}$, $\Lambda'_a = \{0\} \cup \mathrm{spec}_{pp}(H^a)$, and $\Lambda_1 = \Lambda = \cup_{\# a=2} \Lambda'_a$, $\Lambda' = \Lambda'_1 = \Lambda \cup \mathrm{spec}_{pp}(H)$.

Let ψ_α denote the normalized L^2 eigenfunctions of H^a, and let ϵ_α be the bound state energy in ψ_α: $(H^a - \epsilon_\alpha)\psi_\alpha = 0$. If ϵ_α is not an eigenvalue of a proper subsystem of H^a, then $\psi_\alpha \in e^{-\mu_\alpha |w_a|} L^2(X^a)$ for some $\mu_\alpha > 0$ given by the next threshold above ϵ_α (see [8]). We call such a bound state α a non-threshold bound state.

Usually the Poisson operator and the scattering matrices are considered as operators on functions on unit spheres in appropriate spaces. When we investigate the real-frequency behavior of the exponential solutions that we construct in the next section, it will be convenient to consider the Poisson operators and S-matrices as operators acting on functions on spheres of different radii. Thus, we replace the unit sphere in X_a by the sphere $\mathbb{S}_a(\sqrt{\lambda - \epsilon_\alpha})$ as the parameterization space for the Poisson operators; here

$$\mathbb{S}_a(\sigma) = \{\xi_a \in X_a : \ |\xi_a| = \sigma\}.$$

The regular and singular parts of $\mathbb{S}_a(\sigma)$ are defined analogously to those of C_a.

The thus normalized forward Poisson operator of H_a in channel α is given by

$$\begin{aligned}\widetilde{\mathcal{P}}_{\alpha,+}(\lambda) g = c_\alpha \psi_\alpha(w^a) \int_{\mathbb{S}_a(\sqrt{\lambda - \epsilon_\alpha})} e^{-i w_a \cdot \omega_a} g \, d\omega_a, \quad g \in \mathcal{C}_c^\infty(\mathbb{S}_{a,\mathrm{reg}}(\sqrt{\lambda - \epsilon_\alpha})),\\ c_\alpha = (\lambda - \epsilon_\alpha)^{\frac{m-1}{4}} e^{-\frac{m-1}{4}\pi i} (2\pi)^{-\frac{m-1}{2}}, \ m = \dim X_a,\end{aligned}$$

where $d\omega_a$ is the standard measure on $\mathbb{S}_a(\sqrt{\lambda - \epsilon_\alpha})$ normalized to have volume equal to that of the unit sphere. The Poisson operator of H in channel α is then

$$\begin{aligned}\mathcal{P}_{\alpha,+}(\lambda) g &= \widetilde{\mathcal{P}}_{\alpha,+}(\lambda) g - R(\lambda + i0)((H - \lambda)\widetilde{\mathcal{P}}_{\alpha,+}(\lambda) g)\\ &= \widetilde{\mathcal{P}}_{\alpha,+}(\lambda) g - R(\lambda + i0) I_a \widetilde{\mathcal{P}}_{\alpha,+}(\lambda) g.\end{aligned}$$

Note that if $\langle w^b \rangle^\mu V_b \in L^\infty(X^b)$, then

$$\langle w_a \rangle^p I_a \widetilde{\mathcal{P}}_{\alpha,+}(\lambda) g \in L^2(X_0), \ g \in \mathcal{C}_c^\infty(\mathbb{S}_{a,\mathrm{reg}}(\sqrt{\lambda - \epsilon_\alpha})), \ p < \mu - 1/2,$$

so the preceeding expression makes $\mathcal{P}_{\alpha,+}(\lambda)$ well-defined for $\mu > 1$. There is some arbitrariness in the normalization of $\widetilde{\mathcal{P}}_{\alpha,+}(\lambda)$. The present definition is adopted because of its connection with the asymptotic behavior of $\widetilde{\mathcal{P}}_{\alpha,+}(\lambda) g$ at infinity, see [18].

The backward Poisson operator is defined similarly, with

$$\widetilde{\mathcal{P}}_{\alpha,-}(\lambda) g = \overline{c_\alpha} \psi_\alpha(w^a) \int_{\mathbb{S}_a(\sqrt{\lambda - \epsilon_\alpha})} e^{i w_a \cdot \omega_a} g \, d\omega_a, \quad g \in \mathcal{C}_c^\infty(\mathbb{S}_{a,\mathrm{reg}}(\sqrt{\lambda - \epsilon_\alpha})), \tag{2.1}$$

$$\begin{aligned}\mathcal{P}_{\alpha,-}(\lambda) g &= \widetilde{\mathcal{P}}_{\alpha,-}(\lambda) g - R(\lambda - i0)((H - \lambda)\widetilde{\mathcal{P}}_{\alpha,-}(\lambda) g)\\ &= \widetilde{\mathcal{P}}_{\alpha,-}(\lambda) g - R(\lambda - i0) I_a \widetilde{\mathcal{P}}_{\alpha,-}(\lambda) g.\end{aligned}$$

The scattering matrix relates the forward and backward Poisson operators, i.e. connects incoming and outgoing data. Here we only need an expression connecting the S-matrices to the 'Green pairing'.

Proposition 2.1. *Let α and β be channels associated to the clusters a and b respectively, and suppose that $\lambda \notin \Lambda'$. Let $u_+ = \mathcal{P}_{\alpha,+}(\lambda)g_+$, $u_- = \widetilde{\mathcal{P}}_{\beta,-}(\lambda)g_-$, $g_+ \in \mathcal{C}_c^\infty(\mathbb{S}_{a,reg}(\sqrt{\lambda-\epsilon_\alpha}))$, $g_- \in \mathcal{C}_c^\infty(C_{b,reg}(\sqrt{\lambda-\epsilon_\beta}))$. Then*

$$\begin{aligned}\langle u_+, I_b u_-\rangle &= \langle u_+, (H-\lambda)u_-\rangle - \langle (H-\lambda)u_+, u_-\rangle\\ &= 2i\sqrt{\lambda-\epsilon_\beta}(\langle \mathcal{S}_{\alpha\beta+}(\lambda)g_+, g_-\rangle - \delta_{\alpha\beta}\langle g_+, \widetilde{\mathcal{S}}_{\beta\beta-}(\lambda)g_-\rangle)\\ &= 2i\sqrt{\lambda-\epsilon_\beta}\langle (\mathcal{S}_{\alpha\beta+}(\lambda) - \delta_{\alpha\beta}\mathcal{S}_+(\lambda))g_+, g_-\rangle.\end{aligned} \tag{2.2}$$

where the L^2 pairings on the spheres are with respect to the standard measures normalized to have the volume of the unit sphere, $\delta_{\alpha\beta}$ is the Kronecker delta function, and $\mathcal{S}_{\beta\beta-}(\lambda)$ is the free scattering matrix on X_a at energy $\lambda-\epsilon_\beta$, hence it is a constant multiple of pull-back by the antipodal map on $\mathbb{S}_b(\sqrt{\lambda-\epsilon_\beta})$.

Proof. In each of the two relevant microlocal regions, namely incoming and outgoing, one of the two functions u_+ and u_- has trivial asymptotics. Thus, the calculation of [18, Section 3] applies separately in each region. □

With the current normalization, the S-matrix is geometric, i.e. under the free evolution particles incoming at direction ω exit in the opposite direction $-\omega$. We now introduce the relative S-matrix (relative to free motion) as follows. Let p^* denote pull-back by the antipodal map, and let

$$\mathcal{S}^\sharp_{\alpha\beta+}(\lambda) = \frac{1}{2i\sqrt{\lambda-\epsilon_\beta}}\left(\mathcal{S}_{\alpha\beta+}(\lambda) - \delta_{\alpha\beta}\widetilde{\mathcal{S}}_+(\lambda)\right)p^*. \tag{2.3}$$

If α, β are 2-clusters, and $V_b \in S^{-s}(X^b)$, $s>0$, is a symbol, then the kernel of $\mathcal{S}^\sharp_{\alpha\beta+}(\lambda)$ is conormal to the diagonal (in the sense that it is smooth for $\alpha\neq\beta$, conormal for $\alpha=\beta$), and if V_b is Schwartz, $\mathcal{S}^\sharp_{\alpha\beta+}(\lambda)$ is a smoothing operator, i.e. it has a smooth kernel, as was proved by Skibsted [15]. If $V_b \in \mathcal{S}(X^b)$ for all b, this can be seen from (2.2), for $I_a\widetilde{\mathcal{P}}_{\alpha,-}(\lambda) : \mathcal{C}^{-\infty}(\mathbb{S}_a(\sqrt{\lambda-\epsilon_\alpha})) \to \mathcal{S}(X_0)$. In general, one needs to construct a better approximation for $\mathcal{P}_{\alpha+}(\lambda)$ (better than $\widetilde{\mathcal{P}}_{\alpha+}(\lambda)$); this is what Skibsted did in [15]. If $\langle w^b\rangle^\mu V_b \in L^\infty(X^b)$, with $\mu > \dim X_a$, we may take $g_\pm$ to be delta distributions directly (without using Skibsted's construction, hence without a symbolic assumption), $g_+ = \delta_\omega$, $g_- = \delta_{\omega'}$, $\omega \in \mathbb{S}_a(\sqrt{\lambda-\epsilon_\alpha})$, $\omega' \in \mathbb{S}_b(\sqrt{\lambda-\epsilon_\beta})$. Writing

$$U_\rho = (\mathrm{Id} - R(\lambda+i0)I_a)u^0_\rho,\ \lambda = \rho^2+\epsilon_\alpha, \tag{2.4}$$

we thus deduce the following.

Corollary 2.2. *For $\lambda\notin\Lambda'$, $\langle w^b\rangle^\mu V_b \in L^\infty(X^b)$, $\mu > \dim X_a$,*

$$\mathcal{S}^\sharp_{\alpha\beta+}(\lambda,\omega,\omega') = \int_{\mathbb{R}^n} I_b U_\omega \overline{e^{iw_b\cdot\omega'}\psi_\beta} = \int_{\mathbb{R}^n} I_b U_\omega e^{-iw_b\cdot\omega'}\overline{\psi_\beta}, \tag{2.5}$$

hence $\mathcal{S}^\sharp_{\alpha\beta+}$ has a continuous kernel.

3. Exponential eigenfunctions for three-body Hamiltonians

In this section we construct exponential solutions of $(H-\lambda)u=0$ in the three-body setting. First, for $\rho\in\mathbb{C}(X_a)$, i.e. $\operatorname{Re}\rho, \operatorname{Im}\rho \in X_a$, let

$$u^0_\rho = u^0_{\alpha,\rho} = e^{i\rho\cdot w_a}\psi_\alpha(w^a).$$

Thus, u^0_ρ is an 'exponential eigenfunction' of H_a, namely

$$(H_a - \lambda)u^0_\rho = 0, \qquad \rho\cdot\rho = \lambda - \epsilon_\alpha.$$

We assume everywhere that $\dim X_a \geq 2$.

For the Hamiltonian H, we then seek exponential solutions u of the form

$$u = u_\rho = e^{i\rho\cdot w_a}(\psi_\alpha(w^a) + v),\ \rho\cdot\rho = \lambda - \epsilon_\alpha,\ \rho \in \mathbb{C}(X_a), \tag{3.1}$$

where v is supposed to be 'small', and $\mathbb{C}(X_a)$ denotes the complexification of X_a, i.e. $\operatorname{Re}\rho, \operatorname{Im}\rho \in X_a$. Substituting into $(H-\lambda)u = 0$, we obtain

$$(\Delta + 2\rho\cdot D_w + V_a + I_a - \epsilon_\alpha)v = -I_a\psi_\alpha. \tag{3.2}$$

The right hand side decays at infinity since ψ_α does so in X^a, and I_a decays away from $\cup_{\#b=2,\ b\neq a} C_b$. More precisely, we have the following lemma.

Lemma 3.1. *Suppose that $\mu > 0$. There exists $C > 0$ with the following property. If $\langle w^b\rangle^\mu V_b \in L^\infty(X^b)$ for all two-clusters b with $b \neq a$ then*

$$|I_a| \leq C\langle w_a\rangle^{-\mu}\langle w^a\rangle^\mu \sup|\langle w^b\rangle^\mu V_b| \tag{3.3}$$

Proof. As $X_a \cap X_b = \{0\}$ for $b \neq a$, $X^a \oplus X^b = X_0$, hence for some $C' > 0$

$$\langle w^a\rangle\langle w^b\rangle \geq C'\langle w\rangle.$$

Thus, for $\mu > 0$,

$$\langle w^b\rangle^{-\mu} \leq C\langle w_a\rangle^{-\mu}\langle w^a\rangle^\mu, \tag{3.4}$$

proving the lemma. □

Corollary 3.2. *Let $\mu > 0$ and V_b as above. Then*

$$I_a\psi_\alpha \in L^2_p(X_0),\ p < \mu - \frac{\dim X_a}{2}. \tag{3.5}$$

Thus, we need to construct a right inverse $G(\rho)$ to

$$P(\rho) = \Delta + 2\rho\cdot D_w + V - \epsilon_\alpha$$

that can be applied to elements of $L^2_p(X_0)$. Once this is done,

$$u(\rho) = e^{i\rho\cdot w_a}(\psi_\alpha(w^a) - G(\rho)I_a\psi_\alpha)$$

is the solution to the original problem. Below we write

$$\begin{aligned} P_0(\rho) &= \Delta + 2\rho\cdot D_w - \epsilon_\alpha,\\ P_a(\rho) &= \Delta + 2\rho\cdot D_w + V_a - \epsilon_\alpha\\ &= (\Delta_{w_a} + 2\rho\cdot D_{w_a} - \epsilon_\alpha) + (\Delta_{w^a} + 2\rho\cdot D_{w^a} + V_a(w^a)). \end{aligned} \tag{3.6}$$

Since a right inverse $G_a(\rho)$ of $P_a(\rho)$ can be constructed explicitly, perturbation theory will give the existence of $G(\rho)$, provided that I_a is small.

It is convenient to represent ρ as

$$\rho = z\nu + \rho_\perp,\ \rho_\perp, \nu \in X_a,\ z \in \mathbb{C},\ \rho_\perp\cdot\nu = 0.$$

We will take $|\rho_\perp|$ sufficiently small. To see why the size of $\rho_\perp$ matters, consider

$$G_0(\rho) = \mathcal{F}^{-1}(|\xi|^2 + 2\rho\cdot\xi - \epsilon_\alpha)^{-1}\mathcal{F},$$

so $P_0(\rho)G_0(\rho) = \mathrm{Id}$ e.g. on Schwartz functions. Thus, on the Fourier transform side $G_0(\rho)$ acts via multiplication by $(|\xi|^2 + 2\rho\cdot\xi - \epsilon_\alpha)^{-1}$. For $z \notin \mathbb{R}$, this distribution is conormal to

$$\begin{aligned} S(\rho) &= \{\xi \in X_0 :\ |\xi|^2 + 2\,\mathrm{Re}\,\rho\cdot\xi - \epsilon_\alpha = 0,\ \mathrm{Im}\,\rho\cdot\xi = 0\} \\ &= \{\xi \in X_0 :\ (\xi + \rho_\perp)^2 = \rho_\perp^2 + \epsilon_\alpha,\ \nu\cdot\xi = 0\}. \end{aligned} \tag{3.7}$$

Note that $S(\rho)$ actually depends only on $\rho_\perp$ and ν, not on z. Now, for $|\rho_\perp| < \sqrt{-\epsilon_\alpha}$ (note that $\epsilon_\alpha < 0$), $S(\rho) = \emptyset$, so $P_0(\rho)$ is elliptic 'at infinity' in a sense discussed by Melrose [10], namely as an element of $\mathrm{Diff}^2_{\mathrm{sc}}(\overline{X_0})$, $\overline{X_0}$ being the radial compactification of X_0.

Below we assume that

$$|\rho_\perp| < \sqrt{-\epsilon_\alpha}, \tag{3.8}$$

where $P_0(\rho)$ is elliptic. This does *not* mean that $P(\rho)$ itself is elliptic; indeed $P_a(\rho)$ cannot be such thanks to the bound state ψ_α. For

$$\lambda_1 \in [|\rho_\perp|^2 + \epsilon_\alpha, 0) \setminus \Lambda'_a, \tag{3.9}$$

let $e_a(\lambda_1)$ be the orthogonal projection to the L^2 eigenfunctions of H^a with eigenvalue $\leq \lambda_1$,

$$e_a(\lambda_1) = \sum_{\alpha':\ \epsilon_{\alpha'} \leq \lambda_1} (\psi_{\alpha'} \otimes \overline{\psi_{\alpha'}}) \in \mathcal{B}(L^2(X^a), L^2(X^a)),$$

and let E_a be its extension to X_0 via tensoring by Id_{X_a}, so

$$E_a(\lambda_1) = \sum_{\alpha'} \mathrm{Id}_{X_a} \otimes (\psi_{\alpha'} \otimes \overline{\psi_{\alpha'}}). \tag{3.10}$$

Since eigenvalues of H^a can only accumulate at $\Lambda_a = \{0\}$, e_a is finite rank. We also let

$$\lambda_0 = \inf\,(\Lambda'_a \cap (\lambda_1, +\infty)) > \lambda_1. \tag{3.11}$$

The particular choice of λ_1, provided that it is sufficiently close to 0, does not play a major role in our arguments, so we usually simply write e_a for $e_a(\lambda_1)$, etc.

We restrict the region (3.8) slightly further and work in the region

$$\mathbb{C}(X_a)^\circ_\alpha = \{(z, \nu, \rho_\perp) :\ \mathrm{Im}\, z \neq 0,\ |\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a\}, \tag{3.12}$$

i.e. we also assume that $|\rho_\perp|^2 + \epsilon_\alpha$ is not an eigenvalue of H^a. Again, we do not indicate λ_0 explicitly in the notation.

Since the ranges of E_a and $\mathrm{Id} - E_a$ play a rather different role below, we introduce weighted spaces that reflect this. So for $p \in \mathbb{R}$ we let

$$\begin{aligned} \mathcal{H}_p =& (L^2_p(X_a) \otimes \mathrm{Ran}\, e_a) \oplus (L^2(X_a) \otimes \mathrm{Ran}(\mathrm{Id} - e_a)) \\ &\subset (L^2_p(X_a) \otimes \mathrm{Ran}\, e_a) \oplus L^2(X_0), \end{aligned} \tag{3.13}$$

with e_a considered as a bounded operator on $L^2(X^a)$. Thus, we allow weights on the range of E_a, but not on its orthocomplement. Again, $\mathrm{Ran}\, e_a$ is finite dimensional, hence it is closed in $L^2_r(X^a)$ for all $r \in \mathbb{R}$, while $\mathrm{Ran}(\mathrm{Id} - e_a)$ is closed in $L^2(X^a)$, so $L^2(X_a) \otimes \mathrm{Ran}(\mathrm{Id} - e_a)$ is a closed subspace of $L^2(X_0)$. Thus, for all $p \in \mathbb{R}$, $\mathcal{H}_p$ is a Hilbert space with norms induced on the summands by the $L^2_p(X_a)$ and $L^2(X_0)$ norms respectively.

We start the construction of $G(\rho)$ by analyzing $G_a(\rho)$. In view of (3.6), taking the Fourier transform in X_a makes the invertibility of $P_a(\rho)$ into a question on the behavior of the resolvent of $H^a = \Delta_{w^a} + V_a(w^a)$, uniformly across the spectrum. That is,

$$\mathcal{F}_{X_a} P_a(\rho) \mathcal{F}_{X_a}^{-1} = \Delta_{X^a} + V_a - (\epsilon_\alpha - |\xi_a|^2 - 2\rho \cdot \xi_a),$$

acting pointwise in ξ_a, so

$$\mathcal{F}_{X_a} G_a(\rho) \mathcal{F}_{X_a}^{-1} = R^a(\epsilon_\alpha - |\xi_a|^2 - 2\rho \cdot \xi_a),$$

with $R^a(\sigma) = (H^a - \sigma)^{-1}$, provided we show that this makes sense – the only issues being the behavior for real σ and bounds as $|\xi_a| \to \infty$.

So let

$$F_\rho : \xi_a \mapsto \epsilon_\alpha - |\xi_a|^2 - 2\rho \cdot \xi_a.$$

Thus,

$$\begin{aligned} \operatorname{Im} F_\rho(\xi_a) &= -2(\operatorname{Im} z)(\nu \cdot \xi_a) \\ \operatorname{Re} F_\rho(\xi_a) &= \epsilon_\alpha + \rho_\perp^2 - ((\xi_a)_\perp + \rho_\perp)^2 - (\nu \cdot \xi_a)^2 - 2(\operatorname{Re} z)(\nu \cdot \xi_a) \\ &\leq \epsilon_\alpha + \rho_\perp^2 - (\nu \cdot \xi_a)^2 - 2(\operatorname{Re} z)(\nu \cdot \xi_a) \\ &\leq \epsilon_\alpha + \rho_\perp^2 + (\operatorname{Re} z)^2, \end{aligned} \tag{3.14}$$

where $(\xi_a)_\perp$ is the orthogonal projection of ξ_a to the orthocomplement of the span of ν, and $\nu \cdot \xi_a$ is the component of ξ_a parallel to ν. If $F_\rho(\xi_a)$ is real and $\operatorname{Im} z \neq 0$, then $\nu \cdot \xi_a = 0$, hence

$$\epsilon_\alpha - |\xi_a|^2 - 2\rho \cdot \xi_a = \epsilon_\alpha + \rho_\perp^2 - (\xi_a + \rho_\perp)^2 \leq \epsilon_\alpha + \rho_\perp^2 < 0$$

under our assumptions. In fact,

$$\operatorname{Re} F_\rho(\xi_a) \leq \epsilon_\alpha + \rho_\perp^2 - 2(\operatorname{Re} z)(\nu \cdot \xi_a) = \epsilon_\alpha + \rho_\perp^2 + 2\frac{\operatorname{Re} z}{\operatorname{Im} z} \operatorname{Im} F_\rho(\xi_a),$$

so $\operatorname{Re} F_\rho(\xi_a) > \epsilon_\alpha + \rho_\perp^2$, which holds in particular if $\operatorname{Re} F_\rho(\xi_a) \geq 0$, implies that $\nu \cdot \xi_a$ is non-zero and has the same sign as $-\operatorname{Re} z$. Indeed, we deduce that

$$\operatorname{Re} F_\rho(\xi_a) > \epsilon_\alpha + \rho_\perp^2 \Rightarrow \frac{\operatorname{Re} z}{\operatorname{Im} z} \operatorname{Im} F_\rho(\xi_a) > 0. \tag{3.15}$$

In fact, we deduce the quantitative bound

$$\operatorname{Re} F_\rho(\xi_a) > \epsilon_\alpha + \rho_\perp^2 \Rightarrow \operatorname{Im} F_\rho(\xi_a) > (\operatorname{Re} F_\rho(\xi_a) - (\epsilon_\alpha + \rho_\perp^2)) \frac{|\operatorname{Im} z|}{|\operatorname{Re} z|} \tag{3.16}$$

Moreover, if $\dim X_a \geq 2$, as is assumed throughout this paper, $F_\rho : X_a^* \to \mathbb{C}$ (the latter considered as a 2-dimensional real manifold) has a surjective differential unless $(\xi_a)_\perp = -\rho_\perp$. Indeed, since $d \operatorname{Im} F_\rho$ is nonzero, and is a multiple of $d(\nu \cdot \xi_a)$, $d \operatorname{Im} F_\rho$ and $d \operatorname{Re} F_\rho$ are linearly independent if and only if $d(((\xi_a)_\perp + \rho_\perp)^2) \neq 0$, i.e. if and only if $(\xi_a)_\perp \neq -\rho_\perp$.

Note also that for any fixed ρ there exists $C > 0$ such that

$$\operatorname{Re} F_\rho(\xi_a) < C - |\xi_a|^2/2. \tag{3.17}$$

The structure of $R^a(\sigma)$ corresponds to that of $R_0^a(\sigma)$ and $(\operatorname{Id} + V_a R_0^a(\sigma))^{-1}$, where $R_0^a(\sigma)$ denotes the resolvent of Δ_{X^a}, for $R^a(\sigma) = R_0^a(\sigma)(\operatorname{Id} + V_a R_0^a(\sigma))^{-1}$. It satisfies automatically that for $\operatorname{Re} \sigma < \inf \operatorname{spec} H^a$,

$$\|R^a(\sigma)\|_{\mathcal{B}(L^2, L^2)} \leq (\inf \operatorname{spec} H^a - \operatorname{Re} \sigma)^{-1}.$$

In view of (3.17), for $\phi \in \mathcal{C}_c^\infty(X_a^*)$ identically 1 on a large enough ball, $G_a \mathcal{F}_{X_a}^{-1}(1 - \phi)\mathcal{F}_{X_a}$ is bounded on L^2. Hence, we only need to be concerned about what happens in a compact set in X_a^*. We also mention two other bounds that hold by the selfadjointness of H^a and its spectral properties, namely

$$\begin{aligned}&\|(\mathrm{Id}-e_a)R^a(\sigma)\|_{\mathcal{B}(L^2,L^2)}\le|\operatorname{Re}\sigma-\lambda_0|^{-1},\ \operatorname{Re}\sigma<\lambda_0,\\ &\|R^a(\sigma)\|_{\mathcal{B}(L^2,L^2)}\le|\operatorname{Im}\sigma|^{-1},\ \operatorname{Im}\sigma\neq 0,\end{aligned}\tag{3.18}$$

λ_0 as in (3.11).

Now $R^a(\sigma)$ is analytic in σ for $\operatorname{Im}\sigma\neq 0$, with values in bounded operators $L^2(X^a)\to H^2(X^a)$, and from $\operatorname{Im}\sigma>0$ (and from $\operatorname{Im}\sigma<0$) it extends to be smooth to $\mathbb{C}\setminus[0,+\infty)$ away from the eigenvalues of H^a, where it has a simple pole. With e_a denoting the projection to the L^2 eigenspace of H^a with eigenvalues $\le\lambda_1$, $\lambda_1\in[\rho_\perp^2+\epsilon_\alpha,0)$, and λ_0 given by (3.11) as before, $R^a(\sigma)(\mathrm{Id}-e_a)$ is smooth on $\mathbb{C}\setminus(\lambda_0,+\infty)$. On the range of $e_{a,\epsilon}$, the projection to the eigenspace with eigenvalue ϵ, $R^a(\sigma)$ is multiplication by $(\epsilon-\sigma)^{-1}$. This is a locally integrable function of σ near ϵ (in $\mathbb{C}$!), so the application of $R^a(F_\rho(.))$ to $\hat{u}=\mathcal{F}_{X_a}u$ is well defined, provided that at every point ξ_a with $F_\rho(\xi_a)=\epsilon$, the differential of F_ρ is surjective, i.e. $F_\rho(\xi_a)=\epsilon$ implies that $(\xi_a)_\perp\neq-\rho_\perp$. But if $(\xi_a)_\perp=-\rho_\perp$ and $F_\rho(\xi_a)$ is real, then $\nu\cdot\xi_a=0$, hence $F_\rho(\xi_a)=\epsilon_\alpha+\rho_\perp^2$. Now let

$$\begin{aligned}G_a(\rho)&=\mathcal{F}_{X_a}^{-1}R^a(\epsilon_\alpha-|.|^2-2\rho\cdot.)\mathcal{F}_{X_a}\\ &=\mathcal{F}_{X_a}^{-1}R^a(\epsilon_\alpha-|.|^2-2\rho\cdot.)(\mathrm{Id}-E_a)\mathcal{F}_{X_a}+\mathcal{F}_{X_a}^{-1}R^a(\epsilon_\alpha-|.|^2-2\rho\cdot.)E_a\mathcal{F}_{X_a};\end{aligned}$$

both terms are well defined by the preceeding considerations when applied to functions in $\mathcal{S}(X_a;L^2(X^a))$. Indeed, application of $G_a(\rho)$ to the range of $E_{a,\epsilon}$ is the only issue, and there, with $u=v\otimes\psi_{\alpha'}$, $v\in L^2_p(X_a)$, $\hat{v}=\mathcal{F}_{X_a}v$,

$$(\mathcal{F}_{X_a}E_{a,\epsilon}G_a(\rho)v)(\xi_a,.)=(\epsilon-F_\rho(\xi_a))^{-1}\hat{v}\otimes\psi_{\alpha'},$$

so the mapping properties of $G_a(\rho)$ on $\operatorname{Ran}E_a$ are given by the two-body results of Weder [21]. In particular $E_aG_a(\rho)$ is well defined for functions in $L^2_p(X_a)\otimes\operatorname{Ran}e_a$, $p>0$. Hence we deduce the following result.

Proposition 3.3. *Suppose that $(z,\nu,\rho_\perp)\in\mathbb{C}(X_a)^\circ_\alpha$. The operator*

$$G_a(\rho)=\mathcal{F}_{X_a}^{-1}R^a(\epsilon_\alpha-|.|^2-2\rho\cdot.)\mathcal{F}_{X_a}$$

is a bounded operator $\mathcal{H}_p\to\mathcal{H}_r$ for $p>0$, $r<0$, $r<p-1$. It satisfies

$$\begin{aligned}&P_a(\rho)G_a(\rho)=\mathrm{Id}:\mathcal{H}_p\to\mathcal{H}_p,\\ &G_a(\rho)P_a(\rho)=\mathrm{Id}:\mathcal{H}_p\to\mathcal{H}_p.\end{aligned}\tag{3.19}$$

It is continuous in $\rho\in\mathbb{C}(X_a)^\circ_\alpha$ and analytic in $z\in\mathbb{C}\setminus\mathbb{R}$. Moreover, for $p>0$, $r<0$, $r<p-1$, $\rho_\perp$ fixed, for any $C>0$, $G_a(\rho)$ is uniformly bounded in $\mathcal{B}(\mathcal{H}_p,\mathcal{H}_r)$ in $|\operatorname{Im}z|\ge C|\operatorname{Re}z|$, and s-$\lim_{|z|\to\infty}G_a(\rho)=0$ *as an operator in $\mathcal{B}(\mathcal{H}_p,\mathcal{H}_r)$, provided that $|z|\to\infty$ in the region $|\operatorname{Im}z|\ge C|\operatorname{Re}z|$.*

Proof. All of the claims follow from the previous argument, except the behavior of $G_a(\rho)$ as $\rho\to\infty$. That in turn follows from

$$\begin{aligned}&\lim_{\rho\to\infty}\|E_aG_a(\rho)\|_{\mathcal{B}(\mathcal{H}_p,\mathcal{H}_r)}=0,\ p>0,r<0,r<p-1,\\ &\|(\mathrm{Id}-E_a)G_a(\rho)\|_{\mathcal{B}(\mathcal{H}_p,\mathcal{H}_p)}\le C,\\ &(\mathrm{Id}-E_a)G_a(\rho)\to 0\text{ strongly on }\mathcal{H}_p.\end{aligned}\tag{3.20}$$

The first estimate here is a two-body result, relying on a similar estimate for $\|E_{a,\epsilon}G_a(\rho)\|$ for each $\epsilon \in \operatorname{spec}_{pp}(H^a)$, see [21]. Indeed, with $u = v \otimes \psi_{\alpha'}$, $v \in L^2_p(X_a)$, $\hat{v} = \mathcal{F}_{X_a} v$,

$$(\mathcal{F}_{X_a} E_{a,\epsilon} G_a(\rho) v)(\xi_a, .) = (\epsilon - F_\rho(\xi_a))^{-1} \hat{v} \otimes \psi_{\alpha'}.$$

The uniform estimate for $(\operatorname{Id} - E_a)G_a(\rho)$ holds because $R^a(\sigma)(\operatorname{Id} - e_a)$ is uniformly bounded as an operator on $L^2(X^a)$ as long as σ is uniformly bounded away from $[0, +\infty)$, which holds for $\sigma = F_\rho(\xi_a)$ provided $|\operatorname{Im} z| \geq C|\operatorname{Re} z|$ by (3.16). Then by the Parseval's formula, and with $\hat{u} = \mathcal{F}_{X_a} u$,

$$\begin{aligned} \|G_a(\rho)(\operatorname{Id} - E_a)u\|^2_{L^2(X_0)} &= (2\pi)^{-n} \int_{X_a} \|R^a(F_\rho(\xi_a))\hat{u}(\xi_a, .)\|^2_{L^2(X^a)}\, d\xi_a \\ &\leq (2\pi)^{-n} M^2 \int_{X_a} \|\hat{u}(\xi_a, .)\|^2_{L^2(X^a)}\, d\xi_a = (2\pi)^{-n} M^2 \|u\|^2_{L^2(X_0)}, \\ M &= \sup\{\|R^a(F_\rho(\xi_a))(\operatorname{Id} - e_a)\|_{\mathcal{B}(L^2(X^a), L^2(X^a))} :\ \xi_a \in X_a\}. \end{aligned}$$

Moreover, as $\operatorname{Im} z \to \infty$, $\operatorname{Im} F_\rho(\xi_a) \to \infty$ for almost every ξ_a, namely for ξ_a such that $\xi_a \cdot \nu \neq 0$. But $R^a(\sigma) \to 0$ as a bounded operator on $L^2(X^a)$ as $\operatorname{Im} \sigma \to \infty$ by (3.18). Since

$$\|R^a(F_\rho(\xi_a))\hat{u}(\xi_a, .)\|^2 \leq M^2 \|\hat{u}(\xi_a, .)\|^2_{L^2(X^a)},$$

and $\|\hat{u}(\xi_a, .)\|^2_{L^2(X^a)} \in L^1(X_a)$, the dominated convergence theorem implies that $G_a(\rho)(\operatorname{Id} - E_a) \to 0$ strongly. □

Since

$$P(\rho)G_a(\rho)v = (\operatorname{Id} + I_a G_a(\rho))v,\ v \in \mathcal{H}_p,\ p > 0,$$

we next investigate I_a on $\mathcal{H}_r$.

Lemma 3.4. *Suppose that $\mu > 0$, and $\langle w^b \rangle^\mu V_b \in L^\infty(X^b)$ for all $b \neq a$. The multiplication operator I_a is in $\mathcal{B}(\mathcal{H}_r, \mathcal{H}_p)$ provided that $p \leq r + \mu$, $p \leq \mu$, $r \geq -\mu$.*

Moreover, there exists $C > 0$ (independent of V_b) such that the norm of I_a as such an operator is bounded by $C \max_b \sup(\langle w^b \rangle^\mu |V_b|)$.

Proof. We decompose I_a as a matrix corresponding to the direct sum in (3.13). Since I_a is bounded on $L^2(X_0)$, it follows that $(\operatorname{Id} - E_a)I_a(\operatorname{Id} - E_a) : \mathcal{H}_r \to \mathcal{H}_p$ for all r and p. Moreover, by (3.3), for all α', α'',

$$\langle w_a \rangle^\mu \int_{X^a} I_a \overline{\psi_{\alpha'}} \psi_{\alpha''}\, dw_a$$

is bounded on X_a, hence $E_a I_a E_a : \mathcal{H}_r \to \mathcal{H}_p$ for all r and p with $p \leq r + \mu$. In addition, $\langle w_a \rangle^\mu I_a \overline{\psi_{\alpha'}} \in L^\infty(X_0)$, so $E_a I_a : L^2(X_0) \to \mathcal{H}_p$ for $p \leq \mu$. Similarly, $\langle w_a \rangle^\mu \psi_{\alpha'} I_a \in L^\infty(X_a; L^2(X^a))$, so $I_a E_a : \mathcal{H}_r \to L^2(X_0)$ provided $r \geq -\mu$. As $\operatorname{Id} - E_a$ is a bounded operator on $L^2(X_0)$, this shows that $I_a : \mathcal{H}_r \to \mathcal{H}_p$ as stated. □

Combining the preceeding proposition and lemma we deduce the following.

Corollary 3.5. *Suppose that $\mu > 1$, and $\langle w^b \rangle^\mu V_b \in L^\infty(X^b)$ for all $b \neq a$. Let p satisfy $0 < p < \mu$. Then $I_a G_a(\rho) \in \mathcal{B}(\mathcal{H}_p, \mathcal{H}_p)$, continuous in $\rho \in \mathbb{C}(X_a)^\circ_\alpha$, and analytic in z in this region. Moreover, for $\rho_\perp$ fixed and $C > 0$, $I_a G_a(\rho)$ is uniformly bounded in $|\operatorname{Im} z| \geq C|\operatorname{Re} z|$, $|z| > 1$, and* $\operatorname{s-lim}_{|z| \to \infty} I_a G_a(\rho) = 0$ *inside this region.*

We also need to consider the invertibility properties of $\mathrm{Id} + I_a G_a(\rho)$ on $\mathcal{H}_p$. As $I_a G_a(\rho) \in \mathcal{B}(\mathcal{H}_p, \mathcal{H}_p)$, $\mathrm{Id} + I_a G_a(\rho)$ is invertible and $\|(\mathrm{Id} + I_a G_a(\rho))^{-1}\|_{\mathcal{B}(\mathcal{H}_p,\mathcal{H}_p)} < 2$ provided that $\|I_a G_a(\rho)\|_{\mathcal{B}(\mathcal{H}_p,\mathcal{H}_p)} < 1/2$. In view of the uniform boundedness of $G_a(\rho)$ as a map in $\mathcal{B}(\mathcal{H}_p, \mathcal{H}_p)$, there exists $\delta' > 0$ such that if $\|I_a\|_{\mathcal{B}(\mathcal{H}_r,\mathcal{H}_p)} < \delta'$ then $\|I_a G_a(\rho)\|_{\mathcal{B}(\mathcal{H}_p,\mathcal{H}_p)} < 1/2$. Combining this with the norm estimate of Lemma 3.4 leads to the following theorem.

Theorem 3.6. *Suppose that $\rho_\perp$ satisfies $|\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a$, $\mu > \max(p, 1)$, $p > 0$, $r < 0$, $r < p - 1$. There exists $\delta > 0$ with the following property. Suppose that for all $b \neq a$, $\sup |\langle w^b \rangle^\mu V_b| < \delta$. Then the operator*

$$G(\rho) = G_a(\rho)(\mathrm{Id} + I_a G_a(\rho))^{-1} : \mathcal{H}_p \to \mathcal{H}_r \tag{3.21}$$

satisfies

$$\begin{aligned} P(\rho)G(\rho) &= \mathrm{Id} : \mathcal{H}_p \to \mathcal{H}_p, \\ G(\rho)P(\rho) &= \mathrm{Id} : \mathcal{H}_p \to \mathcal{H}_p. \end{aligned} \tag{3.22}$$

Moreover, $G(\rho)$ is a continuous function of ρ in $\mathbb{C}(X_a)^\circ_\alpha$, and an analytic function of $z \in \mathbb{C} \setminus \mathbb{R}$, and $\operatorname{s-lim}_{|z|\to\infty} G(\rho) = 0$ as a map $\mathcal{H}_p \to \mathcal{H}_r$ provided that $|z| \to \infty$ in $|\operatorname{Im} z| \geq C|\operatorname{Re} z|$, $C > 0$.

Proof. With $G(\rho)$ as in (3.21), $G(\rho) : \mathcal{H}_p \to \mathcal{H}_r$, $p > 0$, $r < 0$, $r < p - 1$, since $G_a(\rho)$ has these mapping properties, and $(\mathrm{Id} + I_a G_a(\rho))^{-1}$ is bounded on $\mathcal{H}_p$. Now,

$$\begin{aligned} P(\rho)G(\rho) &= (P_a(\rho) + I_a)G_a(\rho)(\mathrm{Id} + I_a G_a(\rho))^{-1}, \\ (P_a(\rho) + I_a)G_a(\rho) &= \mathrm{Id} + I_a G_a(\rho) : \mathcal{H}_p \to \mathcal{H}_p, \end{aligned} \tag{3.23}$$

proving the first line of (3.22). The second line follows from the identity

$$(\mathrm{Id} + I_a G_a(\rho))^{-1} = \mathrm{Id} - (\mathrm{Id} + I_a G_a(\rho))^{-1} I_a G_a(\rho),$$

and $G_a(\rho)P_a(\rho) = \mathrm{Id}$ on $\mathcal{H}_p$, $p > 0$. The limiting behavior follows from the uniform boundedness of $(\mathrm{Id} + I_a G_a(\rho))^{-1}$ on $\mathcal{H}_p$, and from

$$G(\rho) = G_a(\rho) - G_a(\rho)(\mathrm{Id} + I_a G_a(\rho))^{-1}(I_a G_a(\rho))$$

with the last factor tending to 0 strongly on $\mathcal{H}_p$, and the other factors remaining bounded. □

Since $I_a \psi_\alpha \in \mathcal{H}_p$ for some $p > 0$ if $\mu > \dim X_a/2$, due the (3.5), we deduce the following corollary.

Corollary 3.7. *Suppose that $\mu > \dim X_a/2$, $|\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a$. Then*

$$u_\rho = u^0_\rho - e^{i\rho\cdot w_a} G(\rho) I_a \psi_\alpha = e^{i\rho\cdot w_a}(\psi_\alpha - G(\rho) I_a \psi_\alpha) \tag{3.24}$$

satisfies $P(\rho)u_\rho = 0$, and

$$u^0_\rho = e^{i\rho\cdot w_a}(\mathrm{Id} + G_a(\rho) I_a) e^{-i\rho\cdot w_a} u_\rho. \tag{3.25}$$

Moreover, $u_\rho - u^0_\rho \to 0$ in $\mathcal{H}_r$, $r < 0$, $r < \mu - 1 - \frac{\dim X_a}{2}$, as $|z| \to \infty$ in $|\operatorname{Im} z| > C|\operatorname{Re} z|$, $C > 0$.

4. Limit as ρ goes to the reals

As ρ becomes real, the structure of the operator $G_a(\rho)$ degenerates since $|\xi|^2 + 2\rho\cdot\xi - \epsilon_\alpha$ becomes real. Many of the details of the following calculations are similar to the corresponding two-body calculations, see e.g. Weder [21]. We proceed as follows.

Recall that for $\rho \in \mathbb{C}(X_a)$, $\operatorname{Im}\rho \neq 0$, there exist unique $\nu \in X_a$, $z \in \mathbb{C}$, $\rho_\perp \in X_a$ such that

$$\rho = z\nu + \rho_\perp, \quad |\nu|^2 = 1,\ \rho_\perp \cdot \nu = 0,\ \operatorname{Im} z > 0.$$

Alternatively, the inequality $\operatorname{Im} z > 0$ can be replaced by $\operatorname{Im} z < 0$. Here we keep $\operatorname{Im} z > 0$ for the sake of definiteness. The behavior as ρ approaches X_a then corresponds to z approaching the real axis. Now, u_ρ solves $(H-\lambda)u_\rho = 0$, $\lambda = \rho^2 + \epsilon_\alpha$, i.e. has energy $\rho^2+\epsilon_\alpha$. Thus, the form of the limit as $\rho \to X_a$ depends on the nature of the spectrum of H near $\lambda = \rho^2 + \epsilon_\alpha$. Here we restrict ourselves to $|\rho_\perp|^2 < -\epsilon_\alpha$. Note that with λ_0 as in (3.11),

$$\lambda < 0 \text{ iff } \rho^2 = z^2 + \rho_\perp^2 < -\epsilon_\alpha;\ \lambda < \lambda_0 \text{ iff } \rho^2 = z^2 + \rho_\perp^2 < \lambda_0 - \epsilon_\alpha. \tag{4.1}$$

Below we consider

$$\operatorname{Re} z \geq 0,\ \operatorname{Im} z \geq 0. \tag{4.2}$$

and show that $u_\rho = u_{\alpha,\rho}$ extends to a continuous function of $(z,\nu,\rho_\perp)$ in

$$\begin{aligned}\mathbb{C}(X_a)^+_\alpha = &\{(z,\nu,\rho_\perp):\ \operatorname{Im} z > 0,\ |\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha,\lambda_0)\setminus\Lambda'_a\}\\ &\cup\{(z,\nu,\rho_\perp):\ |\rho_\perp|^2+\epsilon_\alpha \in (\epsilon_\alpha,\lambda_0)\setminus\Lambda'_a,\ z\geq 0,\\ &\qquad z^2+\rho_\perp^2+\epsilon_\alpha \in (\epsilon_\alpha,\lambda_0)\setminus\Lambda'_a\}.\end{aligned}$$

Remark 4.1. Below we often consider $\rho \in \mathbb{C}(X_a)^+_\alpha$, and keep writing u_ρ even when the projection of ρ to $\mathbb{C}(X_a)$ lies in X_a.

First consider $G_a(\rho)E_a$. It is better to consider this as $\sum_{\epsilon\in\operatorname{spec}_{pp}(H^a)} G_a(\rho)E_{a,\epsilon}$, where $E_{a,\epsilon}$ is projection to the ϵ eigenspace of H^a tensored with the identity map on X_a. Now, for $z \notin \mathbb{R}$,

$$\begin{aligned}\mathcal{F}_{X_a}G_a(\rho)E_{a,\epsilon}\mathcal{F}^{-1}_{X_a}u &= (|\xi_a|^2 + 2\rho\cdot\xi_a - \epsilon_\alpha + \epsilon)^{-1}E_{a,\epsilon}u\\ &= (|\xi_a + \operatorname{Re}\rho|^2 + 2i(\operatorname{Im} z)\nu\cdot\xi_a - \operatorname{Re}\rho^2 - \epsilon_\alpha + \epsilon)^{-1}H(\nu\cdot\xi_a)E_{a,\epsilon}u\\ &\quad + (|\xi_a + \operatorname{Re}\rho|^2 + 2i(\operatorname{Im} z)\nu\cdot\xi_a - \operatorname{Re}\rho^2 - \epsilon_\alpha + \epsilon)^{-1}H(-\nu\cdot\xi_a)E_{a,\epsilon}u,\end{aligned} \tag{4.3}$$

where H is the Heaviside step function, so $H=1$ on $(0,+\infty)$, $H=0$ on $(-\infty,0)$. Thus, if $\operatorname{Im} z > 0$, letting $\operatorname{Im} z \to 0$, yields

$$\begin{aligned}&\lim_{\operatorname{Im} z\to 0}\mathcal{F}_{X_a}G_a(\rho)E_{a,\epsilon}\mathcal{F}^{-1}_{X_a}u\\ &\qquad = (|\xi_a + \operatorname{Re}\rho|^2 - (\operatorname{Re}\rho^2 + \epsilon_\alpha - \epsilon - i0))^{-1}H(\nu\cdot\xi_a)E_{a,\epsilon}u\\ &\qquad\quad + (|\xi_a + \operatorname{Re}\rho|^2 - (\operatorname{Re}\rho^2 + \epsilon_\alpha - \epsilon + i0))^{-1}H(-\nu\cdot\xi_a)E_{a,\epsilon}u.\end{aligned} \tag{4.4}$$

This calculation makes sense for $\mathcal{F}^{-1}_{X_a}u \in \mathcal{H}_s$, $s > 1/2$. Note that if $\epsilon > \lambda = \operatorname{Re}\rho^2 - \epsilon_\alpha$, then $|\xi_a + \operatorname{Re}\rho|^2 - (\operatorname{Re}\rho^2 + \epsilon_\alpha - \epsilon)$ does not vanish for any ξ_a (as it is positive), so in $E_{a,\epsilon}$ the factors with $+i0$ and $-i0$ are the same. Conjugating

$G_a(\rho)$ by $e^{i\operatorname{Re}\rho\cdot w_a}$ replaces ξ_a by $\xi_a+\operatorname{Re}\rho$. Writing $H(-\nu\cdot\xi_a)=1-H(\nu\cdot\xi_a)$ and $\lambda=\operatorname{Re}\rho^2+\epsilon_\alpha$, we thus deduce that

(4.5)
$$\begin{aligned}&\lim_{\operatorname{Im} z\to 0} e^{i\operatorname{Re}\rho\cdot w_a}G_a(\rho)E_{a,\epsilon}e^{-i\operatorname{Re}\rho\cdot w_a}\\ &=\mathcal{F}^{-1}(|\xi|^2-(\lambda-\epsilon+i0))^{-1}\mathcal{F}E_{a,\epsilon}\\ &\quad+\mathcal{F}^{-1}H(\nu\cdot(\xi_a-\operatorname{Re}\rho))((|\xi|^2-(\lambda-\epsilon-i0))^{-1}-(|\xi|^2-(\lambda-\epsilon+i0))^{-1})\mathcal{F}E_{a,\epsilon}\\ &=R_a(\lambda+i0)E_{a,\epsilon}+\mathcal{F}^{-1}H(\nu\cdot(\xi_a-\operatorname{Re}\rho)\mathcal{F}[R_a(\lambda-i0)-R_a(\lambda+i0)]E_{a,\epsilon}.\end{aligned}$$

Now consider $G_a(\rho)(\operatorname{Id}-E_a)$. This is analytic in

$$\{z\in\mathbb{C}:\ z^2\notin[\lambda_1-\epsilon_\alpha-\rho_\perp^2,+\infty)\},\tag{4.6}$$

hence in $\mathbb{C}(X_a)^+_\alpha$. Indeed, since

$$G_a(\rho)(\operatorname{Id}-E_a)=\mathcal{F}^{-1}_{X_a}R^a(\epsilon_\alpha-|\xi_a|^2-2\rho\cdot\xi_a)(\operatorname{Id}-E_a)\mathcal{F}_{X_a},\tag{4.7}$$

we need to analyze the range of the map $F_\rho:\xi_a\mapsto\epsilon_\alpha-|\xi_a|^2-2\rho\cdot\xi_a$, which has been done in (3.14)-(3.15). Thus, when $\operatorname{Im} z\to 0$ and $0\le\operatorname{Re} z<\sqrt{\lambda_1-\epsilon_\alpha-\rho_\perp^2}$ implies $\operatorname{Re} F_\rho<\lambda_1$ by (3.14), and $R^a(\sigma)(\operatorname{Id}-e_a)$ is analytic in $\operatorname{Re}\sigma<\lambda_1$. We only need the additional observation that as $\operatorname{Im} z\to 0$, $\operatorname{Im} F_\rho\to 0$. Thus, in $\operatorname{Im} z>0$, by the dominated convergence theorem,

$$\lim_{\operatorname{Im} z\to 0}G_a(\rho)(\operatorname{Id}-E_a)=\mathcal{F}^{-1}_{X_a}R^a(\epsilon_\alpha-(|\xi_a|^2+2\rho\cdot\xi_a-i0))(\operatorname{Id}-E_a)\mathcal{F}_{X_a}.\tag{4.8}$$

Again, conjugating by $e^{i\rho\cdot w_a}$ replaces ξ_a by $\xi_a+\rho$. Thus,

$$\begin{aligned}&\lim_{\operatorname{Im} z\to 0} e^{i\operatorname{Re}\rho\cdot w_a}G_a(\rho)(\operatorname{Id}-E_a)e^{-i\operatorname{Re}\rho\cdot w_a}\\ &\quad=\mathcal{F}^{-1}_{X_a}R^a(\epsilon_\alpha+\rho^2-(|\xi_a|^2+i0))(\operatorname{Id}-E_a)\mathcal{F}_{X_a}=R_a(\lambda+i0)(\operatorname{Id}-E_a),\end{aligned}\tag{4.9}$$

where we took into account that $\epsilon_\alpha+\rho^2=\lambda$. Note that $R_a(\lambda+i0)(\operatorname{Id}-E_a)=R_a(\lambda-i0)(\operatorname{Id}-E_a)$ as $\lambda<\lambda_0$.

Combining these two results we deduce the following proposition.

Proposition 4.2. *The operator $G_a(\rho):\mathcal{H}_p\to\mathcal{H}_r$ extends continuously (from $\mathbb{C}(X_a)^+_\alpha\cap\mathbb{C}(X_a)^\circ_\alpha$) to $\mathbb{C}(X_a)^+_\alpha$ for $p>1/2$, $r<-1/2$, and it satisfies $P_a(\rho)G_a(\rho)=\operatorname{Id}$, $G_a(\rho)P_a(\rho)=\operatorname{Id}$, on $\mathcal{H}_p$, $p>1/2$. The limit $G_a(\nu,\rho_\perp,z\pm i0)$ satisfies*

$$\begin{aligned}&e^{i\rho\cdot w_a}G_a(\nu,\rho_\perp,z\pm i0)e^{-i\rho\cdot w_a}\\ &\quad=R_a(\rho^2+i0)+\sum_{\alpha'}\mathcal{F}^{-1}_{X_a}H(\nu\cdot(\xi_a-\operatorname{Re}\rho))[(|\xi_a|^2-(\lambda-\epsilon_{\alpha'}-i0))^{-1}\\ &\qquad\qquad-(|\xi_a|^2-(\lambda-\epsilon_{\alpha'}+i0))^{-1}]\mathcal{F}_{X_a}E_{a,\alpha'}.\end{aligned}\tag{4.10}$$

The last term in (4.10) can be written in terms of the Poisson operators in the bound states of H^a, using the following lemma.

Lemma 4.3. *Suppose that $\lambda\notin\Lambda'_a$. Then*

(4.11)
$$\begin{aligned}&\sum_{\alpha'}\mathcal{F}^{-1}_{X_a}H(\nu\cdot(\xi_a-\operatorname{Re}\rho))\\ &\qquad[(|\xi_a|^2-(\lambda-\epsilon_{\alpha'}-i0))^{-1}-(|\xi_a|^2-(\lambda-\epsilon_{\alpha'}+i0))^{-1}]\mathcal{F}_{X_a}E_{a,\alpha'}\\ &=\sum_{\alpha'}\frac{-i}{2\sqrt{\lambda-\epsilon_{\alpha'}}}\widetilde{\mathcal{P}}_{\alpha'-}(\lambda)H(\nu\cdot(.-\rho))\widetilde{\mathcal{P}}_{\alpha'-}(\lambda)^*.\end{aligned}$$

Proof. On $\operatorname{Ran}(\operatorname{Id} - E_a)$ both sides vanish. Thus, it suffices to consider the equation on $\operatorname{Ran}(E_{a,\epsilon})$, $\epsilon \in \operatorname{spec}_{pp}(H^a)$, or simply on $\psi_{\alpha'}$ where α' is a bound state of H^a of energy $\epsilon = \epsilon_{\alpha'}$. Explicitly, on the X_a-Fourier transform side, on this space the equation follows from

$$(\rho - (\lambda - \epsilon - i0))^{-1} - (\rho - (\lambda - \epsilon + i0))^{-1} = -2\pi i \delta_{\rho-(\lambda-\epsilon)}.$$

Indeed, with $\rho = |\xi_a|^2$, hence $d\xi_a = \frac{1}{2}\rho^{\frac{\dim X_a - 2}{2}}\, d\rho\, d\omega_a$, $u = v \otimes \psi_{\alpha'}$, $\hat{v} = \mathcal{F}_{X_a} v$, this gives

$$\begin{aligned}(2\pi)^{-\dim X_a} \int_{X_a} &\left((|\xi_a|^2 - (\lambda - \epsilon - i0))^{-1} - (|\xi_a|^2 - (\lambda - \epsilon + i0))^{-1} \right) \hat{v}(w_a)\, dw_a \\ &= -(2\pi)^{-\dim X_a} \pi i \lambda^{\frac{\dim X_a - 2}{2}} \int_{\mathbb{S}_a(\sqrt{\lambda - \epsilon_\alpha})} e^{i w_a \cdot \omega_a} \hat{v}(\omega_a)\, d\omega_a.\end{aligned}$$

Comparing with the definition of the Poisson operators, namely that the Fourier transform, followed by restriction to $\mathbb{S}_a(\sqrt{\lambda - \epsilon_\alpha})$, is essentially given by $\widetilde{\mathcal{P}}_{\alpha'-}(\lambda)^*$ in view of (2.1), with a similar relation connecting the inverse Fourier transform and $\widetilde{\mathcal{P}}_{\alpha'-}(\lambda)$, (4.11) follows. □

The combination of the preceeding two results yields:

Corollary 4.4. *Suppose that* $\rho \cdot \rho + \epsilon_\alpha = \lambda \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a$, $\rho_\perp^2 + \epsilon_\alpha \notin \Lambda'_a$. *Then*

$$\begin{aligned} e^{i\rho \cdot w_a} G_a(\rho) e^{-i\rho \cdot w_a} = & R_a(\lambda + i0) \\ & - \sum_{\alpha'} \frac{i}{2\sqrt{\lambda - \epsilon_{\alpha'}}} \widetilde{\mathcal{P}}_{\alpha'-}(\lambda) H(\nu \cdot (. - \operatorname{Re} \rho)) \widetilde{\mathcal{P}}_{\alpha'-}(\lambda)^*, \end{aligned}$$

where $\widetilde{\mathcal{P}}_{\alpha'-}(\lambda)$ *is considered as an operator on the sphere of radius* $\lambda - \epsilon_{\alpha'}$.

Proposition 4.2 can be used to show that $G(\rho)$ itself has a limit when ρ becomes real, provided that $\lambda = \rho^2 + \epsilon_\alpha < \lambda_0$.

Theorem 4.5. *Suppose that* $|\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a$, $p > 1/2$, $r < -1/2$, $\mu > \max(p, 1)$. *There exists* $\delta > 0$ *with the following property. Suppose that for all* $b \neq a$, $\sup |\langle w^b \rangle^\mu V_b| < \delta$. *Then the operator* $G(\rho) : \mathcal{H}_p \to \mathcal{H}_r$ *extends continuously to*

$$\{z : \operatorname{Im} z > 0\} \cup \{z \geq 0 : z^2 + \rho_\perp^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'\}, \tag{4.12}$$

and it satisfies $P(\rho)G(\rho) = \operatorname{Id}$, $G(\rho)P(\rho) = \operatorname{Id}$, *on* $\mathcal{H}_p$.

Proof. We only need to show that $(\operatorname{Id} + I_a G_a(\rho))^{-1}$ extends to real ρ as stated, as a bounded operator on $\mathcal{H}_p$, $p > 1/2$. But this follows as in the remarks preceeding Theorem 3.6. □

Corollary 4.6. *Suppose that* $|\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a$, $\mu > (\dim X_a + 1)/2$, V_b *as in Theorem 4.5. Then* u_ρ *extends continuously to* (4.12), *with* $u_\rho - u_\rho^0 \in \mathcal{H}_r$ *for all* $r < 0$, *and* u_ρ *is analytic in* z *in* $\operatorname{Im} z > 0$.

5. The connection between the S-matrix and the exponential solutions

We introduce the analogue of the pairing (2.5) describing the S-matrix via

$$G_{\alpha\alpha'}(\rho, \overline{\rho} + \zeta) = \int_{\mathbb{R}^n} I_a u_\rho \overline{u^0_{\alpha', \overline{\rho} + \zeta}} = \int_{\mathbb{R}^n} I_a u_\rho \overline{\psi_{\alpha'}(w^a)} e^{-i(\rho + \zeta)\cdot w_a}. \tag{5.1}$$

By Corollary 4.6, if $\langle w^b\rangle^\mu V_b \in L^\infty(X^b)$ for all b and for some $\mu > \dim X_a$, then the integral in (5.1) converges for all ρ for which u_ρ exists, and for $\zeta \in \mathbb{R}^n$, since then the real parts of the exponentials cancel, and $I_a\psi_\alpha\overline{\psi_{\alpha'}} \in L^1(X_0)$, and the same holds for $I_a\overline{\psi_{\alpha'}}G(\rho)(I_a\psi_\alpha)$. Other properties of (5.1) follow immediately from Corollary 4.6.

Proposition 5.1. *Suppose that $|\rho_\perp|^2 + \epsilon_\alpha \in (\epsilon_\alpha, \lambda_0) \setminus \Lambda'_a$, $\mu > \dim X_a$, and V_b as in Theorem 4.5. Then $G_{\alpha\alpha'}$ is an analytic function of z in $\mathbb{C}\setminus\mathbb{R}$, and extends to be continuous on* (4.12)*. In addition,*

$$\lim_{|z|\to\infty} G_{\alpha\alpha'}(\rho,\overline{\rho}+\zeta) = \int_{\mathbb{R}^n} I_a\psi_\alpha\overline{\psi_{\alpha'}}e^{-i\zeta\cdot w_a}, \tag{5.2}$$

provided that $|z|\to\infty$ in $|\operatorname{Im} z| > C|\operatorname{Re} z|$, $C > 0$.

Proof. The first two statements are direct consequences of Corollary 4.6. By Corollary 3.7, $G(\rho)(I_a\psi_\alpha) \to 0$ as $|z|\to\infty$ in $\mathcal{H}_r$ for $r < \mu - 1 - \frac{\dim X_a}{2} < 0$. On the other hand,

$$\overline{\psi_{\alpha'}}I_a e^{-i\zeta\cdot w_a} \in L^2_s(X_0) \subset \mathcal{H}_s,\ s > 0,\ s < \mu - \frac{\dim X_a}{2},$$

hence in $\mathcal{H}_{-r}$ provided that $r > \frac{\dim X_a}{2} - \mu$. We can take $r = \mu - 1 - \frac{\dim X_a}{2} - \epsilon$, $\epsilon > 0$ sufficiently small. Thus we conclude that

$$\lim_{|z|\to\infty}\int_{\mathbb{R}^n}(G(\rho)(I_a\psi_\alpha))(I_a\overline{\psi_{\alpha'}}e^{-i\zeta\cdot w_a}) = 0,$$

hence (5.2) follows. □

For fixed ρ real and ζ satisfying

$$\lambda = \rho^2 + \epsilon_\alpha = (\rho+\zeta)^2 + \epsilon_{\alpha''},$$

i.e. the equality of incoming and outgoing energies, we can relate $G_{\alpha\alpha''}(\rho,\rho+\zeta)$, ρ real, to the S-matrices as follows. Under our assumptions,

$$e^{i\rho\cdot w_a}(\operatorname{Id} + G_a(\rho)I_a)e^{-i\rho\cdot w_a}u_\rho = u^0_\rho = (\operatorname{Id} + R_a(\rho^2 + i0)I_a)U_\rho. \tag{5.3}$$

Applying $(\operatorname{Id} + R_a(\rho^2+i0)I_a)^{-1}$ to both sides of (5.3), we deduce that

$$\begin{aligned} U_\rho = u_\rho - \sum_{\alpha'}\frac{i}{2\sqrt{\lambda-\epsilon_{\alpha'}}}&(\operatorname{Id} + R_a(\rho^2+i0)I_a)^{-1}\\ &\widetilde{\mathcal{P}}_{\alpha'-}(\lambda)H(\nu\cdot(.-\operatorname{Re}\rho))\widetilde{\mathcal{P}}_{\alpha'-}(\lambda)^* I_a u_\rho\\ = u_\rho - \sum_{\alpha'}\frac{i}{2\sqrt{\lambda-\epsilon_{\alpha'}}}&\mathcal{P}_{\alpha'-}(\lambda)H(\nu\cdot(.-\operatorname{Re}\rho))\widetilde{\mathcal{P}}_{\alpha'-}(\lambda)^* I_a u_\rho. \end{aligned} \tag{5.4}$$

Integrating against $I_a e^{-i(\rho+\zeta)\cdot w_a}\overline{\psi_{\alpha''}(w^a)}$ yields

$$\begin{aligned} \mathcal{S}^\sharp_{\alpha\alpha''+}(\lambda,\rho,\rho+\zeta) =& G_{\alpha\alpha''}(\rho,\rho+\zeta)\\ &- \sum_{\alpha'}\frac{i}{2\sqrt{\lambda-\epsilon_{\alpha'}}}\int_{\mathbb{S}_a(\sqrt{\lambda-\epsilon_{\alpha'}})}\mathcal{S}^\sharp_{\alpha'\alpha''+}(\lambda,\rho',\rho+\zeta)\\ &\qquad H(\nu\cdot(\rho'-\operatorname{Re}\rho))G_{\alpha\alpha'}(\rho,\rho')\,d\rho'. \end{aligned} \tag{5.5}$$

This is an integral equation for $G_{\alpha\alpha''}$ in terms of $\mathcal{S}^\sharp_{\alpha\alpha''+}(\lambda)$, $\mathcal{S}^\sharp_{\alpha'\alpha''+}(\lambda)$, $\lambda = \rho^2 + \epsilon_\alpha$.

Proposition 5.2. *Suppose that* $\rho^2+\epsilon_\alpha=\lambda=(\rho+\zeta)^2+\epsilon_{\alpha''}\in(-\infty,0)\setminus\Lambda'$, $|\rho_\perp|^2+\epsilon_\alpha\in(\epsilon_\alpha,\lambda_0)\setminus\Lambda'_a$, $\mu>\dim X_a$. *There exists* $\delta>0$ *with the following property.*

Suppose that for all $b\neq a$, $\sup|\langle w^b\rangle^\mu V_b|<\delta$. *Then the pairings* $G_{\alpha\alpha''}(\rho,\rho+\zeta)$ *are determined by the operators* $\mathcal{S}^\sharp_{\alpha'\alpha''+}(\lambda)$ *given for all* α' *and* α''.

Proof. We first discuss the case when the only bound state of H^a is α, or more generally if $\epsilon'_\alpha>\lambda$ for $\alpha'\neq\alpha$, i.e. the total energy λ is just above the ground state energy. Then we get

(5.6)
$$\begin{aligned}\mathcal{S}^\sharp_{\alpha\alpha+}(\lambda,\rho,\rho+\zeta)=&G_{\alpha\alpha}(\rho,\rho+\zeta)\\ &-\frac{i}{2\sqrt{\lambda-\epsilon_\alpha}}\int_{\mathbb{S}(\sqrt{\lambda-\epsilon_\alpha})}\mathcal{S}^\sharp_{\alpha\alpha+}(\lambda,\rho',\rho+\zeta)H(\nu\cdot(\rho'-\rho))\\ &\qquad G_{\alpha\alpha}(\rho,\rho')\,d\rho'.\end{aligned}$$

Now fix ρ, i.e. more precisely fix ν, z, $\rho_\perp$, and consider this as an integral equation for the function $G_{\alpha\alpha}(\rho,.)$. Then this has the form

$$(\mathrm{Id}-T)G_{\alpha\alpha}(\rho,.)=f, \tag{5.7}$$

with $f\in\mathcal{C}^\infty(\mathbb{S}_a(\sqrt{\lambda-\epsilon_\alpha}))$, $T:L^2\to C^0$ with bounded kernel and small norm as a map $L^2\to L^2$ since $\sup_{\omega,\omega'}|\mathcal{S}^\sharp_{\alpha\alpha+}(\lambda,\omega,\omega'))|$ is small by (2.5) (as I_a is small). Hence, $\mathrm{Id}-T$ is invertible, proving the proposition in this case.

In complete generality, we consider the vector Φ whose α' entry is $\Phi_{\alpha'}(.)=G_{\alpha\alpha'}(\rho,.)$. Then we obtain a system of equations of the form

$$(\mathrm{Id}-T)\Phi=f, \tag{5.8}$$

as above. Again, T has small norm, so $\mathrm{Id}-T$ is invertible, proving the proposition. □

Novikov [14] noticed that in two-body scattering, the near-forward values of $G(\rho,\rho+\zeta)$, i.e. the values when the angles between ρ and ν, resp. between $\rho+\zeta$ and ν are small, is determined by $\mathcal{S}^\sharp_{\alpha'\alpha''+}(\rho,\rho')$ where the angle between ρ and ν, resp. and ρ' and ν is small. His observation also applies in the present setting.

This can be understood via linear algebra. Thus, we decompose

$$\begin{aligned}&V=L^2(\mathbb{S}_a(\sqrt{\lambda-\epsilon}))=V_1\oplus V_2,\\ &V_1=L^2(\{\rho'\in\mathbb{S}_a(\sqrt{\lambda-\epsilon}):\ \rho'\cdot\nu\geq\rho\cdot\nu\}),\\ &V_2=L^2(\{\rho'\in\mathbb{S}_a(\sqrt{\lambda-\epsilon}):\ \rho'\cdot\nu\leq\rho\cdot\nu\}),\end{aligned} \tag{5.9}$$

writing the two orthogonal projections as π_1 and π_2. Now $T:V\to V$ vanishes on V_2, so $T=T\pi_1$, while its restriction to V_1, via π_1, is exactly T_1: $T_1=\pi_1T$. Now linear algebra shows that we only need to know T_1 and $f_1=\pi_1f$ to find $\pi_1(\mathrm{Id}-T)^{-1}f$, namely

$$\pi_1(\mathrm{Id}-T)^{-1}f=(\mathrm{Id}_{V_1}-T_1)^{-1}\pi_1f. \tag{5.10}$$

This proves the following proposition.

Proposition 5.3. *Let* ρ, ζ, λ, μ, δ *be as in Proposition 5.2, and suppose that for all* $b\neq a$, $\sup|\langle w^b\rangle^\mu V_b|<\delta$. *Then the pairings* $G_{\alpha\alpha''}(\rho,\rho+\zeta)$ *for* $(\rho+\zeta)\cdot\nu\geq\rho\cdot\nu$

are determined by the S-matrices $\mathcal{S}^\sharp_{\alpha'\alpha''+}(\lambda,\omega,\omega')$ *given for all* α' *and* α''*, evaluated in* $\omega\cdot\nu\geq z=\rho\cdot\nu$*,* $\omega'\cdot\nu\geq z=\rho\cdot\nu$*.*

Note that if ϵ_α is the bottom of the spectrum of H^a, $\rho_\perp$ is sufficiently small, namely $|\rho_\perp|^2<\epsilon'-\epsilon_\alpha$, ϵ' as in Theorem 1.2, $\omega'\in\mathbb{S}_a(\sqrt{\lambda-\epsilon_{\alpha'}})$ then $\omega'\cdot\nu\leq\sqrt{\lambda-\epsilon_{\alpha'}}$ while $z^2=\lambda-\rho_\perp^2-\epsilon_\alpha>\lambda-\epsilon'$ shows that $\rho\cdot\nu=z>\sqrt{\lambda-\epsilon'}$, hence $\omega'\cdot\nu\geq z$ never holds. Thus, under the conditions of Theorem 1.2, only $\mathcal{S}^\sharp_{\alpha\alpha+}$ is needed to determine $G_{\alpha\alpha}(\rho,\rho+\zeta)$ for $|\rho_\perp|<\sqrt{\epsilon'-\epsilon_\alpha}$, $\zeta\cdot\nu\geq 0$.

6. Inverse results: proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2.

Fix a non-empty interval open $I\subset(\epsilon_\alpha,0)$, and let $R=2\sqrt{\sup I-\epsilon_\alpha}$. Let $\zeta\in X_a$ satisfy $|\zeta|<R$, and $\frac{1}{4}|\zeta|^2+\epsilon_\alpha\notin\Lambda'_a$. The last condition excludes a discrete set of values of $|\zeta|^2$ in $[0,R)$. Note that $\frac{1}{4}|\zeta|^2+\epsilon_\alpha<0$.

We now choose $\lambda_1<0$ in (3.9) so that $\frac{1}{4}|\zeta|^2+\epsilon_\alpha<\lambda_1$, and

$$I\cap(\frac{1}{4}|\zeta|^2+\epsilon_\alpha,\lambda_1)\neq\emptyset. \tag{6.1}$$

Let $\rho_\perp=-\zeta/2$, so $\rho_\perp^2+\epsilon_\alpha\in(\epsilon_\alpha,0)\setminus\Lambda'_a$. Let $\nu\in X_a$ be such that $|\nu|^2=1$, $\nu\cdot\zeta=0$. Then for $\rho=z\nu+\rho_\perp$, $z\in\mathbb{C}$, $(\rho+\zeta)^2-\rho^2=2\rho\cdot\zeta+\zeta^2=0$. Having fixed ζ, $\rho_\perp$, ν, consider the energy $\rho^2+\epsilon_\alpha$ when z becomes real. Since $\rho^2=z^2+|\rho_\perp|^2$, as z varies over $[0,\sqrt{\lambda_0-\rho_\perp^2-\epsilon_\alpha})$, the energy varies over $[|\rho_\perp|^2+\epsilon_\alpha,\lambda_0)$. This intersects the given interval I by (6.1) as $\lambda_1<\lambda_0$.

Let $z_0>0$ satisfy $\lambda=\rho_\perp^2+z_0^2+\epsilon_\alpha\in(\epsilon_\alpha,\lambda_0)\setminus\Lambda'$. By Proposition 5.2 the limit of the pairing $G_{\alpha\alpha}(\rho,\rho+\zeta)$ as $z\to z_0$ (in $\operatorname{Im}z>0$) is determined by the scattering matrices $\mathcal{S}^\sharp_{\alpha'\alpha''}(\rho_\perp^2+z_0^2+\epsilon_\alpha)$. Thus, there exists a non-empty open interval of these values z_0 at which $G_{\alpha\alpha}(\rho,\rho+\zeta)$ is determined by $\mathcal{S}^\sharp_{\alpha'\alpha''}(\lambda)$, $\lambda\in I$.

Indeed, as explained in and after the statement of Proposition 5.3, under the assumptions of Theorem 1.2, with $|\zeta|<2\sqrt{\min(\sup I,\epsilon')-\epsilon_\alpha}$, ϵ' denoting the next eigenvalue of H^a or 0, as in the statement of the theorem, one only needs to know $\mathcal{S}^\sharp_{\alpha\alpha}(|\rho|^2+\epsilon_\alpha)$, and in either case the knowledge of the S-matrices in appropriate near-forward regions suffices due the remarks surrounding (5.9).

Since the limit on any open interval in the boundary of its domain determines an analytic function, we deduce that knowing the S-matrix $\mathcal{S}^\sharp_{\alpha'\alpha''}$ in the interval I determines $G_{\alpha\alpha}(\rho,\zeta)$ for all ζ with $|\zeta|<R$.

Now let $z\to\infty$ through imaginary z. By (5.2), $G_{\alpha\alpha}(\rho,\rho+\zeta)$ converges to

$$\int_{\mathbb{R}^n}I_a|\psi_\alpha|^2e^{-i\zeta\cdot w_a}\,dw=\int_{X_a}e^{-i\zeta\cdot w_a}\left(\int_{X^a}I_a|\psi_\alpha|^2\,dw^a\right)dw_a. \tag{6.2}$$

But this is the Fourier transform of $\int_{X^a}I_a|\psi_\alpha|^2\,dw^a$ in X_a, evaluated at ζ. Hence $S_{\alpha'\alpha''}(\lambda)$, $\lambda\in I$, determines the Fourier transform in w_a of the 'effective interaction'

$$\int_{X^a}I_a|\psi_\alpha|^2\,dw^a \tag{6.3}$$

in a ball of radius $2\sqrt{\sup I-\epsilon_\alpha}$, except possibly on the spheres $\frac{1}{4}|\zeta|^2+\epsilon_\alpha\in\Lambda'_a$. However, $\int_{X^a}I_a|\psi_\alpha|^2\,dw^a\in L^1(X_a)$, hence the Fourier transform is continuous, hence it is determined on the whole ball. This completes the proof of Theorems 1.1 and 1.2.

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