

# RESOLVENT NEAR ZERO ENERGY ON RIEMANNIAN SCATTERING (ASYMPTOTICALLY CONIC) SPACES, A LAGRANGIAN APPROACH

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ABSTRACT. We use a Lagrangian regularity perspective to discuss resolvent estimates near zero energy on Riemannian scattering, i.e. asymptotically conic, spaces, and their generalizations. In addition to the Lagrangian perspective we introduce and use a resolved pseudodifferential algebra to deal with zero energy degeneracies in a robust manner.

## 1. INTRODUCTION AND OUTLINE

The purpose of this paper is to describe the low energy behavior of the resolvent on Riemannian scattering spaces  $(X, g)$  of dimension  $n \geq 3$  using a description that focuses on the outgoing radial set by providing Lagrangian regularity estimates. These spaces, introduced by Melrose [10], are Riemannian manifolds which are asymptotic to the ‘large end’ of a cone; one example is asymptotically Euclidean spaces. For  $\sigma \neq 0$ , including the  $|\sigma| \rightarrow \infty$  limit,  $\Delta_g - \sigma^2$  was studied in [18]. We refer to the introduction of that paper for a comparison of this approach, which uses a conjugation to move the outgoing ‘spherical wave’ asymptotics to the zero section followed by second microlocalized at the zero section scattering analysis (which means scattering-b analysis) on the one hand, and the more standard, variable order space approach that in one way or another underlies a number of the proofs of the limiting absorption principle, including in dynamical systems settings [10, 16, 4], on the other. The recent paper [17] also analyzed the  $\sigma \rightarrow 0$  behavior from the variable order perspective; though this paper also used second microlocal techniques, the reason was different: the degeneration of the characteristic set as  $\sigma \rightarrow 0$ . In this paper we provide an alternative treatment to [17] that matches [18]. We recall here that another area in which a Lagrangian regularity (though without second microlocalization) investigation has proved fruitful recently is describing internal waves in fluids, see [3].

The study of the  $\sigma \rightarrow 0$  limit has a long history, going back to the work of Jensen and Kato [9] in the Euclidean setting. More recently Guillarmou and Hassell analyzed this behavior in a series of works [5, 6] via constructing a parametrix for the resolvent family; here we proceed by *directly* obtaining Fredholm estimates. Other recent works on the subject include those of Bony and Häfner [1], Rodnianski and Tao [13] and Müller and Strohmaier [12]; we refer to [17] for more details.

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*This paper is intended as a companion paper to [18], so the reader is advised to read that paper first for a more detailed introduction to the setting, including the b- and scattering pseudodifferential operator algebras and their relation to analysis on  $\mathbb{R}^n$ , as well as for additional references.*

We recall that second microlocal, spaces, see [17, Section 5] in this scattering context, and see [2, 15] in different contexts, play a role in precise analysis at a Lagrangian, or more generally coisotropic, submanifold. As mentioned above, these second microlocal techniques played a role in [17] due to the degeneration of the principal symbol at zero energy, corresponding to the quadratic vanishing of any dual metric function at the zero section; the chosen Lagrangian is thus the zero section, really understood as the zero section at infinity. In a somewhat simpler way than in other cases, this second microlocalization at the zero section is accomplished by simply using the b-pseudodifferential operator algebra of Melrose [11]. In an informal way, this arises by blowing up the zero section of the scattering cotangent bundle at the boundary, though a more precise description (in that it makes sense even at the level of quantization, the spaces themselves are naturally diffeomorphic) is the reverse: blowing up the corner (fiber infinity over the boundary) of the b-cotangent bundle. In [17] this was used to show a uniform version of the resolvent estimates down to zero energy using variable differential order b-pseudodifferential operators. Indeed, the differential order of these, cf. the aforementioned blow-up of the corner, corresponds to the scattering decay order away from the zero section, thus this allows the uniform analysis of the problem to zero energy. However, for this problem the decay order (of the b-ps.d.o.) is also crucial, for it corresponds to the spaces on which the exact zero energy operator is Fredholm of index zero, which, with  $H_b$  denoting weighted b-Sobolev spaces relative to the scattering (metric)  $L^2$ -density, are  $H_b^{\tilde{r}, l} \rightarrow H_b^{\tilde{r}-2, l+2}$  with  $|l+1| < \frac{n-2}{2}$ , where  $\tilde{r}$  is the variable order (which is irrelevant at zero energy since the operator is elliptic in the b-pseudodifferential algebra then). (The more refined, fully 2-microlocal, spaces, corresponding to the blow-up of the corner, have three orders: sc-differential, sc-decay/b-differential and b-decay; using all of these is convenient, as the operators are sc-differential-elliptic, so one can use easily that there are no constraints on regularity in that sense; this modification is not crucial.)

Now, first for  $\sigma \neq 0$  real, one can work in a second microlocal space by simply conjugating the spectral family  $P(\sigma)$  by  $e^{i\sigma/x}$  (this being the multiplier from the right), with the point being that this conjugation acts as a canonical transformation of the scattering cotangent bundle, moving the outgoing radial set to the zero section. Then the second microlocal analysis is simply a refinement of b-analysis. Indeed, note that this conjugation moves  $x^{-1}(x^2D_x + \sigma)$ , resp.  $x^{-1}(xD_{y_j})$ , to  $x^{-1}(x^2D_x) = xD_x$ , resp.  $x^{-1}(xD_{y_j}) = D_{y_j}$ , so the Lagrangian regularity becomes b-differential-regularity indeed. Notice that the conjugate of the simplest model operator

$$P(\sigma) = (x^2D_x)^2 + i(n-1)x(x^2D_x) + x^2\Delta_y - \sigma^2 \in \text{Diff}_{\text{sc}}^2(X) \subset \text{Diff}_b^2(X)$$

is then

$$\begin{aligned} \hat{P}(\sigma) &= e^{-i\sigma/x}P(\sigma)e^{i\sigma/x} = (x^2D_x - \sigma)^2 + i(n-1)x(x^2D_x - \sigma) + x^2\Delta_y - \sigma^2 \\ &= (x^2D_x) - 2\sigma(x^2D_x) + i(n-1)x(x^2D_x) - i(n-1)x\sigma + x^2\Delta_y \in x\text{Diff}_b^2(X), \end{aligned}$$

which has one additional order of vanishing in this b-sense. (This is basically the effect of the zero section of the sc-cotangent bundle being now in the characteristic set.) Moreover, to leading order in terms of the b-decay sense, i.e. modulo  $x^2\text{Diff}_b^2(X)$ , this is the simple first order operator

$$-2\sigma x \left( xD_x + i \frac{n-1}{2} \right).$$

(In general, decay is controlled by the normal operator of a b-differential operator, which arises by setting  $x = 0$  in its coefficients after factoring out an overall weight, and where one thinks of it as acting on functions on  $[0, \infty)_x \times \partial X$ , of which  $[0, \delta_0)_x \times \partial X$  is identified with a neighborhood of  $\partial X$  in  $X$ .) This is non-degenerate for  $\sigma \neq 0$  in that, on suitable spaces, it has an invertible normal operator; of course, this is not an elliptic operator, so some care is required. Notice that terms like  $(x^2D_x)^2$  and  $\sigma x^2D_x$  have the same scattering decay order, i.e. on the front face of the blown up b-corner they are equally important. Thus, one does real principal type plus radial points estimates in this case to conclude a Fredholm statement

$$(1.1) \quad \hat{P}(\sigma) : \{u \in H_b^{\tilde{r}, l} : \hat{P}(\sigma)u \in H_b^{\tilde{r}, l+1}\} \rightarrow H_b^{\tilde{r}, l+1},$$

and here  $l < -1/2$ ,  $\tilde{r} + l > -1/2$  constant work. (Note that  $\tilde{r} + l$  is the scattering decay order away from the zero section.) We refer to [18] for the proof.

Notice that, in terms of the limiting absorption principle, there are two ways to implement this conjugation: one can conjugate either by  $e^{i\sigma/x}$ , where  $\sigma$  is now complex, or by  $e^{i\text{Re}\sigma/x}$ . The former, which we follow, gives much stronger spaces when  $\sigma$  is not real with  $\text{Im}\sigma > 0$  (which is from where we take the limit), as  $e^{i\sigma/x}$  entails an exponentially decaying weight  $e^{-\text{Im}\sigma/x}$ , so if the original operator is applied to  $u$ , the conjugated operator is applied to  $e^{\text{Im}\sigma/x}u$ .

One cannot expect an estimate that does not already hold for the elliptic operator  $\hat{P}(0) = P(0)$ , and in that case one has a Fredholm elliptic estimate in which the b-decay order changes by 2. Nonetheless, between spaces whose b-decay order differs by 2 one might expect non-degenerate, uniform, estimate, as was done in the unconjugated setting in [17]. We might expect that this conjugated approach will be in some ways more restrictive than the unconjugated one because two of the three phenomena constraining orders (incoming and outgoing radial points, and indicial roots, which are points of normal operator non-invertibility) are realized in the b-decay sense, namely both the outgoing radial set and the indicial root phenomena take place here.

This can be remedied by using a resolved version of the b-pseudodifferential algebra which also allows for weights vanishing precisely at  $x = \sigma = 0$ , thus of the form  $(x + |\sigma|)^\alpha$ . We state the estimates for resolved scattering-b Sobolev spaces  $H_{\text{sc}, \text{b}, \text{res}}^{s, r, l}(X)$ , or rather norms (akin to the semiclassical spaces these are really a family of norms on the same space), defined in Section 3 in (3.4). However, we already state that if  $s = r - l$ , this is simply the standard b-Sobolev space  $H_b^{r-l, l}(X)$  with the standard,  $\sigma$ -independent, norm, see (3.5); given  $r, l$  satisfying the hypotheses below, such a choice of  $s$  is always acceptable for  $H_{\text{sc}, \text{b}, \text{res}}^{s, r, l}(X)$ ; then the  $H_{\text{sc}, \text{b}, \text{res}}^{s-2, r+1, l+1}(X)$  norm on  $\hat{P}(\sigma)u$  can be strengthened to  $H_{\text{sc}, \text{b}, \text{res}}^{s, r+1, l+1} = H_b^{r-l, l+1}$  (thus the estimate weakened) which is of the same form, but this is quite lossy; we give below in (1.2) a better (but still lossy) version.

We formulate the general theorem below in Section 2, in Theorem 2.5, where various additional notations are introduced. However, here we state our main theorem for the spectral family  $P(\sigma)$  for an asymptotically conic Laplacian  $\Delta_g$ :

**Theorem 1.1.** *Suppose that  $|l' + 1| < \frac{n-2}{2}$ , and suppose that  $P(0) : H_b^{\infty, l'} \rightarrow H_b^{\infty, l'+2}$  has trivial nullspace, an assumption independent of  $l'$  in this range. Suppose also that either  $r > -1/2$ ,  $l < -1/2$ , or  $r < -1/2$ ,  $l > -1/2$ . Let*

$$\hat{P}(\sigma) = e^{-i\sigma/x} P(\sigma) e^{i\sigma/x}.$$

There exists  $\sigma_0 > 0$  such that

$$\hat{P}(\sigma) : \{u \in H_{sc, b, res}^{s, r, l} : \hat{P}(\sigma)u \in H_{sc, b, res}^{s-2, r+1, l+1}\} \rightarrow H_{sc, b, res}^{s-2, r+1, l+1}$$

is invertible for  $0 < |\sigma| \leq \sigma_0$ ,  $\text{Im } \sigma \geq 0$ , with this inverse being the  $\pm i0$  resolvent of  $P(\sigma)$  corresponding to  $\pm \text{Re } \sigma > 0$ , and we have the estimate

$$\|(x + |\sigma|)^\alpha u\|_{H_{sc, b, res}^{s, r, l}} \leq C \|(x + |\sigma|)^{\alpha-1} \hat{P}(\sigma)u\|_{H_{sc, b, res}^{s-2, r+1, l+1}}$$

for

$$\alpha \in \left( l + 1 - \frac{n-2}{2}, l + 1 + \frac{n-2}{2} \right).$$

*Remark 1.2.* We remark that the estimate implies the following estimate purely in terms of b-Sobolev spaces. Suppose  $s = r - l$ , with  $r, l$  as in the theorem. Then

$$(1.2) \quad \|(x + |\sigma|)^\alpha u\|_{H_b^{s, l}} \leq C \|(x + |\sigma|)^\alpha \hat{P}(\sigma)u\|_{H_b^{s-1, l+2}}.$$

We refer to (3.8) below for its proof, and to (3.9) for a further strengthened (non-resolved) scattering-b, i.e. second microlocal, statement, which nonetheless is still weaker than the main theorem.

Note that in case we want  $l < -1/2$  (so that the b-decay order is low, but b-differentiability/sc-decay is high), one can always (for  $n \geq 2$ , e.g.  $l$  close to  $-1/2$ ) take  $\alpha = 0$  for suitable  $l$ , while if we want  $l > -1/2$  (so that the b-decay order is high, but b-differentiability/sc-decay is low) we can only do this for  $n \geq 4$ , otherwise the lower limit of the  $\alpha$  interval is  $> 0$ .

We remark that in spite of the earlier indication that two of the three phenomena constraining orders take place at the same location, for the  $l < -1/2$  case in 3 dimensions one may expect, as we prove, an optimal result with  $\alpha = 0$  because the decay order of spherical waves (which is the threshold order for the outgoing Lagrangian) and that of the zero energy Green's function (which is the beginning of the interval allowed by the indicial roots) are the same:  $x = r^{-1}$ .

Note also the estimate can be rewritten as

$$\|(1 + x/|\sigma|)^\alpha u\|_{H_{sc, b, res}^{s, r, l}} \leq C \|(1 + x/|\sigma|)^\alpha (x + |\sigma|)^{-1} \hat{P}(\sigma)u\|_{H_{sc, b, res}^{s-2, r+1, l+1}}.$$

The structure of this paper is the following. In Section 2 we describe the structure of the general class of operators we are considering. In Section 3 we introduce the resolved pseudodifferential algebras that allow for precise estimates down to  $\sigma = 0$ . In Section 4 we obtain symbolic estimates whose errors gain in the scattering differential and decay orders. In Section 5 we remove those errors using a normal operator estimate that is obtained by reducing to the non-zero spectral parameter case analyzed in [18] via a rescaling argument. Finally in Section 6 we discuss what happens in a simple case when  $P(0)$  does have a non-trivial nullspace.

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## 2. THE OPERATOR

First recall the framework in which the  $\sigma \rightarrow 0$  behavior was analyzed in the unconjugated setting in [17]. To start with, we have a scattering metric  $g \in S^0(\text{sc}T^*X \otimes_s \text{sc}T^*X)$  for which there is an actually conic metric  $g_0 = x^{-4} dx^2 + x^{-2} h$ ,  $h$  a Riemannian metric on  $\partial X$ , to which  $g$  is asymptotic in the sense that  $g - g_0 \in S^{-\tilde{\delta}}(X, \text{sc}T^*X \otimes_s \text{sc}T^*X)$ ,  $\tilde{\delta} > 0$ . Then

$$P(\sigma) = P(0) + \sigma Q - \sigma^2,$$

$$P(0) \in S^{-2}\text{Diff}_b^2(X), \quad Q \in S^{-2-\tilde{\delta}}\text{Diff}_b^1(X), \quad P(0) - \Delta_g \in S^{-2-\tilde{\delta}}\text{Diff}_b^2(X)$$

thus also

$$P(0) - \Delta_{g_0} \in S^{-2-\tilde{\delta}}\text{Diff}_b^2(X).$$

In the present paper we obtain more precise information than in [17], but under assumptions which are stronger on the highest (second) order terms, though more relaxed on the lower order terms, namely  $Q$ , as well as  $\sigma^2$  terms (though we recall that in [17]  $Q$  was allowed to smoothly depend on  $\sigma$ ). Thus, we take  $\tilde{\delta} = 1$ , and we impose the existence of leading terms, so

$$g - g_0 \in x\mathcal{C}^\infty(X, \text{sc}T^*X \otimes_s \text{sc}T^*X) + S^{-1-\delta}(X, \text{sc}T^*X \otimes_s \text{sc}T^*X), \quad \delta > 0,$$

for the metric. We allow a more general form for the operator in terms of the coefficient of  $\sigma^2$ :

$$P(\sigma) = P(0) + \sigma Q - \sigma^2(1 - R),$$

and we take

$$(2.1) \quad \begin{aligned} P(0) - \Delta_g &\in x^2\text{Diff}_b^1(X) + S^{-2-\delta}\text{Diff}_b^1(X) \subset x\text{Diff}_{\text{sc}}^1(X) + S^{-1-\delta}\text{Diff}_{\text{sc}}^1(X), \\ Q &\in x\text{Diff}_{\text{sc}}^1(X) + S^{-1-\delta}\text{Diff}_{\text{sc}}^1(X) \\ &= x\mathcal{C}^\infty(X) + S^{-1-\delta}(X) + x^2\text{Diff}_b^1(X) + S^{-2-\delta}\text{Diff}_b^1(X), \\ R &\in x\mathcal{C}^\infty(X) + S^{-1-\delta}(X), \end{aligned}$$

thus also

$$P(0) - \Delta_{g_0} \in x^2\text{Diff}_b^1(X) + S^{-2-\delta}\text{Diff}_b^2(X).$$

Note that the membership of  $P(0) - \Delta_g$  in  $x\text{Diff}_{\text{sc}}^1(X) + S^{-1-\delta}\text{Diff}_{\text{sc}}^1(X)$  is the condition used in [18] and indeed any  $P(\sigma)$  satisfying the requirements here satisfies those of [18]; the requirement here is stronger as it rules out terms in  $x\mathcal{C}^\infty(X) + S^{-1-\delta}(X)$ , such as Coulomb type potentials. However, such  $x\mathcal{C}^\infty(X) + S^{-1-\delta}(X)$  terms are allowed in  $Q$  and  $R$  due to the prefactor  $\sigma$  or  $\sigma^2$  present in front of them.

In [18] there was no need for stronger assumptions for the skew-adjoint parts of operators, essentially because they are subprincipal in terms of sc-decay, so while they affect the statements (via shifting threshold regularity values), they can be

handled. Here, for our more delicate problem, we also demand the stronger statements that

$$(2.2) \quad \frac{1}{2i}(P(0) - P(0)^*) - \beta_I x \left( x^2 D_x + i x \frac{n-2}{2} \right) - \beta'_I x^2 \in S^{-2-\delta} \text{Diff}_b^1(X) \subset S^{-1-\delta} \text{Diff}_{\text{sc}}^1(X)$$

for some  $\beta_I, \beta'_I \in \mathcal{C}^\infty(X)$  (which can simply be thought of as functions on  $\partial X$ , as the  $x\mathcal{C}^\infty(X)$  terms can be absorbed into the right hand side, and the  $i x \frac{n-2}{2}$  term is included in  $\beta_I$  as opposed to  $\beta'_I$  since  $x(x^2 D_x + i x \frac{n-2}{2})$  is formally self-adjoint modulo terms that can be incorporated into the right hand side), and

$$(2.3) \quad \frac{1}{2i}(Q - Q^*) - \gamma_I x \in S^{-1-\delta}(X) + S^{-2-\delta} \text{Diff}_b^1(X) = S^{-1-\delta} \text{Diff}_{\text{sc}}^1(X),$$

for suitable  $\gamma_I \in \mathcal{C}^\infty(X)$ , and

$$(2.4) \quad \frac{1}{2i}(R - R^*) \in S^{-1-\delta}(X).$$

As,  $P(\sigma) \in \Psi_b^{2,0}$  only, and in the usual sense the normal operator in  $\Psi_b^{2,0}$  is simply  $-\sigma^2$  as  $P(\sigma) + \sigma^2 \in \Psi_b^{2,-2}$ . Thus, in [17] we instead considered the ‘effective normal operator’, quotienting the operator by  $S^{-2-\delta} \text{Diff}_b^2(X)$ , which under the assumptions of [17] yields

$$\tilde{N}(P(\sigma)) = N(P(0)) - \sigma^2 = \Delta_{g_0} - \sigma^2,$$

so

$$P(\sigma) - \tilde{N}(P(\sigma)) \in S^{-2-\delta} \text{Diff}_b^2(X);$$

this difference was irrelevant for the analysis of b-decay. Here, due to our weaker assumptions on  $Q$  as well as  $P(0)$ , the aforementioned extended normal operator would in fact also include the leading order terms of  $Q$  as they are in  $x^2 \text{Diff}_b^1(X)$ , and it would also include the  $x^2 \text{Diff}_b^1(X)$  terms from  $P(0)$  as well as more than just the leading order terms from  $R$ .

From the Lagrangian perspective we consider a conjugated version of  $P(\sigma)$ . Thus, let

$$\hat{P}(\sigma) = e^{-i\sigma/x} P(\sigma) e^{i\sigma/x}.$$

Since conjugation by  $e^{i\sigma/x}$  is well-behaved in the scattering, but not in the b-sense, it is actually advantageous to first perform the conjugation in the scattering setting, and then convert the result to a b-form. The principal symbol of  $\hat{P}$  in the scattering decay sense is simply a translated version, by  $d(\sigma/x)$ , of that of  $P$  (this corresponds to  $e^{-i\sigma/x}(x^2 D_x) e^{i\sigma/x} = x^2 D_x - \sigma$ ), which is  $\tau^2 + \mu^2 - \sigma^2$ , thus it is  $\tau^2 + \mu^2 - 2\sigma\tau$ . However, we need more precise information, thus we perform the computation explicitly.

**Proposition 2.1.** *We have*

$$(2.5) \quad \hat{P}(\sigma) = \hat{P}(0) + \sigma \hat{Q} + \sigma^2 \hat{R} - 2\sigma \left( x^2 D_x + i \frac{n-1}{2} x + \frac{\hat{\beta} - \hat{\gamma}}{2} x \right)$$

with  $\hat{\beta}, \hat{\gamma} \in \mathcal{C}^\infty(X) + S^{-\delta}(X)$ ,  $\text{Im } \hat{\beta}|_{\partial X} = \beta_I$ ,  $\text{Im } \hat{\gamma}|_{\partial X} = \gamma_I$ ,

$$\hat{P}(0) \in x^2 \text{Diff}_b^2(X) + S^{-2-\delta} \text{Diff}_b^2(X),$$

$$\hat{Q} \in x^2 \text{Diff}_b^1(X) + S^{-2-\delta} \text{Diff}_b^1(X),$$

$$\hat{R} \in x\mathcal{C}^\infty(X) + S^{-1-\delta}(X);$$

and  $\text{Im } \hat{Q} \in S^{-1-\delta}(X) + S^{-2-\delta}\text{Diff}_b^1(X)$ ,  $\text{Im } \hat{R} \in S^{-1-\delta}(X)$ .

*Remark 2.2.* Notice that  $\hat{P}(\sigma) \in x\text{Diff}_b^2(X)$  (modulo the faster decaying  $S^{-2-\delta}\text{Diff}_b^2(X)$ ), unlike  $P(\sigma)$  which is merely in  $\text{Diff}_b^2(X)$  (modulo the faster decaying  $S^{-1-\delta}\text{Diff}_b^2(X)$ ) due to the  $\sigma^2$  term; this one order decay improvement plays a key role below.

Also, in [18, Equation (3.5)] one has  $\hat{P}(0) = P(0) - xa'$  with the notation there (so  $xa'$  there is  $x^2a'$  here, see the notation in the proof below); under our present assumptions of it being  $O(x^2)$ , we do not need to remove the  $a'$  term from  $P(0)$ .

*Proof.* In general we have

$$\begin{aligned} P(0) &= (1 + xa_{00})(x^2D_x)^2 + \sum_j xa_{0j}((x^2D_x)(xD_{y_j}) + (xD_{y_j})(x^2D_x)) \\ &\quad + \sum_{i,j} a_{ij}(xD_{y_i})(xD_{y_j}) \\ &\quad + (i(n-1) + \beta + a_0)x(x^2D_x) + \sum_j xa_j(xD_{y_j}) + x^2a', \end{aligned}$$

and

$$Q = b_0x(x^2D_x) + \sum_j xb_j(xD_{y_j}) + \gamma x + b'x$$

with  $a_{00}, a_{0j}, a_j, a', b_0, b_j \in \mathcal{C}^\infty(X) + S^{-\delta}(X)$ ,  $a_0, b' \in S^{-\delta}(X)$ ,  $a_{ij} \in \mathcal{C}^\infty(X) + S^{-1-\delta}(X)$ ,  $\text{Im } a_j, \text{Im } b_0, \text{Im } b_j \in S^{-\delta}(X)$ ,  $a' - \beta' \in S^{-\delta}(X)$ ,  $\text{Im } \beta' = \beta'_I + \frac{n-2}{2}$ ,  $\beta, \gamma \in \mathcal{C}^\infty(X)$  (which can be considered as functions on  $\partial X$  due to the  $a_0$  and  $b'$  terms, they are singled out rather than included in  $a_0, b'$  due to their role below) with  $\text{Im } \beta = \beta_I$ ,  $\text{Im } \gamma = \gamma_I$ , and with  $b_0, b_j, b'$  smoothly depending on  $\sigma$ . Let

$$\hat{\beta} = \beta + a_0, \quad \hat{\gamma} = \gamma + b'.$$

As  $e^{-i\sigma/x}(x^2D_x)e^{i\sigma/x} = x^2D_x - \sigma$ , this gives

$$\begin{aligned} &e^{-i\sigma/x}P(0)e^{i\sigma/x} \\ &= (1 + xa_{00})(x^2D_x - \sigma)^2 + \sum_j xa_{0j}((x^2D_x - \sigma)(xD_{y_j}) + (xD_{y_j})(x^2D_x - \sigma)) \\ &\quad + \sum_{i,j} a_{ij}(xD_{y_i})(xD_{y_j}) + (i(n-1) + \beta + a_0)x(x^2D_x - \sigma) \\ &\quad + \sum_j xa_j(xD_{y_j}) + x^2a', \end{aligned}$$

and

$$e^{-i\sigma/x}Qe^{i\sigma/x} = b_0x(x^2D_x - \sigma) + \sum_j xb_j(xD_{y_j}) + \gamma x + b'x.$$

Combining the terms, including  $R$ , gives

$$(2.6) \quad \hat{P}(\sigma) = \hat{P}(0) + \sigma\hat{Q} + \sigma^2\hat{R} - 2\sigma\left(x^2D_x + i\frac{n-1}{2}x + \frac{\hat{\beta} - \hat{\gamma}}{2}x\right)$$

with

$$\begin{aligned}\hat{P}(0) &= P(0) \in x^2\text{Diff}_b^2(X) + S^{-2-\delta}\text{Diff}_b^2(X), \\ \hat{Q} &= Q - 2xa_{00}(x^2D_x) - 2\sum_j xa_{0j}(xD_{y_j}) - x\hat{\gamma} \\ &\in x^2\text{Diff}_b^1(X) + S^{-2-\delta}\text{Diff}_b^1(X), \\ \hat{R} &= R + xa_{00} - xb_0 \in x\mathcal{C}^\infty(X) + S^{-1-\delta}(X);\end{aligned}$$

note that  $\text{Im } \hat{R}, \text{Im } \hat{Q}$  are also as stated.  $\square$

We also remark that the principal symbol of  $\hat{P}(0)$  vanishes quadratically at the scattering zero section,  $\tau = 0, \mu = 0, x = 0$ , hence the subprincipal symbol makes sense directly there (without taking into account contributions from the principal symbol, working with half-densities, etc.), and this in turn vanishes. It is convenient to summarize this, including positivity properties of  $\hat{P}(0)$  here, as this will be helpful when considering non-real  $\sigma$  below. Note that this result already appears in [18]; the stronger assumptions in our case do not affect the statement.

**Lemma 2.3** (cf. Lemma 3.2 of [18]). *The operator  $\hat{P}(0)$  is non-negative modulo terms that are either sub-sub-principal or subprincipal but with vanishing contribution at the scattering zero section, in the sense that it has the form*

$$(2.7) \quad \hat{P}(0) = \sum_j T_j^* T_j + \sum_j T_j^* T_j' + \sum_j T_j^\dagger T_j + T''$$

where  $T_j \in x\text{Diff}_b^1(X) + S^{-2-\delta}\text{Diff}_b^1(X)$ ,  $T_j', T_j^\dagger \in x\mathcal{C}^\infty(X) + S^{-1-\delta}(X)$ ,  $T'' \in x^2\mathcal{C}^\infty(X) + S^{-2-\delta}(X)$ . Moreover,

$$(2.8) \quad \hat{Q} = \sum_j T_j^* \tilde{T}_j' + \sum_j \tilde{T}_j^\dagger T_j + \tilde{T}''$$

with  $\tilde{T}_j', \tilde{T}_j^\dagger \in x\mathcal{C}^\infty(X) + S^{-1-\delta}(X)$ ,  $\tilde{T}'' \in x^2\mathcal{C}^\infty(X) + S^{-2-\delta}(X)$ .

The standard normal operator of  $\hat{P}(\sigma)$ , which arises by considering the operator  $x^{-1}\hat{P}(\sigma)$  and freezing the coefficients at the boundary,

$$(2.9) \quad \begin{aligned}N(\hat{P}(\sigma)) &= -2\sigma\left(x^2D_x + i\frac{n-1}{2}x + \frac{\beta-\gamma}{2}x\right) + \sigma^2x\varpi \\ &= -2\sigma\left(x^2D_x + i\frac{n-1}{2}x + \frac{\beta-\gamma}{2}x - \sigma\frac{\varpi}{2}x\right), \\ \varpi &= (x^{-1}R)|_{\partial X} + a_{00}|_{\partial X} - b_0|_{\partial X}, \quad \beta = \hat{\beta}|_{\partial X}, \quad \gamma = \hat{\gamma}|_{\partial X},\end{aligned}$$

degenerates at  $\sigma = 0$ , corresponding to  $\hat{P}(\sigma)$  being in  $x^2\text{Diff}_b^2(X) + \sigma x\text{Diff}_b^1(X)$  (modulo faster decaying terms with symbolic coefficients), so the definiteness of the operator at  $\sigma = 0$  still arises from  $x^2\text{Diff}_b^2(X)$ . Hence, we need to use an effective normal operator even with this approach, which thus again will not be dilation invariant. However, we shall use a joint scaling in  $(x, \sigma)$ , in which sense it is well behaved.

Before proceeding, we remark that the actual normal operator, (2.9), is  $x$  times the normal vector field to the boundary plus a scalar, which, for  $\sigma \neq 0$ , corresponds to the asymptotic behavior of the solutions of  $\hat{P}(\sigma)v \in \mathcal{C}^\infty(X)$  being

$$x^{(n-1-i(\beta-\gamma)+i\sigma\varpi)/2}\mathcal{C}^\infty(\partial X),$$



modulo faster decaying terms. This corresponds to the asymptotics

$$e^{i\sigma/x} x^{(n-1-i(\beta-\gamma)+i\sigma\varpi)/2} \mathcal{C}^\infty(\partial X)$$

for solutions of  $P(\sigma)u \in \dot{\mathcal{C}}^\infty(X)$  for  $\sigma \neq 0$ . This indicates that we can remove the contribution of  $\varpi, \beta, \gamma$  to leading decay order by conjugating the operator by  $x^{(-i(\beta-\gamma)+i\sigma\varpi)/2}$ , but we do not do this here. We also remark that  $\varpi$  is real by our assumptions. Notice that in the case of Kerr spacetimes, if we factor out the coefficient of  $\partial_t^2$  from the operator,  $a_{00}|_{\partial X} = -4m$  and  $(x^{-1}R)|_{\partial X} = 0$ ,  $b_0|_{\partial X} = 0$ , so our conjugating factor is  $e^{i\sigma(r+2m \log r)}$ , which is asymptotically exactly  $e^{i\sigma r_*}$ ,  $r_*$  the logarithmically modified radial function (a version of the Regge-Wheeler radial tortoise coordinate function). If we do not factor this coefficient out, then  $a_{00}|_{\partial X} = -2m$ ,  $\hat{R} - (-2mx) \in x^2 \mathcal{C}^\infty(X)$ , and  $b_0|_{\partial X} = 0$ , so we obtain the same conclusion.

For us, the key normal operator is the one associated to the front face of the blow up of  $x = \sigma = 0$ . In the present case it captures  $\hat{P}(\sigma)$  modulo

$$x(x + \sigma)(x^\delta + \sigma) S^0 \text{Diff}_b^2(X) \subset \Psi_{b, \text{res}}^{2, -1, -2-\delta, 0}(X),$$

with the latter class of pseudodifferential operators introduced in the next section. Note that this operator does not quite encapsulate the standard normal operator since in the second, b-decay, order there is no gain over the a priori given membership of  $\hat{P}(\sigma)$ , but we can use the smallness of  $\sigma$  (we are interested in the zero energy limit after all) to deal with this, see the proof of Proposition 5.3.

**Definition 2.4.** We define the *effective normal operator*  $N_0(\hat{P}(\sigma))$  as  $\hat{P}(\sigma)$  modulo  $x(x + \sigma)(x^\delta + \sigma) S^0 \text{Diff}_b^2(X)$ .

The nice feature is that starting from sc-differential operators, which we consider as b-operators with decaying coefficients, and conjugating them by exponentials, we lose decay but gain a factor of  $\sigma$ , so if it were not for the overall  $x$  vanishing in the space to be quotiented out,  $x(x + \sigma)(x^\delta + \sigma) S^0 \text{Diff}_b^2(X)$ , any term in  $S^{-2-\delta} \text{Diff}_b^2(X)$  in the unconjugated operator  $P(\sigma)$  would automatically give a trivial contribution to the effective normal operator. As is, in  $P(0)$  more structure is needed, which is the reason for giving the requirements in (2.1) in the stated form.

We then have

$$N_0(\hat{P}(\sigma)) = \Delta_{g_0} + \beta x^2 \left( x D_x + i \frac{n-2}{2} \right) + x^2 \beta' - 2\sigma \left( x^2 D_x + i \frac{n-1}{2} x + \frac{\beta-\gamma}{2} x \right)$$

in the sense that

$$N_0(\hat{P}(\sigma)) - \hat{P}(\sigma) \in x(x + \sigma)(x^\delta + \sigma) \text{Diff}_b^2(X) \subset \Psi_{b, \text{res}}^{2, -1, -2-\delta, 0}(X).$$

More precisely, we identify  $X$  near  $\partial X$  with  $[0, x_0)_x \times \partial X$ , as usual for the standard normal operator, regard  $N_0(\hat{P}(\sigma))$  as an operator on the cone over  $\partial X$ ,  $[0, \infty)_x \times \partial X$ , and the requirement is that *evaluated on*  $\sigma$ -dependent families supported in  $x < x_0$ , with output restricted to the same region, the difference of  $N_0(\hat{P}(\sigma))$  and  $\hat{P}(\sigma)$  has the desired form, i.e. is given by an operator family with the indicated properties. A key point is that  $N_0(\hat{P}(\sigma))$  is dilation invariant jointly in  $(x, \sigma)$ , which we shall use in Section 5.

In addition to the extended normal operator, we also need to consider the *standard* normal operator  $N(\hat{P}(0))$  of  $\hat{P}(0)$  as an operator in

$$x^2 \text{Diff}_b^2(X) + S^{-2-\delta} \text{Diff}_b^2(X),$$

thus quotienting out by  $S^{-2-\delta}\text{Diff}_b^2(X)$ . Correspondingly, we keep more information (for this term) than for  $N(\hat{P}(\sigma))$ , since there the quotient is by  $S^{-1-\delta}\text{Diff}_b^2(X)$  but on the other hand this is simply the extended normal operator  $N_0(\hat{P}(0))$  of  $\hat{P}(0)$ , namely it is

$$\Delta_{g_0} + \beta x^2 \left( xD_x + i\frac{n-2}{2} \right) + x^2 \beta' \in x^2 \text{Diff}_b^2(X),$$

modulo  $S^{-2-\delta}\text{Diff}_b^2(X)$ .

For normal operator purposes it is convenient to work with  $L_b^2$  instead of the metric  $L^2$ -space,  $L_{g_0}^2$ ; this is given by the density  $\frac{dx}{x} dh_0$ ,  $h_0 = h|_{\partial X}$  the metric on the cross section of the asymptotic cone, so

$$L_b^2 = x^{-n/2} L_{g_0}^2.$$

Let

$$\Delta_b = x^{-(n+2)/2} \Delta_{g_0} x^{(n-2)/2} \in \text{Diff}_b^2(X);$$

as  $x^{-1} \Delta_{g_0} x^{-1}$  is symmetric with respect to the  $L_{g_0}^2$ -inner product,  $\Delta_b$  is symmetric with respect to the  $L_b^2$  inner product. Explicitly, with  $\Delta_{\partial X} = \Delta_{h_0}$ ,

$$\begin{aligned} \Delta_b &= x^{n/2} D_x x^{-n+3} D_x x^{n/2-1} + \Delta_{\partial X} \\ &= \left( D_x x + i\frac{n}{2} \right) x^{-n/2+2} D_x x^{n/2-1} + \Delta_{\partial X} \\ (2.10) \quad &= \left( D_x x + i\frac{n}{2} \right) \left( xD_x - i\frac{n-2}{2} \right) + \Delta_{\partial X} \\ &= (xD_x)^2 + \Delta_{\partial X} + \left( \frac{n-2}{2} \right)^2; \end{aligned}$$

notice that this is a positive definite operator on  $L_b^2$  for  $n \geq 3$ , since on the Mellin transform side it is multiplication by a positive (operator valued) function. The full conjugated and re-normalized operator (in that  $x^2$  is factored out) is

$$x^{-(n+2)/2} \left( \Delta_{g_0} + \beta x^2 \left( xD_x + i\frac{n-2}{2} \right) + x^2 \beta' \right) x^{(n-2)/2} = \Delta_b + \beta(xD_x) + \beta'.$$

Mellin transforming in  $x$  we obtain the elliptic family

$$\tau_b^2 + \beta \tau_b + \beta' + \Delta_{\partial X} + \left( \frac{n-2}{2} \right)^2$$

of operators on  $\partial X$ , which is also elliptic in the large parameter sense (in  $\tau_b$ , with  $\text{Im } \tau_b$  bounded), invertible for large  $|\tau_b|$  with  $\text{Im } \tau_b$  bounded, so the inverse is a meromorphic family. Its poles are called the *indicial roots*. If  $\beta, \beta'$  are constant scalars, this is invertible whenever

$$-\left( \frac{n-2}{2} \right)^2 - \tau_b^2 - \beta \tau_b - \beta'$$

is not an eigenvalue of  $\Delta_{\partial X}$ ; if  $\beta, \beta'$  are constant non-scalar and have a joint eigenspace decomposition ( $\beta$  is assumed to be skew-adjoint below!), then one can effectively replace them by the eigenvalues. Thus, the indicial roots are of the form

$$\frac{1}{2} \left( -\beta \pm \sqrt{\beta^2 - 4 \left( \lambda + \left( \frac{n-2}{2} \right)^2 + \beta' \right)} \right),$$

with  $\lambda$  an eigenvalue of  $\Delta_{\partial X}$ , which for  $\beta = 0, \beta' = 0$  reduces to

$$\pm i \sqrt{\lambda + \left( \frac{n-2}{2} \right)^2}.$$

This means that for  $\beta = 0$ ,  $\beta' = 0$  one has a ‘central interval’  $(-\frac{n-2}{2}, \frac{n-2}{2})$  such that if  $\text{Im } \tau_b$  is in the interval, then the Mellin transformed normal operator is invertible. This corresponds to invertibility of the original operator on weighted  $L_b^2$  spaces  $x^\ell L_b^2$ , where  $|\ell| < \frac{n-2}{2}$ . This means that the unconjugated operator  $N(\hat{P}(0))$  is invertible from weighted spaces

$$x^{\ell+(n-2)/2} L_b^2 = x^{\ell-1} L_{g_0}^2$$

to spaces with two additional orders of decay and two b-derivatives, namely  $x^{l'+1} H_b^2$  (recall that we are using the  $g_0$ -density for these spaces), which means the domain space has weight

$$(2.11) \quad l' = \ell - 1 \in \left( -1 - \frac{n-2}{2}, -1 + \frac{n-2}{2} \right).$$

In general, for the simplicity of discussion, and as this covers already the most interesting case, we assume that  $\beta$  is skew-adjoint and  $\beta'$  is sufficiently small. If  $\text{Re } \beta' > \frac{\beta^2}{4} - \left(\frac{n-2}{2}\right)^2$  (which includes  $\beta = 0$ ,  $\beta' = 0$ ), then (2.11) is replaced by

$$(2.12) \quad l' \in \left( -1 + \frac{\text{Im } \beta}{2} - \text{Re} \sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2} + \beta', \right. \\ \left. -1 + \frac{\text{Im } \beta}{2} + \text{Re} \sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2} + \beta' \right),$$

where the right hand side contains the sub-interval where  $\beta'$  is replaced by  $\text{Re } \beta'$ , and then the real part in front of the square roots can be dropped. We call the interval on the right hand side of (2.12) the *central interval for weights for the scattering end*, and we denote it by  $(\nu_-, \nu_+)$ . We refer to Remark 5.5 for an explanation of the role of this particular weight interval free from the negatives of the imaginary parts of the indicial roots.

In Section 5 we rescale  $N_0(\hat{P}(\sigma))$  using the dilation invariance. This amounts to introducing  $X = x/|\sigma|$ ,  $\hat{\sigma} = \sigma/|\sigma|$ , in terms of which

$$N_0(\hat{P}(\sigma)) = \sigma^2 \left( \Delta_{g_0} + \beta X^2 \left( X D_X + i \frac{n-2}{2} \right) + \beta' X^2 \right. \\ \left. - 2\hat{\sigma} \left( X^2 D_X + i \frac{n-1}{2} X + \frac{\beta - \gamma}{2} X \right) \right)$$

and  $\Delta_0$  is the Laplacian of the exact conic metric  $\frac{dX^2}{X^4} + \frac{h_0}{X^2}$ . Since  $\sigma^{-2} N_0(\hat{P}(\sigma))$  is homogeneous with respect to  $X$  dilations, it can be globally Mellin transformed, and one can consider this as simultaneously resolving the scattering end,  $X = 0$ , as well as the conic point,  $X^{-1} = 0$ . The normalization of the Mellin transform

$$(\mathcal{M}u)(\tau_b, y) = \int_0^\infty X^{-i\tau_b} u(X, y) \frac{dX}{X}$$

corresponding to the choice of  $X$ , as opposed to the choice of  $X^{-1}$ ,

$$(\mathcal{M}^c u)(\tau_b^c, y) = \int_0^\infty X^{i\tau_b^c} u(X, y) \frac{dX}{X},$$

with superscript  $c$  standing for the conic point, involves taking the negative of the dual variable (as the transforms use powers of  $X$  vs.  $X^{-1}$ , so  $\tau_b^c = -\tau_b$  identifies the transforms) hence the indicial roots for the scattering end are the negatives of the indicial roots at the conic point. When  $\beta = 0$  and  $\beta' = 0$ , the central interval

at the conic point is  $(1 - \frac{n-2}{2}, 1 + \frac{n-2}{2})$  as is familiar from analysis of the Laplacian on spaces with conic singularities, and this interval indeed is the negative of the range (2.11).

The version of the theorem corresponding to our more general operators is:

**Theorem 2.5.** *Suppose that  $\beta$  is skew-symmetric and  $\operatorname{Re} \beta' > \frac{\beta^2}{4} - \left(\frac{n-2}{2}\right)^2$ . Suppose that  $(\nu_-, \nu_+)$  is the central interval for weights at the scattering end,  $l' \in (\nu_-, \nu_+)$ , see (2.12). Suppose that  $P(0) : H_{\mathfrak{b}}^{\infty, l'} \rightarrow H_{\mathfrak{b}}^{\infty, l'+2}$  has trivial nullspace, an assumption independent of  $l'$  in this range. Suppose also that either*

$$r > -1/2 + \operatorname{Im}(\beta + \gamma)/2, \quad l < -1/2 + \operatorname{Im}(\beta - \gamma)/2,$$

or

$$r < -1/2 + \operatorname{Im}(\beta + \gamma)/2, \quad l > -1/2 + \operatorname{Im}(\beta - \gamma)/2.$$

Let

$$\hat{P}(\sigma) = e^{-i\sigma/x} P(\sigma) e^{i\sigma/x}.$$

There exists  $\sigma_0 > 0$  such that

$$\hat{P}(\sigma) : \{u \in H_{\operatorname{sc}, \mathfrak{b}, \operatorname{res}}^{s, r, l} : \hat{P}(\sigma)u \in H_{\operatorname{sc}, \mathfrak{b}, \operatorname{res}}^{s-2, r+1, l+1}\} \rightarrow H_{\operatorname{sc}, \mathfrak{b}, \operatorname{res}}^{s-2, r+1, l+1}$$

is invertible for  $0 < |\sigma| \leq \sigma_0$ ,  $\operatorname{Im} \sigma \geq 0$ , with this inverse being the  $\pm i0$  resolvent of  $P(\sigma)$  corresponding to  $\pm \operatorname{Re} \sigma > 0$ , and we have the estimate

$$\|(x + |\sigma|)^\alpha u\|_{H_{\operatorname{sc}, \mathfrak{b}, \operatorname{res}}^{s, r, l}} \leq C \|(x + |\sigma|)^{\alpha-1} \hat{P}(\sigma)u\|_{H_{\operatorname{sc}, \mathfrak{b}, \operatorname{res}}^{s-2, r+1, l+1}}$$

for

$$\alpha \in (l - \nu_+, l - \nu_-).$$

*Remark 2.6.* We note that Remark 1.2 remains valid in the present more general setting, including its proof; the only way the proof is affected is via inputting the estimate of Theorem 2.5 instead of Theorem 1.1.

### 3. RESOLVED B-ALGEBRA

In this section we introduce the space  $\Psi_{\mathfrak{b}, \operatorname{res}}^{m, l, \nu, \delta}(X)$  of resolved b-pseudodifferential operators and prove its basic properties. We explicitly consider  $\sigma \geq 0$  for notational simplicity; for  $\sigma \leq 0$  one simply replaces  $\sigma$  by  $|\sigma|$  at various points below, while for  $\sigma$  with  $\operatorname{Im} \sigma \geq 0$  the blow up discussed below is that of  ${}^{\mathfrak{b}}T_{\partial X}^* X \times \{0\}$  in  ${}^{\mathfrak{b}}T^* X \times \{\sigma \in \mathbb{C} : \operatorname{Im} \sigma \geq 0\}$ .

We recall that the b-pseudodifferential algebra is discussed in detail in Melrose's book [11]; the companion paper [18] as well as [17, Section 2] have a brief summary of its properties, while [16, Section 6] has a detailed presentation relating it to Hörmander's uniform pseudodifferential algebra [8, Chapter 18.1].

At the phase space level, the resolved algebra simply blows up  ${}^{\mathfrak{b}}T^* X \times [0, 1)_\sigma$  at the corner  ${}^{\mathfrak{b}}T_{\partial X}^* X \times \{0\}$ ; since the corner is given by  $x = 0$ ,  $\sigma = 0$ , projectively this amounts to the introduction of  $x/\sigma$  and  $\sigma/x$  as smooth variables, where bounded. In order to work uniformly at fiber infinity, it is best to consider the blow up

$${}^{\mathfrak{b}, \operatorname{res}}T(X, [0, 1)) = \overline{{}^{\mathfrak{b}}T^* X \times [0, 1)}; \overline{{}^{\mathfrak{b}}T_{\partial X}^* X \times \{0\}}$$

of  $\overline{{}^{\mathfrak{b}}T_{\partial X}^* X \times \{0\}}$  in  $\overline{{}^{\mathfrak{b}}T^* X \times [0, 1)}$ , see Figure 1. Here  $\overline{{}^{\mathfrak{b}}T^* X}$  is the fiber radially compactified b-cotangent bundle, i.e. the fibers are compactified, being vector spaces, to balls, thus to manifolds with boundary, see [10] for the compactification in the

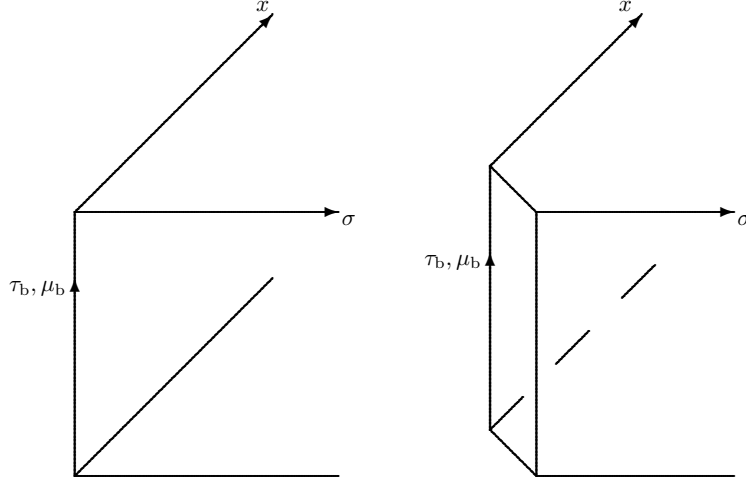


FIGURE 1. The resolved b-cotangent bundle, on the right, obtained by blowing up the corner  ${}^bT^*_{\partial X}X \times \{0\}$  of  ${}^bT^*X \times [0, 1]_{\sigma}$ , shown on the left.

scattering setting, and [17, Section 5] for a discussion connecting the scattering and b-settings in the context of second microlocalization at the zero section. Examples of defining functions of the lift of  $x = 0$ , the front face (i.e. the lift of  $x = 0, \sigma = 0$ ), resp. the lift of  $\sigma = 0$  are

$$(1 + \sigma/x)^{-1} = \frac{x}{x + \sigma}, \quad x + \sigma, \quad \text{resp.} \quad (1 + x/\sigma)^{-1} = \frac{\sigma}{x + \sigma};$$

a defining function of fiber infinity is  $(\tau_b^2 + |\mu_b|^2)^{-1/2}$ . Note that b-vector fields on the total space lift to b-vector fields on its resolution since a boundary face is being resolved; in particular  $x\partial_x, \partial_y, \tilde{\rho}^{-1}\partial_{\tau_b}, \tilde{\rho}^{-1}\partial_{\mu_b}, \tilde{\rho}$  a defining function of fiber infinity, thus  $\tilde{\rho}^{-1}$  equivalent to the larger of  $\tau_b, \mu_b$ , i.e. b-vector fields on the fibers over fixed  $\sigma$ , lift to such, as does  $\sigma\partial_{\sigma}$ . Thus, a conormal family of symbols on the resolved space is also a conormal family on the original, unresolved, space, and thus can be regarded as a family of b-symbols bounded by an appropriate power of  $\sigma$ , thus quantized, etc. We write the symbol orders as

$$S^{m, l, \nu, \delta}({}^b, \text{res}T^*(X, [0, 1])),$$

where  $m$  is the b-differential order,  $l$  is the order at (the lift of)  $x = 0$ ,  $\nu$  is the order at the front face (i.e. the lift of  $x = \sigma = 0$ ) and  $\delta$  is the order at (the lift of)  $\sigma = 0$ . Since  $\sigma$  is a parameter (is commutative), one can easily arrange that the last order is  $\delta = 0$ , but it can be useful to have some flexibility. A typical example of an elliptic symbol of order  $m, l, \nu, \delta$  is then

$$(\tau_b^2 + \mu_b^2)^{m/2}(1 + \sigma/x)^l(x + \sigma)^{-\nu}(1 + x/\sigma)^{\delta} = (\tau_b^2 + \mu_b^2)^{m/2}(x + \sigma)^{l-\nu+\delta}x^{-l}\sigma^{-\delta}.$$

Correspondingly, the relationship between  $S^{m, l, \nu, \delta}({}^b, \text{res}T^*(X, [0, 1]))$  and the symbol space  $S^{m, \alpha, \beta}({}^bT^*(X; [0, 1]))$  (symbols on  ${}^bT^*X \times [0, 1]$ , of order  $\alpha$  at  $x = 0$ ,  $\beta$  at  $\sigma = 0$ ) is

$$(3.1) \quad S^{m, l, \nu, \delta}({}^b, \text{res}T^*(X, [0, 1])) \subset S^{m, \alpha, \beta}({}^bT^*(X; [0, 1])), \quad \alpha \leq l, \quad \beta \leq \delta, \quad \alpha + \beta \leq \nu.$$

Explicitly, cf. [17, Section 2], the quantization map, giving  $A \in \Psi_{\text{b,res}}^{m,l,\nu,\delta}(X)$ , is

$$(3.2) \quad Au(x, y) = (2\pi)^{-n} \int e^{i(\frac{x-x'}{x}\tau_{\text{b}}+(y-y')\mu_{\text{b}})} \tilde{\psi}\left(\frac{x-x'}{x'}\right) a(x, y, \tau_{\text{b}}, \mu_{\text{b}}, \sigma) u(x', y') d\tau_{\text{b}} d\mu_{\text{b}} \frac{dx' dy'}{x'},$$

with  $\tilde{\psi}$  of compact support in  $(-1/2, 1/2)$ , identically 1 near 0, which may be regarded as a member of  $\Psi_{\text{b}}^{m,\alpha,\beta}(X)$  with  $\alpha, \beta$  as above, though this is imprecise unless  $\alpha = l$ ,  $\beta = \delta$  and  $\alpha + \beta = \nu$ , i.e.  $\nu = l + \delta$ .

It is useful to note here that

$$\frac{x' + \sigma}{x + \sigma} = \frac{x' - x}{x + \sigma} + 1,$$

and

$$\left| \frac{x' - x}{x + \sigma} \right| \leq \left| \frac{x' - x}{x} \right|,$$

with  $\frac{x'-x}{x+\sigma}$  having the same sign as  $\frac{x'-x}{x}$ , so over compact subsets of the *b-front face* (the lift of  $x = x' = 0$  to the b-double space, which is the space resulting from blowing up this submanifold, i.e.  $\partial X \times \partial X$ , in  $X \times X$ ), where  $\frac{x'-x}{x}$  is in a compact subset of  $(-1, \infty)$  (cf.  $\tilde{\psi}$  in (3.2)),  $\frac{x'-x}{x+\sigma}$  is in the same region. Thus, conjugating the localized in compact subsets of the front face (family) b-algebra by powers of  $x + \sigma$  is an isomorphism, hence one can indeed work with the standard family b-algebra with  $(x + \sigma)^{\tilde{\nu}}$ -weights. Notice that the full weight

$$(1 + \sigma/x)^l (x + \sigma)^{-\nu} (1 + x/\sigma)^\delta$$

can be rewritten in terms of a power of  $x + \sigma$ , times powers of  $x$  and  $\sigma$ , so as the latter two are well-behaved as far as the conjugation is concerned, so is the total weight.

Indeed,

$$x' < x \Rightarrow \frac{x' - x}{x} \leq \frac{x' - x}{x + \sigma} \leq 0$$

and

$$x' > x \Rightarrow 0 \leq \frac{x' - x}{x + \sigma} \leq \frac{x' - x}{x}$$

show that  $\frac{x'+\sigma}{x+\sigma}$  is controlled by  $\frac{x'}{x} = \frac{x'-x}{x} + 1$ , so even in the small (family) b-algebra, with infinite order vanishing on the side faces, i.e. as  $\frac{x'}{x}$ , resp.  $\frac{x}{x'}$ , tend to 0, so equivalently the reciprocals  $\frac{x}{x'}$ , resp.  $\frac{x'}{x}$ , tend to  $\infty$ , analogous statements hold.

It is useful to ‘complete’ the resolved b-algebra by order  $-\infty$  in the differential sense terms. For this recall that for the unresolved family, order  $-\infty$  operators, in  $\Psi_{\text{b}}^{-\infty,l}(X)$ , have Schwartz kernels which are conormal at the b-front face (of order  $l$ , interpreted as a right b-density) and vanish to infinite order at the left and right faces ( $x = 0$ , resp.  $x' = 0$ , lifted to the b-double space), with smooth (or conormal) behavior in  $\sigma$ . The new resolution is then that of the corner given by the b-front face, locally defined by  $x + x'$ , at  $\sigma = 0$ , which introduces coordinates  $\frac{x+x'}{\sigma}, \sigma$  where  $\sigma$  is relatively large, and  $\frac{\sigma}{x+x'}, x + x'$  where  $x + x'$  is such, together with  $\frac{x}{x'}$  or  $\frac{x'}{x}$ , as well as  $y, y'$ ; see Figure 2. The class of distributions giving the Schwartz kernel is conormal ones to all boundary hypersurfaces with infinite order vanishing

at the lifts of the left and right faces. Again, as a corner is being blown up, the property of being conormal does not change, though orders are affected, so the new class of operators is still a subclass of the family b-pseudodifferential operators. Note that this blow up indeed corresponds to the one at the symbol level: the quantization map for the family (3.2) gives a Schwartz kernel which is the inverse Fourier transform in the fiber variables of the b-cotangent bundle to the fibers of the b-double space over a fixed point  $(x, y)$  on the boundary, relative to the diagonal  $\frac{x-x'}{x} = 0, y - y' = 0$ , localized by  $\tilde{\psi}$ : the blow-up in the cotangent bundle, i.e. that of  $x = \sigma = 0$ , commutes with the inverse Fourier transform, considered as mapping to the local product space in  $x, y, \sigma, \frac{x-x'}{x}, y - y'$ . The conjugation invariance by powers of  $x + \sigma$  then follows from the previous paragraph.

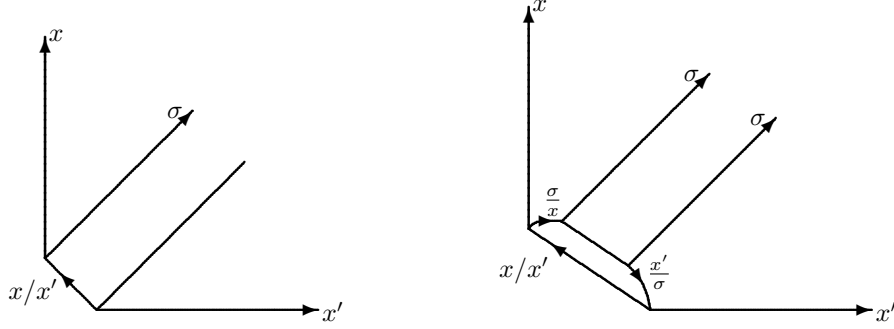


FIGURE 2. The resolved b-double space, on the right, obtained by blowing up the corner given by the b-front face at  $\sigma = 0$  of the b-double space times  $[0, 1]_\sigma$ , shown on the left.

The standard composition rules hold, including full asymptotic expansions. Thus, using symbols on the resolved space, one can define a resolved b-wave front set,  $\text{WF}'_{\text{b,res}}(A)$ , resp.  $\text{WF}_{\text{b,res}}(u)$ , of operator families, resp. distributions, with both being a subset of fiber infinity of the resolved space, which we denote by  ${}^{\text{b,res}}S^*(X, [0, 1])$ , with the main point being that points with various finite values of  $x/\sigma$  are now distinguished for  $x = 0, \sigma = 0$ . Finally,  $A \in \Psi_{\text{b,res}}^{m', l', l', 0}(X)$  acts on the standard b-Sobolev spaces since it lies in a continuous family of b-operators in  $\Psi_{\text{b}}^{m', l'}(X)$ , cf. (3.1) taken with  $\beta = 0$ , giving estimates

$$\|Au\|_{H_{\text{b}}^{\bar{r}-m', l-l'}} \leq C\|u\|_{H_{\text{b}}^{\bar{r}, l}},$$

with  $C$  independent of  $\sigma$ ; more generally  $A \in \Psi_{\text{b,res}}^{m', l', l'+k', k'}(X)$  gives estimates

$$\|Au\|_{H_{\text{b}}^{\bar{r}-m', l-l'}} \leq C|\sigma|^{-k'}\|u\|_{H_{\text{b}}^{\bar{r}, l}},$$

with uniform  $C$ .

With this,  $\hat{P}(\sigma)$  is elliptic in  $\Psi_{\text{b,res}}^{2, -1, -2, 0}(X)$  away from the lift of  $x = 0$  (i.e. away from  $x/\sigma = 0$ ), thus elliptic where  $x, S = \sigma/x$  (together with  $y, \tau_{\text{b}}, \mu_{\text{b}}$ ) are valid coordinates, at least for  $x$  small, since its principal symbol is  $x^2(\tau_{\text{b}}^2 + \mu_{\text{b}}^2)$ . Notice that this is a key advantage of working with the resolved space: on the front face, both  $\sigma x(xD_x)$  and  $x^2(xD_x)^2$  have the same decay order,  $-2$  (i.e. 2 orders of

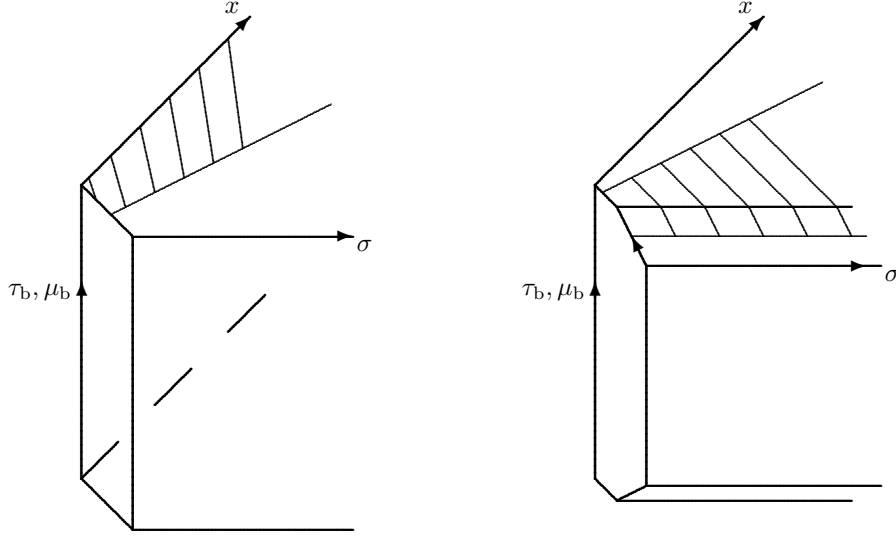


FIGURE 3. Microsupport of the operator  $B_1$  in (3.3) on the resolved b-cotangent bundle on the left, resp. the operator  $B_1$  in (3.10) on the scattering-b resolved cotangent bundle on the right. Both are shown as shaded regions.

decay), while in the decay sense the former dominates at  $x/\sigma = 0$  (order  $-1$ ) and the latter at  $\sigma/x = 0$  (order  $0$ ), though of course only the latter matters in the standard principal symbol sense (order  $2$ ). In particular, elliptic estimates hold in this region:

$$(3.3) \quad \begin{aligned} & \|(1+x/\sigma)^\delta B_1 u\|_{H_b^{\bar{r},l}} \\ & \leq C(\|(1+x/\sigma)^\delta (x+\sigma)^{-2} B_3 \hat{P}(\sigma) u\|_{H_b^{\bar{r}-2,l}} + \|(1+x/\sigma)^\delta u\|_{H_b^{-N,l}}), \end{aligned}$$

with  $B_1, B_3 \in \Psi_{b,\text{res}}^{0,0,0,0}(X)$  with wave front set away from the lift of  $x = 0$  (i.e. away from  $x/\sigma = 0$ ),  $B_3$  elliptic on a neighborhood of  $\text{WF}'_{b,\text{res}}(B_1)$ , and where by a careful arrangement of support properties of  $B_3$ , one could also replace  $(x+\sigma)^{-2}$  by  $x^{-2}$ . See Figure 3.

We in fact have

$$\hat{P}(\sigma) \in \Psi_{b,\text{res}}^{2,-2,-2,0}(X) + \Psi_{b,\text{res}}^{1,-1,-2,0}(X),$$

rather than merely in  $\Psi_{b,\text{res}}^{2,-1,-2,0}(X)$ , which means that there is a degeneracy at the lift of  $x = 0$ , i.e. at  $X = 0$ , where  $X = x/\sigma$ . This is, however, fixed by second microlocal considerations, which take the form of a resolution of fiber infinity at  $X = x/\sigma = 0$ , see Figure 4. This introduces a (rescaled, by  $\sigma$ ) scattering momentum variable,

$$\tau_{\text{res}} = \tau_b X = \tau/\sigma, \quad \mu_{\text{res}} = \mu_b X = \mu/\sigma$$

in the interior of the new front face (via  $|(\tau_b, \mu_b)|^{-1}$ ,  $X$ ,  $\hat{\tau}_b = \tau_b/|(\tau_b, \mu_b)|$ ,  $\hat{\mu}_b = \mu_b/|(\tau_b, \mu_b)$  being smooth nearby prior to the blow-up, giving  $|(\tau_b, \mu_b)|^{-1}/X$ ,  $X$ ,  $\hat{\tau}_b, \hat{\mu}_b$  smooth after the blow-up, with the first quotient away from  $0, \infty$  in the



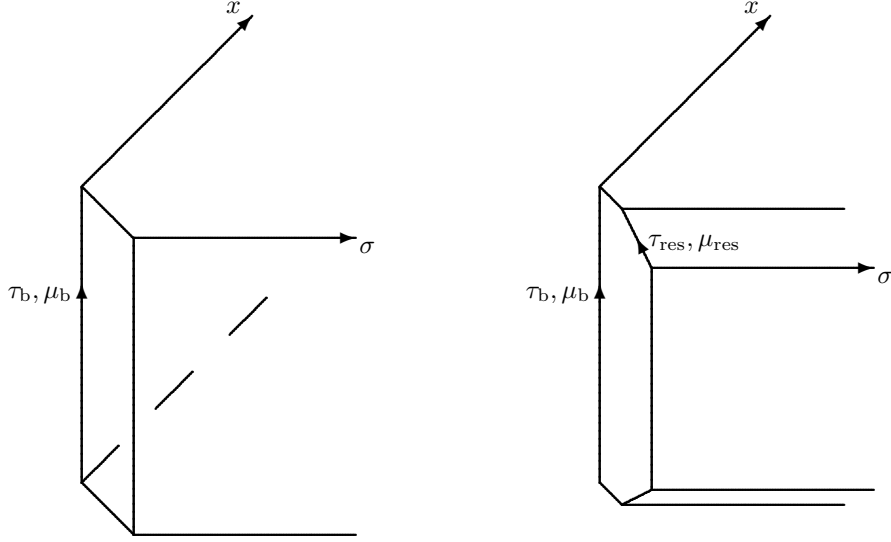


FIGURE 4. The resolved b-cotangent bundle on the left, and its scattering-b resolution on the right obtained by blowing up the corner  $x/\sigma = 0$  at fiber infinity (nearest horizontal edges) of the resolved b-cotangent bundle. At the pseudodifferential operator level the symbolic calculus works at resolved b-fiber infinity which is the top (as well as bottom!) face on both pictures, as well as new face on the right picture, which corresponds to rescaled sc-decay.

interior). Since this is a blow-up of a corner, the conormal spaces are unchanged, but now one can allow different orders at fiber infinity (which is now the sc-differential order) and at the new sc-front face, which is the sc-decay order. One obtains  $\Psi_{sc,b,res}^{s,r,l,\nu,\delta}(X)$  this way, with

$$(\tau_{res}^2 + \mu_{res}^2 + 1)^{s/2} ((\tau_b^2 + \mu_b^2)^{-1} + x^2/\sigma^2)^{-r/2} (1 + \sigma/x)^l (x + \sigma)^{-\nu} (1 + x/\sigma)^\delta$$

being a typical elliptic symbol. Note that  $\tau_{res}, (\mu_{res})_j$  are the principal symbols of  $\sigma^{-1}x^2D_x, \sigma^{-1}xD_{y_j} \in \Psi_{sc,b,res}^{1,0,0,0,1}(X)$  (indeed in  $\Psi_{sc,b,res}^{1,0,-1,0,1}(X)$ ) which are singular as (non-resolved!) scattering vector fields at  $\sigma = 0$ .

This pseudodifferential space also gives rise to scattering-b-resolved Sobolev space  $H_{sc,b,res}^{s,r,l}(X)$  family, namely this is a family of Sobolev spaces on  $X$  depending on  $\sigma$ , which are the same as a topological vector space for  $\sigma \neq 0$ , but with a  $\sigma$ -dependent norm. Before defining it in general, in the special case when  $r = s+l$ , this is simply the b-Sobolev space family (i.e. with elements depending on  $\sigma$ )  $H_b^{s,l}(X)$ , with norm defined independently of  $\sigma$ . In general, it is the scattering-b-Sobolev space  $H_{sc,b}^{s,r,l}(X)$ , but with a  $\sigma$ -dependent norm:

$$(3.4) \quad \|u\|_{H_{sc,b,res}^{s,r,l}}^2 = \|Au\|_{L^2}^2 + \|u\|_{H_b^{-N,l}}^2,$$

where  $A \in \Psi_{\text{sc,b,res}}^{s,r,l,l,0}(X)$  is elliptic in the first two senses (sc-differentiability and sc-decay) and where  $N$  is sufficiently large so that  $s \geq -N$ ,  $r \geq -N + l$ . In particular,

$$(3.5) \quad H_{\text{sc,b,res}}^{s,s+l,l}(X) = H_{\text{b}}^{s,l}(X),$$

as one can use an appropriate elliptic element  $A$  of  $\Psi_{\text{b}}^{s,l}(X)$  in the definition in this case. With this definition, we have for

$$A \in \Psi_{\text{sc,b,res}}^{s',r',l',l'+k',k'}(X) = |\sigma|^{-k'} \Psi_{\text{sc,b,res}}^{s',r',l',l',0}(X)$$

that

$$(3.6) \quad \|Au\|_{H_{\text{sc,b,res}}^{s-s',r-r',l-l'}} \leq C|\sigma|^{-k'} \|u\|_{H_{\text{sc,b,res}}^{s,r,l}}.$$

To give more feel for these spaces, in particular for the meaning of the differential order, we also remark that if  $V \in \mathcal{V}_{\text{b}}(X)$ , then  $\frac{x}{x+\sigma}V \in \Psi_{\text{sc,b,res}}^{1,0,-1,0,0}(X) \subset \Psi_{\text{sc,b,res}}^{1,0,0,0,0}(X)$ , and moreover at the resolved scattering fiber-infinity, corresponding to the first order, at each point there is such a  $V$  that is elliptic. Thus, for instance  $\|u\|_{H_{\text{sc,b,res}}^{s,r,l}}$  is an equivalent norm to

$$(3.7) \quad \sum_j \left\| \frac{x}{x+\sigma} V_j u \right\|_{H_{\text{sc,b,res}}^{s-1,r,l}} + \|u\|_{H_{\text{sc,b,res}}^{s-1,r,l}},$$

where the  $V_j$  span  $\mathcal{V}_{\text{b}}(X)$  (so at each point one of them is elliptic); in local coordinates one can take these as  $x D_x, D_{y_k}$ , so roughly speaking, the differential regularity is in terms of  $\frac{x}{x+\sigma} x D_x, \frac{x}{x+\sigma} D_{y_k}$ .

In part in order to become more familiar with these spaces, we make some further remarks, in particular showing how the main Theorem 1.1 proves Remark 1.2 and its strengthened version. Thus, if  $A \in \Psi_{\text{sc,b,res}}^{s-1,r+1,l+1,l+1,0}(X)$ , then  $A(x+\sigma)^{-1} \in \Psi_{\text{sc,b,res}}^{s-1,r+1,l+1,l+2,0}(X) \subset \Psi_{\text{sc,b,res}}^{s-1,r+1,l+2,l+2,0}(X)$ , hence for such appropriate elliptic  $A$  we have, using (3.6),

$$\|(x+\sigma)^{-1}u\|_{H_{\text{sc,b,res}}^{s-1,r+1,l+1}}^2 = \|A(x+\sigma)^{-1}u\|_{L^2}^2 + \|u\|_{H_{\text{b}}^{s-N,l+1}}^2 \leq C\|u\|_{H_{\text{sc,b,res}}^{s-1,r+1,l+2}}^2.$$

Therefore, Theorem 1.1 gives, with the first inequality being that of the theorem, for  $s = r - l$ ,

$$(3.8) \quad \begin{aligned} \|(x+\sigma)^\alpha u\|_{H_{\text{b}}^{s,l}} &= \|(x+\sigma)^\alpha u\|_{H_{\text{sc,b,res}}^{s,r,l}} \\ &\leq C\|(x+\sigma)^{\alpha-1} \hat{P}(\sigma)u\|_{H_{\text{sc,b,res}}^{s-2,r+1,l+1}} \leq C\|(x+\sigma)^{\alpha-1} \hat{P}(\sigma)u\|_{H_{\text{sc,b,res}}^{s-1,r+1,l+1}} \\ &\leq C'\|(x+\sigma)^\alpha \hat{P}(\sigma)u\|_{H_{\text{sc,b,res}}^{s-1,r+1,l+2}} = C'\|(x+\sigma)^\alpha \hat{P}(\sigma)u\|_{H_{\text{b}}^{s-1,l+2}}, \end{aligned}$$

proving Remark 1.2.

This can be strengthened (made less lossy relative to the main theorem) by combining the argument with (3.7). Thus, with  $V_j$  as there, taking  $s = r - l + 1$ ,

we have

$$\begin{aligned}
& \sum_j \left\| \frac{x}{x+\sigma} V_j(x+\sigma)^\alpha u \right\|_{H_b^{s-1,l}} + \|(x+\sigma)^\alpha u\|_{H_b^{s-1,l}} \\
&= \sum_j \left\| \frac{x}{x+\sigma} V_j(x+\sigma)^\alpha u \right\|_{H_{\text{sc,b, res}}^{s-1,r,l}} + \|(x+\sigma)^\alpha u\|_{H_{\text{sc,b, res}}^{s-1,r,l}} \\
&\leq C'' \|(x+\sigma)^\alpha u\|_{H_{\text{sc,b, res}}^{s,r,l}} \\
&\leq C \|(x+\sigma)^{\alpha-1} \hat{P}(\sigma) u\|_{H_{\text{sc,b, res}}^{s-2,r+1,l+1}} \\
&\leq C' \|(x+\sigma)^\alpha \hat{P}(\sigma) u\|_{H_{\text{sc,b, res}}^{s-2,r+1,l+2}} \\
&= C' \|(x+\sigma)^\alpha \hat{P}(\sigma) u\|_{H_b^{s-2,l+2}},
\end{aligned}$$

and in the first term  $(x+\sigma)^\alpha$  can be commuted to the front (up to changing constants) if one wishes. In particular, on the left hand side, one can estimate  $\|x V_j(x+\sigma)^\alpha u\|_{H_{\text{sc,b}}^{s-1,r,l}}$ , which means  $\|(x+\sigma)^\alpha u\|_{H_{\text{sc,b}}^{s,r,l}} = \|(x+\sigma)^\alpha u\|_{H_{\text{sc,b}}^{s,s-1+l,l}}$ , with  $H_{\text{sc,b}}$  the standard second microlocal space (with  $\sigma$ -independent norm) as in [17, Section 5]; this gains an extra sc-derivative relative to Remark 1.2; it gives

$$(3.9) \quad \|(x+\sigma)^\alpha u\|_{H_{\text{sc,b}}^{s,s+l-1,l}} \leq C \|(x+\sigma)^\alpha \hat{P}(\sigma) u\|_{H_{\text{sc,b}}^{s-2,s+l,l+2}}.$$

We now turn to our operator  $\hat{P}(\sigma)$  and how it fits within our resolved algebra. In this scattering-b-resolved algebra we have

$$\hat{P}(\sigma) \in \Psi_{\text{sc,b, res}}^{2,0,-1,-2,0}(X)$$

with principal symbol in the first, sc-differential, sense at  $X=0$  being

$$x^2(\tau_b^2 + \mu_b^2) = \sigma^2(\tau_{\text{res}}^2 + \mu_{\text{res}}^2),$$

and the dual metric function in general (including away from  $X=0$ ). This is elliptic at the lift of fiber infinity at, thus near,  $X=0$ , where  $\sigma=0$  defines the base-front-face (the penultimate order),  $(\tau_{\text{res}}^2 + \mu_{\text{res}}^2)^{-1/2}$  the scattering fiber infinity (the first order), while the third (b-decay) and last ( $\sigma/x=0$  behavior) orders are irrelevant. Now, even in the sc-decay sense, we have ellipticity near sc-fiber-infinity, for in that sense the principal symbol is

$$x^2(\tau_b^2 + \mu_b^2) - 2\sigma x \tau_b = \sigma^2(\tau_{\text{res}}^2 + \mu_{\text{res}}^2 - 2\tau_{\text{res}}) = \sigma^2((\tau_{\text{res}} - 1)^2 + \mu_{\text{res}}^2 - 1),$$

which is elliptic for sufficiently large  $(\tau_{\text{res}}, \mu_{\text{res}})$ . Thus, we have microlocal elliptic estimates

$$(3.10) \quad \|B_1 u\|_{H_{\text{sc,b, res}}^{s,r,l}} \leq C(|\sigma|^{-2} \|B_3 \hat{P}(\sigma) u\|_{H_{\text{sc,b, res}}^{s-2,r,l}} + \|u\|_{H_{\text{sc,b, res}}^{-N,-N,-N}}),$$

for  $B_1$  microlocalizing in a neighborhood of sc-fiber infinity, made trivial near the lift of  $X=0$  (the b-front face), with  $B_3$  similar, but elliptic on the wave front set of  $B_1$ ; see Figure 3.

In combination these two elliptic estimates (3.3)-(3.10) give

$$(3.11) \quad \begin{aligned} \|(1+x/\sigma)^\delta B_1 u\|_{H_{\text{sc,b, res}}^{s,r,l}} &\leq C(\|(1+x/\sigma)^\delta (x+\sigma)^{-2} B_3 \hat{P}(\sigma) u\|_{H_{\text{sc,b, res}}^{s-2,r,l}} \\ &\quad + \|(1+x/\sigma)^\delta u\|_{H_b^{-N,l}}). \end{aligned}$$

## 4. SYMBOLIC ESTIMATES

We now turn to symbolic estimates at  $X = x/\sigma = 0$ . This is a simple extension of the argument for the limiting absorption principle as presented in [18].

Since from the standard conjugated scattering picture [18] we already know that the zero section has radial points, the only operator that can give positivity microlocally in a symbolic commutator computation is the weight. Recall that the actual positive commutator estimates utilize the computation of

$$(4.1) \quad i(\hat{P}(\sigma)^* A - A\hat{P}(\sigma)) = i(\hat{P}(\sigma)^* - \hat{P}(\sigma))A + i[\hat{P}(\sigma), A]$$

with  $A = A^*$ , so for non-formally-self-adjoint  $\hat{P}(\sigma)$  there is a contribution from the skew-adjoint part

$$\text{Im } \hat{P}(\sigma) = \frac{1}{2i}(P(\sigma) - P(\sigma)^*)$$

of  $\hat{P}(\sigma)$ , relevant for us when  $\sigma$  is not real or when  $\sigma$  is real but  $\beta, \gamma \neq 0$ ; here the notation ‘ $\text{Im } \hat{P}(\sigma)$ ’ is motivated by the fact that its principal symbol is actually  $\text{Im } \hat{p}(\sigma)$ , with  $\hat{p}(\sigma)$  being the principal symbol of  $\hat{P}(\sigma)$ . It is actually a bit better to rewrite this, with

$$\text{Re } \hat{P}(\sigma) = \frac{1}{2}(P(\sigma) + P(\sigma)^*)$$

denoting the self-adjoint part of  $\hat{P}(\sigma)$ , as

$$(4.2) \quad i(\hat{P}(\sigma)^* A - A\hat{P}(\sigma)) = (\text{Im } \hat{P}(\sigma)A + A \text{Im } \hat{P}(\sigma)) + i[\text{Re } \hat{P}(\sigma), A].$$

If  $A \in \Psi_{\text{b,res}}^{2\bar{r}-1, 2l+1, 2\nu+2, -\infty}$ ,  $\hat{P}(\sigma) \in \Psi_{\text{b,res}}^{2, -1, -2, 0}(X)$  implies that the second term (the commutator) is a priori in  $\Psi_{\text{b,res}}^{2\bar{r}, 2l, 2\nu, -\infty}$ . Here we are setting the last order to  $-\infty$  since we are working near  $x/\sigma = 0$  (for away from there we already have elliptic estimates!), so it plays no role; this also means that one can simply use  $\sigma^{-2\nu+2l-1}$ , resp.  $\sigma^{-2\nu+2l}$ , as the weight capturing the front face behavior for  $A$ , resp. the second term, if  $x^{-2l-1}$ , resp.  $x^{-2l}$  is used as the spatial weight. Via the usual quadratic form argument this thus estimates  $u$  in  $|\sigma|^{\nu-l} H_{\text{b}}^{\bar{r}, l}$  in terms of  $|\sigma|^{\nu-l-1} \hat{P}(\sigma)u$  in  $H_{\text{b}}^{\bar{r}-1, l+1}$ , assuming non-degeneracy.

However, we in fact have

$$\hat{P}(\sigma) \in \Psi_{\text{b,res}}^{2, -2, -2, 0}(X) + \Psi_{\text{b,res}}^{1, -1, -2, 0}(X),$$

which means that the second term of (4.2) (the commutator) will be in

$$\Psi_{\text{b,res}}^{2\bar{r}, 2l-1, 2\nu, -\infty} + \Psi_{\text{b,res}}^{2\bar{r}-1, 2l, 2\nu, -\infty},$$

hence will degenerate as an element of  $\Psi_{\text{b,res}}^{2\bar{r}, 2l, 2\nu, -\infty}$ . This is fixed by second microlocal considerations, namely considering

$$(4.3) \quad A \in \Psi_{\text{b,res}}^{2\bar{r}-1, 2l+1, 2\nu+2, -\infty}(X) = \Psi_{\text{sc,b,res}}^{2\bar{r}-1, 2(\bar{r}+l), 2l+1, 2\nu+2, -\infty}(X),$$

and using that

$$\hat{P}(\sigma) \in \Psi_{\text{sc,b,res}}^{2, 0, -1, -2, 0}(X)$$

so that the commutator lies in

$$[\text{Re } \hat{P}(\sigma), A] \in \Psi_{\text{sc,b,res}}^{2\bar{r}, 2(\bar{r}+l)-1, 2l, 2\nu, -\infty}(X),$$

for the algebra is commutative to leading order in the first two orders (namely sc-differentiability and sc-decay). In fact, we modify this somewhat by taking an appropriate  $S \in \Psi_{\text{b,res}}^{-1,0,0,0}(X) = \Psi_{\text{sc,b,res}}^{-1,-1,0,0}(X)$  and considering

$$(4.4) \quad i[\text{Re } \hat{P}(\sigma), A] + AS\hat{P}(\sigma) + \hat{P}(\sigma)^*SA \in \Psi_{\text{sc,b,res}}^{2\tilde{r},2(\tilde{r}+l)-1,2l,2\nu,-\infty}(X),$$

with the last two terms having principal symbol  $2 \text{Re } \hat{p}(\sigma) \hat{s}a$ , if  $a$  is the principal symbol of  $A$ , and  $\hat{s}$  of  $S$ , where  $S$  will be chosen in a manner that cancels an indefinite term near the scattering zero section. We remark that from the second microlocal perspective, the rescaled sc-differential order is irrelevant in view of the elliptic estimate (3.11), but a byproduct of the particular choice of  $s$  is that the principal symbol of the commutator  $[\text{Re } \hat{P}(\sigma), A]$  in the sc-differential sense is also cancelled at  $x = 0$ .

On the other hand, in general in the first term

$$\text{Im } \hat{P}(\sigma) \in \Psi_{\text{b,res}}^{1,-1,-2,0}(X) = \Psi_{\text{sc,b,res}}^{1,0,-1,-2,0}(X),$$

so the first term of (4.2) is in  $\Psi_{\text{b,res}}^{2\tilde{r},2l,2\nu,-\infty}(X)$ , so is the same order,  $2l$ , in the b-decay sense, as well as in the resolved front face sense (order  $2\nu$ ), as the modified commutator, but is actually bigger, order  $2(\tilde{r} + l)$ , in scattering decay sense. However, when  $\sigma$  is real, then

$$\text{Im } \hat{P}(\sigma) \in \Psi_{\text{sc,b,res}}^{1,-1,-1,-2,0}(X),$$

so with  $A$  as in (4.3)

$$\text{Im } \hat{P}(\sigma)A + A \text{Im } \hat{P}(\sigma) \in \Psi_{\text{sc,b,res}}^{2\tilde{r},2(\tilde{r}+l)-1,2l,2\nu,-\infty}(X),$$

which has the same orders as  $[\text{Re } \hat{P}(\sigma), A]$  and (4.4); we make some further adjustments to  $S$  to obtain a definite sign.

Now, going back to the issue of the zero section consisting of radial points, we compute the principal symbol of the second term of (4.1) (which is the only term when  $\sigma$  is real and  $P(\sigma) = P(\sigma)^*$ ) when

$$A \in \Psi_{\text{b,res}}^{2\tilde{r}-1,2l+1,-2\tilde{\nu}+2l+1,-\infty}(X)$$

is the weight (as mentioned above, only this can give positivity) times a cutoff in  $x/\sigma$ , i.e.

$$x^{-2l-1}(x+\sigma)^{2\tilde{\nu}}(\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{\tilde{r}-1/2}\phi(x/\sigma), \quad \nu = -\tilde{\nu} + l - 1/2,$$

with  $\phi \geq 0$  supported near 0, identically 1 in a smaller neighborhood of 0. Computationally it is better to take the equivalent, in view of the support of  $\phi$ ,

$$a = x^{-2l-1}\sigma^{2\tilde{\nu}}(\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{\tilde{r}-1/2}\phi(x/\sigma),$$

since  $\sigma$  commutes with all operators — the weight  $(x+\sigma)^{2\tilde{\nu}}$  has an equivalent effect as long as  $\text{supp } \phi$  is taken sufficiently small.

**Lemma 4.1.** *The principal symbol of*

$$(\text{Im } \hat{P}(\sigma)A + A \text{Im } \hat{P}(\sigma)) + i[\text{Re } \hat{P}(\sigma), A] \in \Psi_{\text{sc,b,res}}^{2\tilde{r},2(\tilde{r}+l)-1,2l,-2\tilde{\nu}+2l-1,-\infty}(X)$$

for real  $\sigma$ , suppressing the  $\phi(x/\sigma)$  factor as well as terms involving its derivative, is

$$(4.5) \quad \begin{aligned} x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2}\sigma^{2\tilde{\nu}} & \left( 4\sigma \left( \left( l + \tilde{r} - \frac{\operatorname{Im}(\beta - \gamma)}{2} \right) \tau_b^2 \right. \right. \\ & \left. \left. + \left( l + 1/2 - \frac{\operatorname{Im}(\beta - \gamma)}{2} \right) \mu_b^2 \right) \right. \\ & \left. - 4x \left( l + \tilde{r} - \frac{\operatorname{Im}\beta}{2} \right) \tau_b (\tau_b^2 + \mu_b^2) \right). \end{aligned}$$

*Remark 4.2.* The cutoff factor  $\phi(x/\sigma)$  contributes an additional term to the commutator, but as it is supported in the elliptic region, this is estimated by the elliptic estimate, so henceforth can be ignored.

*Proof.* Since the principal symbol of  $\operatorname{Re} \hat{P}(\sigma)$  in the joint sc-differential-sc-decay sense is

$$\operatorname{Re} \hat{p}(\sigma) = x^2(\tau_b^2 + \mu_b^2) - 2x \operatorname{Re} \sigma \tau_b = x^2(\tau_b^2 + \mu_b^2) - 2x\sigma\tau_b,$$

we compute

$$(4.6) \quad \begin{aligned} & \{x^2(\tau_b^2 + \mu_b^2) - 2x\sigma\tau_b, x^{-2l-1}(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2}\} \\ & = (2x^2\tau_b - 2x\sigma)(-2l-1)x^{-2l-1}(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \\ & \quad - (2x^2(\tau_b^2 + \mu_b^2) - 2x\sigma\tau_b)x^{-2l-1}2(\tilde{r}-1/2)\tau_b(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2}. \end{aligned}$$

Expanding and rearranging,

$$(4.7) \quad \begin{aligned} & = 4\sigma x^{-2l}(l+1/2)(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \\ & \quad + 4\sigma(\tilde{r}-1/2)x^{-2l}\tau_b^2(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} \\ & \quad - 4x^{-2l+1}(l+1/2)\tau_b(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \\ & \quad - 4(\tilde{r}-1/2)x^{-2l+1}\tau_b(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \\ & = x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} \left( 4\sigma \left( (l+1/2)(\tau_b^2 + \mu_b^2) + (\tilde{r}-1/2)\tau_b^2 \right) \right. \\ & \quad \left. - 4x(l+\tilde{r})\tau_b(\tau_b^2 + \mu_b^2) \right) \\ & = x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} \left( 4\sigma \left( (l+\tilde{r})\tau_b^2 + (l+1/2)\mu_b^2 \right) \right. \\ & \quad \left. - 4x(l+\tilde{r})\tau_b(\tau_b^2 + \mu_b^2) \right). \end{aligned}$$

On the other hand, we have an additional term  $\operatorname{Im} \hat{P}(\sigma)A + A \operatorname{Im} \hat{P}(\sigma)$ ; by Proposition 2.1 its principal symbol for real  $\sigma$  is

$$\begin{aligned} & (2x\tau_b \operatorname{Im} \beta - 2\sigma \operatorname{Im}(\beta - \gamma))xa \\ & = 2x^{-2l+1}\tau_b(\operatorname{Im} \beta)\sigma^{2\tilde{\nu}}(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2}\phi(x/\sigma) \\ & \quad - 2 \operatorname{Im}(\beta - \gamma)x^{-2l}\sigma^{2\tilde{\nu}+1}(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2}\phi(x/\sigma). \end{aligned}$$

Thus, the total expression, suppressing the  $\phi(x/\sigma)$  factor as well as terms with its derivatives,

$$\begin{aligned} x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2}\sigma^{2\tilde{\nu}} & \left( 4(\operatorname{Re} \sigma) \left( \left( l + \tilde{r} - \frac{\operatorname{Im}(\beta - \gamma)}{2} \right) \tau_b^2 \right. \right. \\ & \left. \left. + \left( l + 1/2 - \frac{\operatorname{Im}(\beta - \gamma)}{2} \right) \mu_b^2 \right) \right. \\ & \left. - 4x \left( l + \tilde{r} - \frac{\operatorname{Im} \beta}{2} \right) \tau_b (\tau_b^2 + \mu_b^2) \right), \end{aligned}$$

proving the lemma.  $\square$

*Remark 4.3.* In an analogue of Remark 4.2 of [18], we record the impact of having an additional regularizer factor, namely replacing  $a$  by

$$a^{(\epsilon)} = a f_\epsilon,$$

as is standard in positive commutator estimates, including at radial points, see the references in [18]. The slightly delicate issue at radial points is the limitation of regularizability, which was not a problem in [18] since we work in a neighborhood of the radial set at the zero section there, and the second microlocal setup means that the only potential issue (from the perspective of regularization actually needed and limited from the purely scattering, as opposed to the second microlocal, setting) amounts to getting additional b-decay, which is irrelevant for the symbolic considerations. However, in the present setting the two radial sets are simultaneously considered, naturally in view of the  $\sigma \rightarrow 0$  limit (since the radial sets ‘collide’ in the limit), and thus the limitations of regularizability are relevant.

As in [18, Remark 4.2], we can take the regularizer of the form

$$f_\epsilon(\tau_b^2 + \mu_b^2), \quad f_\epsilon(s) = (1 + \epsilon s)^{-K/2},$$

where  $K > 0$  fixed and  $\epsilon \in [0, 1]$ , with the interesting behavior being the  $\epsilon \rightarrow 0$  limit. Note that  $f_\epsilon(\tau_b^2 + \mu_b^2)$  is a symbol of order  $-K$  for  $\epsilon > 0$ , but is only uniformly bounded in symbols of order 0, converging to 1 in symbols of positive order. Then

$$s f'_\epsilon(s) = -\frac{K}{2} \frac{\epsilon s}{1 + \epsilon s} f_\epsilon(s),$$

and  $0 \leq \frac{\epsilon s}{1 + \epsilon s} \leq 1$ , so in particular  $s f'_\epsilon(s)/f_\epsilon(s)$  is bounded. *Just as in* [18, Remark 4.2], the effect of this is to add an overall factor of  $f_\epsilon(\tau_b^2 + \mu_b^2)$  to (4.5) and (4.6) as well as the subsequent expressions in the above proof, and replace every occurrence of  $\tilde{r}$ , other than those in the exponent, by

$$(4.8) \quad \tilde{r} + (\tau_b^2 + \mu_b^2) \frac{f'_\epsilon(\tau_b^2 + \mu_b^2)}{f_\epsilon(\tau_b^2 + \mu_b^2)} = \tilde{r} - \frac{K}{2} \frac{\epsilon(\tau_b^2 + \mu_b^2)}{1 + \epsilon(\tau_b^2 + \mu_b^2)}.$$

It is useful to rewrite the last term in the outermost parentheses of (4.5) since it comes with an indefinite sign due to the factor of  $\tau_b$ .

**Lemma 4.4.** *Let*

$$\hat{s} = 2 \left( l + \tilde{r} - \frac{\operatorname{Im} \beta}{2} \right) (\tau_b^2 + \mu_b^2)^{-1} \tau_b,$$

and let  $S \in \Psi_{\text{sc,b,res}}^{-1,-1,0,0,0}(X) = \Psi_{\text{b,res}}^{-1,0,0,0}(X)$  with principal symbol  $\hat{s}$ . Then for real  $\sigma$  the principal symbol of

$$\begin{aligned} & i(\hat{P}(\sigma)^* A - A\hat{P}(\sigma)) + AS\hat{P}(\sigma) + \hat{P}(\sigma)^* SA \\ &= (\text{Im } \hat{P}(\sigma)A + A \text{Im } \hat{P}(\sigma)) + i[\text{Re } \hat{P}(\sigma), A] + AS\hat{P}(\sigma) + \hat{P}(\sigma)^* SA \\ &\in \Psi_{\text{sc,b,res}}^{2\tilde{r}, 2(\tilde{r}+l)-1, 2l, -2\tilde{\nu}+2l-1, -\infty}(X) \end{aligned}$$

at  $x = 0$  is, suppressing the factor  $\phi(x/\sigma)$  as well as terms with its derivatives,

$$(4.9) \quad \begin{aligned} & \sigma^{2\tilde{\nu}+1} x^{-2l} (\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{\tilde{r}-3/2} 4 \left( - \left( l + \tilde{r} - \frac{\text{Im}(\beta + \gamma)}{2} \right) \tau_{\text{b}}^2 \right. \\ & \quad \left. + \left( l + 1/2 - \frac{\text{Im}(\beta - \gamma)}{2} \right) \mu_{\text{b}}^2 \right). \end{aligned}$$

*Remark 4.5.* See [18, Remark 4.6] for a discussion of this choice of  $\hat{s}$ , including both the advantages and the disadvantages, in the context of the more general operators considered in [18].

Moreover, the analogue of the conclusion remains valid with a regularizer as in Remark 4.3, i.e.  $a$  replaced by  $a^{(\epsilon)}$ , and correspondingly  $A$  by  $A^{(\epsilon)}$ , provided in the definition of  $\hat{s}$  as well as in the conclusion,  $\tilde{r}$  is replaced by (4.8) (except in the exponent), and in the conclusion an overall factor of  $f_{\epsilon}(\tau_{\text{b}}^2 + \mu_{\text{b}}^2)$  is added.

*Proof.* We add to  $\sigma^{2\tilde{\nu}}$  times (4.6) the product of

$$\begin{aligned} 2\hat{s}a &= 4 \left( l + \tilde{r} - \frac{\text{Im} \beta}{2} \right) (\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{-1} \tau_{\text{b}} a \\ &= 4\sigma^{2\tilde{\nu}} x^{-2l-1} (\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{\tilde{r}-3/2} \left( l + \tilde{r} - \frac{\text{Im} \beta}{2} \right) \tau_{\text{b}} \phi(x/\sigma) \end{aligned}$$

and the principal symbol of  $\text{Re } \hat{P}(\sigma)$ , namely

$$x^2(\tau_{\text{b}}^2 + \mu_{\text{b}}^2) - 2\sigma x \tau_{\text{b}}.$$

We obtain, dropping the factor  $\phi$ ,

$$\begin{aligned} & \sigma^{2\tilde{\nu}} x^{-2l} (\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{\tilde{r}-3/2} \left( 4\sigma \left( \left( l + \tilde{r} - \frac{\text{Im}(\beta - \gamma)}{2} \right) \tau_{\text{b}}^2 \right. \right. \\ & \quad \left. \left. + \left( l + 1/2 - \frac{\text{Im}(\beta - \gamma)}{2} \right) \mu_{\text{b}}^2 \right) \right. \\ & \quad \left. - 8 \left( l + \tilde{r} - \frac{\text{Im} \beta}{2} \right) \sigma \tau_{\text{b}}^2 \right) \\ &= \sigma^{2\tilde{\nu}+1} x^{-2l} (\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{\tilde{r}-3/2} 4 \left( - \left( l + \tilde{r} - \frac{\text{Im}(\beta + \gamma)}{2} \right) \tau_{\text{b}}^2 \right. \\ & \quad \left. + \left( l + 1/2 - \frac{\text{Im}(\beta - \gamma)}{2} \right) \mu_{\text{b}}^2 \right), \end{aligned}$$

proving the lemma.  $\square$

For  $l, \tilde{r}$  with  $l + \tilde{r} - \frac{\text{Im}(\beta + \gamma)}{2} > 0$ ,  $l + 1/2 - \frac{\text{Im}(\beta - \gamma)}{2} < 0$ , or with both terms having the opposite sign, we thus obtain a positive commutator estimate.

**Proposition 4.6.** *We have*

$$\begin{aligned} & \| (1 + x/\sigma)^{\alpha} u \|_{H_{\text{sc,b,res}}^{s, \tilde{r}+l-1/2, l}} \\ & \leq C (\| (1 + x/\sigma)^{\alpha} (x + \sigma)^{-1} \hat{P}(\sigma) u \|_{H_{\text{sc,b,res}}^{s-2, \tilde{r}+l+1/2, l+1}} + \| (1 + x/\sigma)^{\alpha} u \|_{H_{\text{sc,b,res}}^{-N, -N, l}}), \end{aligned}$$



provided that  $l + \tilde{r} - \frac{\text{Im}(\beta+\gamma)}{2} > 0$ ,  $l + 1/2 - \frac{\text{Im}(\beta-\gamma)}{2} < 0$ , or vice versa.

The estimate is valid in the sense that if  $(1+x/\sigma)^\alpha u \in H_{\text{sc,b,res}}^{s',\tilde{r}'+l-1/2,l}$  for some  $s', \tilde{r}'$  with  $\tilde{r}'$  satisfying the inequality in place of  $\tilde{r}$ , and with  $(1+x/\sigma)^\alpha(x+\sigma)^{-1}\hat{P}(\sigma)u \in H_{\text{sc,b,res}}^{s-2,\tilde{r}+l+1/2,l+1}$  then  $(1+x/\sigma)^\alpha u \in H_{\text{sc,b,res}}^{s,\tilde{r}+l-1/2,l}$  and the estimate holds.

*Proof.* At first we discuss the argument for sufficiently regular  $u$ . Concretely,  $u$  with  $(1+x/\sigma)^\alpha u \in H_{\text{sc,b,res}}^{s,\tilde{r}+l-1/2,l}$  suffices (so the left hand side is a priori finite). Indeed, even in this case there is one subtlety, for at first sight half an order additional regularity is needed, see the proof of Proposition 4.10 [18], as well as the references given there, [16, Proof of Proposition 5.26] and [7, Lemma 3.4], to make sense of  $\hat{P}(\sigma)^*A$  applied to  $u$  and paired with  $u$ , but this is easily overcome by a simple regularization argument (which does *not* have limitations unlike the more serious regularization discussed below) given in the references.

Recalling that

$$A \in \Psi_{\text{b,res}}^{2\tilde{r}-1,2l+1,-2\tilde{\nu}+2l+1,-\infty}(X) = \Psi_{\text{sc,b,res}}^{2\tilde{r}-1,2(\tilde{r}+l),2l+1,-2\tilde{\nu}+2l+1,-\infty}(X),$$

where we take  $\tilde{\nu} = \alpha - 1/2$ , Lemma 4.4 gives, with  $S \in \Psi_{\text{sc,b,res}}^{-1,-1,0,0,0}(X) = \Psi_{\text{b,res}}^{-1,0,0,0}(X)$  with principal symbol  $\hat{s}$ , that

$$(4.10) \quad \begin{aligned} & i(\hat{P}(\sigma)^*A - A\hat{P}(\sigma)) + AS\hat{P}(\sigma) + \hat{P}(\sigma)^*SA \\ &= (\text{Im } \hat{P}(\sigma)A + A\text{Im } \hat{P}(\sigma)) + i[\text{Re } \hat{P}(\sigma), A] + AS\hat{P}(\sigma) + \hat{P}(\sigma)^*SA = \pm B^*B + F \end{aligned}$$

in  $\Psi_{\text{sc,b,res}}^{2\tilde{r},2(\tilde{r}+l)-1,2l,-2\tilde{\nu}+2l-1,-\infty}(X)$  with

$$\begin{aligned} B &\in \Psi_{\text{sc,b,res}}^{\tilde{r}-1/2,\tilde{r}+l-1/2,l,l-\tilde{\nu}-1/2,-\infty}(X) = \Psi_{\text{b,res}}^{\tilde{r}-1/2,l,l-\tilde{\nu}-1/2,-\infty}(X), \\ F &\in \Psi_{\text{sc,b,res}}^{2\tilde{r},2\tilde{r}+2l-2,2l,2l-2\tilde{\nu}-1,-\infty}(X), \end{aligned}$$

and with the principal symbol of  $B$  given by, up to a factor involving  $\phi$ , by the square root of (4.9). Note that  $F$  only drops an order in the sc-decay relative to (4.10), but is actually higher order in the sc-differentiability sense as the symbolic computation was performed at  $x = 0$  and as the principal symbol in the sc-decay sense vanishes there but not nearby. However, we already have elliptic estimates at sc-fiber infinity, so this is of no relevance; by virtue of (3.11) we have

$$\begin{aligned} |\langle Fu, u \rangle| &\leq C' \|\sigma^{\tilde{\nu}+1/2}u\|_{H_{\text{sc,b,res}}^{\tilde{r},\tilde{r}+l-1,l}}^2 \\ &\leq C(|\sigma|^{-2} \|\sigma^{\tilde{\nu}+1/2}\hat{P}(\sigma)u\|_{H_{\text{sc,b,res}}^{\tilde{r}-2,\tilde{r}+l-1,l}}^2 + \|\sigma^{\tilde{\nu}+1/2}u\|_{H_{\text{sc,b,res}}^{-N,\tilde{r}+l-1,l}}^2) \\ &\leq C(|\sigma|^{-2} \|\sigma^{\tilde{\nu}+1/2}\hat{P}(\sigma)u\|_{H_{\text{b}}^{\tilde{r}-1,l}}^2 + \|\sigma^{\tilde{\nu}+1/2}u\|_{H_{\text{sc,b,res}}^{-N,\tilde{r}+l-1,l}}^2). \end{aligned}$$

Thus, computing (4.10) applied to  $u$  and paired with  $u$  yields

$$\pm \|Bu\|^2 + \langle Fu, u \rangle = 2\text{Im}\langle \hat{P}(\sigma)u, Au \rangle + 2\text{Re}\langle \hat{P}(\sigma)u, SAu \rangle,$$

so

$$\|\sigma^{\tilde{\nu}+1/2}B_1u\|_{H_{\text{b}}^{\tilde{r}-1/2,l}}^2 \leq C(\|\sigma^{\tilde{\nu}-1/2}B_3\hat{P}(\sigma)u\|_{H_{\text{b}}^{\tilde{r}-1/2,l+1}}^2 + \|\sigma^{\tilde{\nu}+1/2}u\|_{H_{\text{b}}^{-N,l}}^2)$$

with  $B_1, B_3 \in \Psi_{\text{b, res}}^{0,0,0,0}(X) = \Psi_{\text{sc, b, res}}^{0,0,0,0}(X)$  microsupported near  $x/\sigma = 0$ . In fact, one can even use a cutoff with differential supported near sc-fiber infinity, i.e. localizing to a compact region in  $(\tau_{\text{res}}, \mu_{\text{res}})$ ; this again gives an error term we can already estimate by elliptic estimates.

Thus, combined with the existing elliptic estimates, this proves the proposition in the weaker sense of a priori having a sufficiently regular  $u$ .

In order to obtain the full result, we need to regularize, replacing  $A$  by  $A^{(\epsilon)}$ . The main impact of this is that in (4.9)  $\tilde{r}$  is replaced by (4.8) (except in the exponent) and an overall factor of  $f_\epsilon(\tau_{\text{b}}^2 + \mu_{\text{b}}^2)$  is added, so that (4.9) is replaced by

$$(4.11) \quad \begin{aligned} & \sigma^{2\tilde{\nu}+1} x^{-2l} (\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{\tilde{r}-3/2} 4 \left( - \left( l + \tilde{r} - \frac{\text{Im}(\beta + \gamma)}{2} - \frac{K}{2} \frac{\epsilon(\tau_{\text{b}}^2 + \mu_{\text{b}}^2)}{1 + \epsilon(\tau_{\text{b}}^2 + \mu_{\text{b}}^2)} \right) \tau_{\text{b}}^2 \right. \\ & \quad \left. + \left( l + 1/2 - \frac{\text{Im}(\beta - \gamma)}{2} \right) \mu_{\text{b}}^2 \right) f_\epsilon(\tau_{\text{b}}^2 + \mu_{\text{b}}^2). \end{aligned}$$

Here we need  $K = 2(\tilde{r} - \tilde{r}')$ : the regularized operator improves  $K$  b-differentiability, thus sc-decay and sc-differentiability, orders, but in the quadratic form both slots have a  $u$  which needs improved regularity. Since  $0 \leq \frac{\epsilon(\tau_{\text{b}}^2 + \mu_{\text{b}}^2)}{1 + \epsilon(\tau_{\text{b}}^2 + \mu_{\text{b}}^2)} \leq 1$ , this still gives the desired definite sign, and the rest of the argument can proceed essentially unchanged. We refer to [16, Section 5.4.7], [14, Proof of Proposition 2.3], as well as earlier work going back to [10] and including [7, Theorem 1.4] for the concrete implementation.  $\square$

We now turn to the case of not necessarily real  $\sigma$ . *We remark that the regularization issues and the ways of dealing with them are completely analogous to the real  $\sigma$  case, and we will not comment on these explicitly.*

Near the scattering zero section the term  $-2\sigma\tau$  is the most important part of the principal symbol since the other terms vanish quadratically at the zero section, so it is useful to consider

$$\tilde{P}(\sigma) = \sigma^{-1} \hat{P}(\sigma) \in \Psi_{\text{sc, b, res}}^{2,0,-1,-1,1}(X)$$

so the principal symbol is

$$\tilde{p}(\sigma) = \sigma^{-1} \hat{p}(\sigma) = -2\tau + \bar{\sigma} |\sigma|^{-2} (\tau^2 + \mu^2),$$

hence

$$\text{Re } \tilde{p}(\sigma) = -2\tau + (\text{Re } \sigma) |\sigma|^{-2} (\tau^2 + \mu^2),$$

$$\text{Im } \tilde{p}(\sigma) = -(\text{Im } \sigma) |\sigma|^{-2} (\tau^2 + \mu^2).$$

Thus,  $\text{Im } \tilde{p}(\sigma) \leq 0$  if  $\text{Im } \sigma \geq 0$ , which means one can propagate estimates forwards along the Hamilton flow of  $\text{Re } \tilde{p}(\sigma)$ ; similarly, if  $\text{Im } \sigma \leq 0$ , one can propagate estimates backwards along the Hamilton flow of  $\text{Re } \tilde{p}(\sigma)$ . We have

**Lemma 4.7.** *We have*

$$(4.12) \quad \begin{aligned} & H_{\text{Re } \tilde{p}(\sigma)} x^{-2l-1} (\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{\tilde{r}-1/2} \\ & = x^{-2l} (\tau_{\text{b}}^2 + \mu_{\text{b}}^2)^{\tilde{r}-3/2} \left( 4((l + \tilde{r})\tau_{\text{b}}^2 + (l + 1/2)\mu_{\text{b}}^2) - 4 \frac{\text{Re } \sigma}{|\sigma|^2} x(l + \tilde{r})\tau_{\text{b}} (\tau_{\text{b}}^2 + \mu_{\text{b}}^2) \right) \\ & = x^{-2(l+\tilde{r})+1} (\tau^2 + \mu^2)^{\tilde{r}-3/2} \left( 4((l + \tilde{r})\tau^2 + (l + 1/2)\mu^2) - 4 \frac{\text{Re } \sigma}{|\sigma|^2} (l + \tilde{r})\tau (\tau^2 + \mu^2) \right). \end{aligned}$$

*Proof.* We compute

$$\begin{aligned} & \left\{ \frac{\operatorname{Re} \sigma}{|\sigma|^2} x^2 (\tau_b^2 + \mu_b^2) - 2x\tau_b, x^{-2l-1} (\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \right\} \\ &= (2 \frac{\operatorname{Re} \sigma}{|\sigma|^2} x^2 \tau_b - 2x) (-2l-1) x^{-2l-1} (\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \\ & \quad - (2 \frac{\operatorname{Re} \sigma}{|\sigma|^2} x^2 (\tau_b^2 + \mu_b^2) - 2x\tau_b) x^{-2l-1} 2(\tilde{r}-1/2) \tau_b (\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2}. \end{aligned}$$

Expanding and rearranging,

$$\begin{aligned} &= 4(l+1/2)x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \\ & \quad + 4(\tilde{r}-1/2)x^{-2l}\tau_b^2(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} \\ & \quad - 4\frac{\operatorname{Re} \sigma}{|\sigma|^2}(l+1/2)x^{-2l+1}\tau_b(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \\ & \quad - 4\frac{\operatorname{Re} \sigma}{|\sigma|^2}(\tilde{r}-1/2)x^{-2l+1}\tau_b(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \\ &= x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} \left( 4((l+1/2)(\tau_b^2 + \mu_b^2) + (\tilde{r}-1/2)\tau_b^2) \right. \\ & \quad \left. - 4\frac{\operatorname{Re} \sigma}{|\sigma|^2}x(l+\tilde{r})\tau_b(\tau_b^2 + \mu_b^2) \right) \\ &= x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} \left( 4((l+\tilde{r})\tau_b^2 + (l+1/2)\mu_b^2) - 4\frac{\operatorname{Re} \sigma}{|\sigma|^2}x(l+\tilde{r})\tau_b(\tau_b^2 + \mu_b^2) \right). \end{aligned}$$

Writing  $\tau = x\tau_b$ ,  $\mu = x\mu_b$  proves the lemma.  $\square$

**Lemma 4.8.** *Let*

$$\hat{s}_0 = 2(l+\tilde{r})(\tau_b^2 + \mu_b^2)^{-1}\tau_b,$$

and let  $S_0 \in \Psi_{\text{sc,b,res}}^{-1,-1,0,0,0}(X) = \Psi_{\text{b,res}}^{-1,0,0,0}(X)$  with principal symbol  $\hat{s}_0$ , and let

$$A_0 \in \Psi_{\text{b,res}}^{2\tilde{r}-1,2l+1,2\nu+2,-\infty}(X) = \Psi_{\text{sc,b,res}}^{2\tilde{r}-1,2(\tilde{r}+l),2l+1,2l+1,-\infty}(X),$$

have principal symbol

$$x^{-2l-1}(\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2}.$$

Then the principal symbol of

$$i[\operatorname{Re} \hat{P}(\sigma), A_0] + A_0 S_0 \hat{P}(\sigma) + \hat{P}(\sigma)^* S_0 A_0$$

in  $\Psi_{\text{sc,b,res}}^{2\tilde{r},2(\tilde{r}+l)-1,2l,2l,-\infty}(X)$  at  $x=0$  is

$$(4.13) \quad x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} 4(- (l+\tilde{r})\tau_b^2 + (l+1/2)\mu_b^2).$$

*Proof.* Adding to (4.12)

$$4(l+\tilde{r}) \operatorname{Re} \tilde{p}(\sigma) x^{-2l-1} \tau_b (\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2}$$

we get

$$(4.14) \quad \begin{aligned} & x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} \left( 4((l+\tilde{r})\tau_b^2 + (l+1/2)\mu_b^2) - 8\tau_b^2(l+\tilde{r}) \right) \\ &= x^{-2l}(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} 4(- (l+\tilde{r})\tau_b^2 + (l+1/2)\mu_b^2), \end{aligned}$$

proving the lemma.  $\square$

Before proceeding, we state a more precise structure result for  $\text{Im } \tilde{P}(\sigma)$  which is crucial as the contribution of  $\text{Im } \tilde{P}(\sigma)$  to the operator we compute is higher order than the commutator itself. This result states that  $\text{Im } \tilde{P}(\sigma)$  has the same sign as  $-(\text{Im } \sigma)$ , in the operator theoretic sense, modulo terms we can otherwise control or are irrelevant.

**Lemma 4.9.** *We have*

$$\begin{aligned} \text{Im } \tilde{P}(\sigma) &= -(\text{Im } \sigma)T(\sigma) + W(\sigma) \in \Psi_{\text{sc,b,res}}^{2,0,-1,-1,1}(X), \\ T(\sigma) &\in \Psi_{\text{sc,b,res}}^{2,0,-1,-1,2}(X), \quad W(\sigma) \in \Psi_{\text{sc,b,res}}^{1,-1,-1,-1,1}(X) \end{aligned}$$

with

$$T(\sigma) = T = \sum_j T_j^2 + \sum_j T_j T'_j + \sum_j T'_j T_j + T''_j$$

with  $T_j = T_j^* \in \Psi_{\text{sc,b,res}}^{1,0,-1,0,1}(X)$  (where  $T_j$  is  $|\sigma|^{-1}$  times the  $T_j$  of (2.7)),  $T'_j = (T'_j)^* \in \Psi_{\text{sc,b,res}}^{0,-1,-1,0,1}(X)$ ,  $T''_j = (T''_j)^* \in \Psi_{\text{sc,b,res}}^{0,-2,-2,0,2}(X)$ ,  $W = W^*$ , so  $T'_j, W$  are one order lower than  $T$  in terms of sc-decay,  $T''_j$  two orders lower, and where  $W(\sigma)$  has principal symbol

$$\text{Im} \left( x^2 \tau_b \beta \frac{\bar{\sigma}}{|\sigma|^2} - x(\beta - \gamma) + x\sigma\varpi \right).$$

*Remark 4.10.* While we could factor out  $\text{Im } \sigma$  from the  $\text{Im}(x\sigma\varpi)$  term in  $W$  by our assumptions, it is only  $O(x)$ , so could not be absorbed in  $T''$ .

*Proof.* This is an immediate consequence of (2.5), (2.7) and (2.8).  $\square$

**Proposition 4.11.** *Suppose  $\beta$  is skew-symmetric. There exists  $\sigma_0 > 0$  such that for  $|\text{Im } \sigma| < \sigma_0$  we have*

$$\begin{aligned} &\|(1 + x/|\sigma|)^\alpha u\|_{H_{\text{sc,b,res}}^{s,\tilde{r}+l-1/2,l}} \\ &\leq C(\|(1 + x/|\sigma|)^\alpha (x + |\sigma|)^{-1} \hat{P}(\sigma)u\|_{H_{\text{sc,b,res}}^{s-2,\tilde{r}+l+1/2,l+1}} + \|(1 + x/|\sigma|)^\alpha u\|_{H_{\text{sc,b,res}}^{-N,-N,l}}), \end{aligned}$$

provided that

$$(4.15) \quad 0 \leq \text{Im } \sigma, \quad l + \tilde{r} - \frac{\text{Im}(\beta + \gamma)}{2} > 0, \quad l + 1/2 - \frac{\text{Im}(\beta - \gamma)}{2} < 0,$$

or if all inequalities in the constraints are reversed.

*This estimate holds in the same sense as Proposition 4.6.*

*Proof.* Let

$$A \in \Psi_{\text{b,res}}^{2\tilde{r}-1,2l+1,-2\tilde{\nu}+2l+1,-\infty}(X) = \Psi_{\text{sc,b,res}}^{2\tilde{r}-1,2(\tilde{r}+l),2l+1,-2\tilde{\nu}+2l+1,-\infty}(X)$$

with principal symbol

$$a = x^{-2l-1} |\sigma|^{2\tilde{\nu}} (\tau_b^2 + \mu_b^2)^{\tilde{r}-1/2} \phi(x/|\sigma|)^2,$$

with  $\phi \geq 0$  as above. Notice that (4.13) is  $\leq 0$  if  $l + 1/2 < 0$ ,  $l + \tilde{r} > 0$  and  $\geq 0$  if  $l + 1/2 > 0$ ,  $l + \tilde{r} < 0$ , matching the sign of the principal symbol of  $\text{Im } \tilde{P}(\sigma)A + A\tilde{P}(\sigma)$  term if  $\text{Im } \sigma \geq 0$ , resp.  $\text{Im } \sigma \leq 0$ . Note also that the subprincipal terms of  $\tilde{P}(\sigma)$

arising from  $\beta, \gamma$  and  $\varpi$ , namely the  $W$  term in Lemma 4.9, yield a contribution to  $\text{Im } \tilde{P}(\sigma)A + A \text{Im } \tilde{P}(\sigma)$  in  $\Psi_{\text{sc,b,res}}^{2\tilde{r}, 2(\tilde{r}+l)-1, 2l, -2\tilde{\nu}+2l, -\infty}(X)$  which has principal symbol

$$\begin{aligned} & 2 \text{Im} \left( x^2 \tau_b \beta \frac{\bar{\sigma}}{|\sigma|^2} - x(\beta - \gamma) + x\sigma\varpi \right) a \\ &= 2x^2 \tau_b (\text{Im } \beta) \frac{\text{Re } \sigma}{|\sigma|^2} a - 2x^2 \tau_b \text{Re } \beta \frac{\text{Im } \sigma}{|\sigma|^2} a - 2x \text{Im}(\beta - \gamma) a + 2x(\text{Im } \sigma)\varpi a. \end{aligned}$$

Assuming that  $\text{Re } \beta = 0$ , adding to this

$$\begin{aligned} & -2(\text{Im } \beta)(\text{Re } \tilde{p}(\sigma))\tau_b(\tau_b^2 + \mu_b^2)^{-1} a \\ &= -2(\text{Im } \beta)(\text{Re } \tilde{p}(\sigma))x^{-2l-1}\tau_b(\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2}\sigma^{2\tilde{\nu}}\phi(x/|\sigma|) \end{aligned}$$

we obtain

$$\begin{aligned} & 4x(\text{Im } \beta)(\tau_b^2 + \mu_b^2)^{-1}\tau_b^2 a - 2x \text{Im}(\beta - \gamma) a + 2x(\text{Im } \sigma)\varpi a \\ &= 2x \left( (\text{Im}(\beta + \gamma) + (\text{Im } \sigma)\varpi)\tau_b^2 - (\text{Im}(\beta - \gamma) - (\text{Im } \sigma)\varpi)\mu_b^2 \right) (\tau_b^2 + \mu_b^2)^{-1} a. \end{aligned}$$

Let

$$\hat{s} = \hat{s}_0 - (\text{Im } \beta)\tau_b(\tau_b^2 + \mu_b^2)^{-1}$$

be the principal symbol of  $S \in \Psi_{\text{sc,b,res}}^{-1, -1, 0, 0, 0}(X) = \Psi_{\text{b,res}}^{-1, 0, 0, 0}(X)$ . We can arrange that

$$A = A_1^2, \quad A_1 = A_1^*, \quad A_1 \in \Psi_{\text{sc,b}}^{\tilde{r}-1/2, \tilde{r}+l, l+1/2, -\tilde{\nu}+l+1/2, -\infty}(X)$$

by choosing  $A_1$  first with the desired principal symbol. Then

$$\begin{aligned} TA + AT &= TA_1^2 + A_1^2 T = 2A_1 T A_1 + [T, A_1]A_1 + A_1[A_1, T] \\ &= 2A_1 T A_1 + [[T, A_1], A_1], \end{aligned}$$

and now the second term is two orders lower than first due to the double commutator. Combined with (4.13) this gives that

$$\begin{aligned} (4.16) \quad & i(\hat{P}(\sigma)^* A - A \hat{P}(\sigma)) + A S \hat{P}(\sigma) + \hat{P}(\sigma)^* S A \\ &= (\text{Im } \hat{P}(\sigma)A + A \text{Im } \hat{P}(\sigma)) + i[\text{Re } \hat{P}(\sigma), A] + A S \hat{P}(\sigma) + \hat{P}(\sigma)^* S A \end{aligned}$$

in  $\Psi_{\text{sc,b,res}}^{2\tilde{r}+1, 2(\tilde{r}+l), 2l, 2l-2\tilde{\nu}, -\infty}(X)$  is of the form

$$\begin{aligned} (4.17) \quad & 2(\text{Im } \sigma)A_1 T A_1 = 2(\text{Im } \sigma) \sum_j A_1^* T_j^* T_j A_1 + 2(\text{Im } \sigma)A_1^* T_j^* T_j' A_1 \\ & \quad + 2(\text{Im } \sigma)A_1^* (T_j')^* T_j A_1, \end{aligned}$$

plus a term in  $\Psi_{\text{sc,b,res}}^{2\tilde{r}-1, 2(\tilde{r}+l)-1, 2l, 2l-2\tilde{\nu}, -\infty}(X)$  whose principal symbol is

$$\begin{aligned} (4.18) \quad & \sigma^{2\tilde{\nu}} x^{-2l} (\tau_b^2 + \mu_b^2)^{\tilde{r}-3/2} 4 \left( - \left( l + \tilde{r} - \frac{\text{Im}(\beta + \gamma) - (\text{Im } \sigma)\varpi}{2} \right) \tau_b^2 \right. \\ & \quad \left. + \left( l + 1/2 - \frac{\text{Im}(\beta - \gamma) - (\text{Im } \sigma)\varpi}{2} \right) \mu_b^2 \right) \end{aligned}$$

times  $\phi(x/\sigma)^2$ , plus a term localized where we have elliptic estimates, arising from  $\phi'$ . For suitable small  $\text{Im } \sigma$ , the  $(\text{Im } \sigma)\varpi$  terms can be absorbed in the definite

$$l + \tilde{r} - \frac{\text{Im}(\beta + \gamma)}{2}, \quad l + 1/2 - \frac{\text{Im}(\beta - \gamma)}{2}$$

terms, so (4.18) is of the form  $b^2$  with

$$b = \sigma^{\tilde{\nu}} x^{-l} (\tau_b^2 + \mu_b^2)^{\tilde{r}/2 - 3/4} 2 \left( - \left( l + \tilde{r} - \frac{\operatorname{Im}(\beta + \gamma) - (\operatorname{Im} \sigma) \overline{\omega}}{2} \right) \tau_b^2 + \left( l + 1/2 - \frac{\operatorname{Im}(\beta - \gamma) - (\operatorname{Im} \sigma) \overline{\omega}}{2} \right) \mu_b^2 \right)^{1/2} \phi(x/\sigma).$$

Thus, (4.18) can be written as  $B^*B + F$  with the principal symbol of symbol of  $B$  being  $b$ ,

$$B \in \Psi_{\text{sc,b,res}}^{\tilde{r}-1/2, \tilde{r}+l-1/2, l, l-\tilde{\nu}-1/2, -\infty}(X) = \Psi_{\text{b,res}}^{\tilde{r}-1/2, l, l-\tilde{\nu}-1/2, -\infty}(X),$$

$$F \in \Psi_{\text{sc,b,res}}^{2\tilde{r}, 2\tilde{r}+2l-2, 2l, 2l-2\tilde{\nu}-1, -\infty}(X).$$

Now, applying both sides of (4.16) to  $u$  and pairing with  $u$  we have

$$(4.19) \quad \begin{aligned} & 2(\operatorname{Im} \sigma) \sum_j \|T_j A_1 u\|^2 + 4(\operatorname{Im} \sigma) \operatorname{Re} \langle T_j A_1 u, T_j' A_1 u \rangle + \|Bu\|^2 + \langle Fu, u \rangle \\ & = 2 \operatorname{Im} \langle \hat{P}(\sigma) u, Au \rangle + 2 \operatorname{Re} \langle \hat{P}(\sigma) u, SAu \rangle. \end{aligned}$$

The terms from the ‘cross terms’ in (4.17) can be estimated via Cauchy-Schwartz:

$$|(\operatorname{Im} \sigma) \langle T_j A_1 u, T_j' A_1 u \rangle| \leq \operatorname{Im} \sigma (\epsilon \|T_j A_1 u\|^2 + \epsilon^{-1} \|T_j' A_1 u\|^2),$$

and now, for sufficiently small  $\epsilon > 0$ , the  $T_j$  term can be absorbed into  $\|T_j A_1 u\|^2$  arising from the first term of the right hand side of (4.17), while the second term, with  $|\operatorname{Im} \sigma|^{1/2}$  included, has

$$|\operatorname{Im} \sigma|^{1/2} T_j' A_1 \in \Psi_{\text{sc,b,res}}^{\tilde{r}-1/2, \tilde{r}+l-1, l-1/2, -\tilde{\nu}+l, -\infty}(X) = \Psi_{\text{b,res}}^{\tilde{r}-1/2, l-1/2, -\tilde{\nu}+l, -\infty}(X)$$

which, for  $\phi$  with sufficiently small support, can be absorbed into  $\|Bu\|^2$  as  $B \in \Psi_{\text{b,res}}^{\tilde{r}-1/2, l, l-\tilde{\nu}, -\infty}(X)$ , so the orders are the same, but there is extra vanishing in  $x/\sigma$  in the strong b-decay sense. Finally, as  $b$  is an elliptic multiple of  $x^{1/2} a_1$ , we can estimate  $|\langle \hat{P}(\sigma) u, Au \rangle|$ , modulo terms like the one given by  $F$ , by

$$\|x^{-1/2} A_1 \hat{P}(\sigma) u\| \|Bu\| \leq \epsilon^{-1} \|x^{-1/2} A_1 \hat{P}(\sigma) u\|^2 + \epsilon \|Bu\|^2,$$

and now for sufficiently small  $\epsilon$ , the last term on the right hand side can be absorbed in  $\|Bu\|^2$ . This, combined with the existing elliptic estimates, gives the desired estimate with  $-N$  replaced by a half order improvement, but an iterative argument gives the full conclusion.

The case of reversed inequalities is similar.  $\square$

## 5. NORMAL OPERATOR

We now turn to normal operators.

We recall that  $(\nu_-, \nu_+)$  is the central weight interval for the scattering end, see (2.12), and in particular for  $\beta = 0, \beta' = 0$  we have

$$(\nu_-, \nu_+) = \left( -1 - \frac{n-2}{2}, -1 + \frac{n-2}{2} \right).$$

As explained in the paragraph following (2.12), a dilation invariant operator can be considered from the perspective of either ‘end’ of the dilation orbit, which concretely for the rescaled variable  $X = x/\sigma$  means either from the scattering end, where  $X \rightarrow 0$ , or the conic point end, where  $X^{-1} \rightarrow 0$ , and as the Mellin transform exponent changes sign under this perspective change, the weights, including the central weight interval, also change sign. In part of this section the conic point

perspective plays a bigger role, so we also use the corresponding indicial roots; we write

$$(\nu_-^c, \nu_+^c) = (-\nu_+, -\nu_-)$$

for the corresponding interval, so for  $\beta = 0$ ,  $\beta' = 0$ , we have

$$(\nu_-^c, \nu_+^c) = \left(1 - \frac{n-2}{2}, 1 + \frac{n-2}{2}\right),$$

and in general

$$(5.1) \quad (\nu_-^c, \nu_+^c) = \left(1 - \frac{\operatorname{Im} \beta}{2} - \operatorname{Re} \sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}, \right. \\ \left. 1 - \frac{\operatorname{Im} \beta}{2} + \operatorname{Re} \sqrt{-\frac{\beta^2}{4} + \left(\frac{n-2}{2}\right)^2 + \beta'}\right),$$

where we recall that  $\beta$  is skew-symmetric and  $\operatorname{Re} \beta' > \frac{\beta^2}{4} - \left(\frac{n-2}{2}\right)^2$ .

The key point is to show that

**Proposition 5.1.** *Let  $0 < x'_0 < x_0$ . Suppose that  $\tilde{r}, l \in \mathbb{R}$ ,*

$$\alpha \in \left(l - \nu_+, l - \nu_-\right) = \left(l + \nu_-^c, l + \nu_+^c\right),$$

and

$$(5.2) \quad \text{either } \tilde{r} + l > -1/2 + \frac{\operatorname{Im}(\beta + \gamma)}{2}, \quad l < -1/2 + \frac{\operatorname{Im}(\beta - \gamma)}{2}, \\ \text{or } \tilde{r} + l < -1/2 + \frac{\operatorname{Im}(\beta + \gamma)}{2}, \quad l > -1/2 + \frac{\operatorname{Im}(\beta - \gamma)}{2}.$$

There is  $C > 0$  such that for distributions  $v$  supported in  $x \leq x'_0$ ,  $v \in H_{\mathbf{b}}^{\tilde{r}, l}$ ,

$$(5.3) \quad \|(1 + x/|\sigma|)^\alpha v\|_{H_{\mathbf{b}}^{\tilde{r}, l}} \leq C \|(1 + x/|\sigma|)^\alpha (x + |\sigma|)^{-1} N_0(\hat{P}(\sigma))v\|_{H_{\mathbf{b}}^{\tilde{r}-1, l+1}},$$

with  $C$  independent of  $\sigma$  with  $\operatorname{Im} \sigma \geq 0$ ,  $\sigma \neq 0$ . The same is true if we take  $\tilde{r}_0 \in \mathbb{R}$ ,  $\tilde{r}_0 < \tilde{r}$ , satisfying the same inequalities as  $\tilde{r}$ , and only assume that  $v \in H_{\mathbf{b}}^{\tilde{r}_0, l}$ .

Similarly, with  $\tilde{r} + l$  replaced by  $r$  in (5.2), there is  $C > 0$  such that for distributions  $v$  supported in  $x \leq x'_0$ ,  $v \in H_{\text{sc}, \mathbf{b}, \text{res}}^{s, r, l}$ ,

$$(5.4) \quad \|(1 + x/|\sigma|)^\alpha v\|_{H_{\text{sc}, \mathbf{b}, \text{res}}^{s, r, l}} \leq C \|(1 + x/|\sigma|)^\alpha (x + |\sigma|)^{-1} N_0(\hat{P}(\sigma))v\|_{H_{\text{sc}, \mathbf{b}, \text{res}}^{s-2, r+1, l+1}},$$

with  $C$  independent of  $\sigma$  with  $\operatorname{Im} \sigma \geq 0$ ,  $\sigma \neq 0$ , and the conclusion also holds if we take  $s_0, r_0 \in \mathbb{R}$ ,  $s_0 < s$ ,  $r_0 < r$ , satisfying the same inequalities as  $r$ , and only assume that  $v \in H_{\text{sc}, \mathbf{b}, \text{res}}^{s_0, r_0, l}$ .

*Remark 5.2.* Notice that  $\alpha = 0$  is acceptable for suitable  $l < -1/2$  for  $n \geq 3$ , while for suitable  $l > -1/2$  only if  $n \geq 4$ .

We postpone the proof of this proposition to the end of this section, and rather start by deducing its consequences. The first is:

**Proposition 5.3.** *For  $\tilde{r}, l, \alpha$  as in Proposition 5.1,  $\tilde{r}'$  arbitrary,  $\delta > 0$  sufficiently small, there is  $C > 0$  such that we have*

$$(5.5) \quad \|(x + |\sigma|)^\alpha u\|_{H_{\text{sc}, \mathbf{b}, \text{res}}^{s, \tilde{r}+l-1/2, l}} \\ \leq C(\|(x + |\sigma|)^{\alpha-1} \hat{P}(\sigma)u\|_{H_{\text{sc}, \mathbf{b}, \text{res}}^{s-2, \tilde{r}+l+1/2, l+1}} + \|(x + |\sigma|)^\alpha x^\delta u\|_{H_{\mathbf{b}}^{\tilde{r}'+1, l}}).$$

*Proof of Proposition 5.3.* For the sake of notational convenience, we assume that  $\sigma > 0$  so that we can avoid writing  $|\sigma|$ ; the general case only needs notational changes.

We first show the proposition under the assumption that  $s \geq \tilde{r} - 1/2$ . Due to Proposition 5.1, with  $v$  having the support properties indicated there, and as

$$\hat{P}(\sigma) - N_0(\hat{P}(\sigma)) \in x(x + \sigma)x^\delta \text{Diff}_b^2(X) + x(x + \sigma)\sigma \text{Diff}_b^2(X),$$

we have

$$\begin{aligned} & \|(1 + x/\sigma)^\alpha v\|_{H_b^{\tilde{r}, l}} \\ & \leq C(\|(1 + x/\sigma)^\alpha (x + \sigma)^{-1} \hat{P}(\sigma)v\|_{H_b^{\tilde{r}-1, l+1}} \\ & \quad + \|(1 + x/\sigma)^\alpha x^\delta v\|_{H_b^{\tilde{r}+1, l}} + \|(1 + x/\sigma)^\alpha \sigma v\|_{H_b^{\tilde{r}+1, l}}). \end{aligned}$$

Furthermore, with  $v = \chi(x)u$ , where  $\chi$  is supported near 0, in  $x \leq x'_0 < x_0$ , and is identically 1 near 0,

$$\hat{P}(\sigma)v = \chi \hat{P}(\sigma)u + [\hat{P}(\sigma), \chi]u,$$

with the second term supported in  $x > 0$  and  $(1 + x/\sigma)^\alpha (x + \sigma)^{-1}$  times it controlled in  $H_b^{\tilde{r}-1, l+1}$  by  $\|(1 + x/\sigma)^\alpha u\|_{H_b^{\tilde{r}, l'}}$ ,  $l'$  arbitrary, so

$$\begin{aligned} & \|(1 + x/\sigma)^\alpha \chi u\|_{H_b^{\tilde{r}, l}} \\ (5.6) \quad & \leq C(\|(1 + x/\sigma)^\alpha (x + \sigma)^{-1} \hat{P}(\sigma)u\|_{H_b^{\tilde{r}-1, l+1}} \\ & \quad + \|(1 + x/\sigma)^\alpha x^\delta u\|_{H_b^{\tilde{r}+1, l}} + \|(1 + x/\sigma)^\alpha \sigma u\|_{H_b^{\tilde{r}+1, l}}). \end{aligned}$$

Thus, starting with the symbolic estimate

$$\begin{aligned} & \|(1 + x/\sigma)^\alpha u\|_{H_{\text{sc}, \text{b}, \text{res}}^{s, \tilde{r}+l-1/2, l}} \\ & \leq C(\|(1 + x/\sigma)^\alpha (x + \sigma)^{-1} \hat{P}(\sigma)u\|_{H_{\text{sc}, \text{b}, \text{res}}^{s-2, \tilde{r}+l+1/2, l+1}} + \|(1 + x/\sigma)^\alpha u\|_{H_{\text{sc}, \text{b}, \text{res}}^{-N, -N, l}}), \end{aligned}$$

and estimating the last term by the above normal operator estimate, with  $\tilde{r}$  replaced by any  $\tilde{r}' \in [-N, \tilde{r} - 3/2]$ , we get, with  $s \geq \tilde{r} - 1/2$  (so that the norm on the left hand side is stronger than that of  $H_b^{\tilde{r}-1/2, l}$ )

$$\begin{aligned} & \|(1 + x/\sigma)^\alpha u\|_{H_{\text{sc}, \text{b}, \text{res}}^{s, \tilde{r}+l-1/2, l}} \\ & \leq C(\|(1 + x/\sigma)^\alpha (x + \sigma)^{-1} \hat{P}(\sigma)u\|_{H_{\text{sc}, \text{b}, \text{res}}^{s-2, \tilde{r}+l+1/2, l+1}} \\ & \quad + \|(1 + x/\sigma)^\alpha x^\delta u\|_{H_b^{\tilde{r}'+1, l}} + \|(1 + x/\sigma)^\alpha \sigma u\|_{H_b^{\tilde{r}'+1, l}}). \end{aligned}$$

Now for  $\tilde{r}'$  small relative to  $\tilde{r}$ , the third term on the right can be absorbed into the left hand side, while the second is relatively compact, so this is an estimate modulo compact errors. Note that as  $\sigma$  powers can be pulled out of the norms, this is equivalent to

$$\begin{aligned} & \|(x + \sigma)^\alpha u\|_{H_{\text{sc}, \text{b}, \text{res}}^{s, \tilde{r}+l-1/2, l}} \\ (5.7) \quad & \leq C(\|(x + \sigma)^{\alpha-1} \hat{P}(\sigma)u\|_{H_{\text{sc}, \text{b}, \text{res}}^{s-2, \tilde{r}+l+1/2, l+1}} + \|(x + \sigma)^\alpha x^\delta u\|_{H_b^{\tilde{r}'+1, l}}), \end{aligned}$$

which completes the proof of the Proposition when  $s \geq \tilde{r} - 1/2$ .



For the general case, we simply use the  $H_{\text{sc,b, res}}$  version of Proposition 5.1, which replaces (5.6) with

$$\begin{aligned} & \|(1+x/\sigma)^\alpha \chi u\|_{H_{\text{sc,b, res}}^{s,r,l}} \\ & \leq C(\|(1+x/\sigma)^\alpha (x+\sigma)^{-1} \hat{P}(\sigma) u\|_{H_{\text{sc,b, res}}^{s-2,r-1,l+1}} \\ & \quad + \|(1+x/\sigma)^\alpha x^\delta u\|_{H_{\text{sc,b, res}}^{s,r+1,l}} + \|(1+x/\sigma)^\alpha \sigma u\|_{H_{\text{sc,b, res}}^{s,r+1,l}}). \end{aligned}$$

□

We can then finish the proof of the main theorem, as we do below, using a variant of the standard compactness considerations to obtain that there is  $\sigma_0 > 0$  such that

$$\|(1+x/\sigma)^\alpha u\|_{H_{\text{sc,b, res}}^{s,\tilde{r}+l-1/2,l}} \leq C\|(1+x/\sigma)^\alpha (x+\sigma)^{-1} \hat{P}(\sigma) u\|_{H_{\text{sc,b, res}}^{s-2,\tilde{r}+l+1/2,l+1}},$$

or equivalently

$$(5.8) \quad \|(x+\sigma)^\alpha u\|_{H_{\text{sc,b, res}}^{s,\tilde{r}+l-1/2,l}} \leq C\|(x+\sigma)^{\alpha-1} \hat{P}(\sigma) u\|_{H_{\text{sc,b, res}}^{s-2,\tilde{r}+l+1/2,l+1}},$$

hold for  $0 < |\sigma| \leq \sigma_0$  provided  $\hat{P}(0)$  has trivial nullspace. Taking into account that, by [18], for  $\sigma \neq 0$ ,  $\hat{P}(\sigma)$  is Fredholm of index zero on the spaces stated in Theorem 1.1 and Theorem 2.5 (and is indeed shown to be even invertible if  $P(\sigma) = P(\sigma)^*$  for  $\sigma$  real), and that in that case the weight factors inside the norms are bounded above and below by positive constants, hence can be dropped, the above estimates, in which the subscript ‘res’, indicating the zero energy behavior, can be dropped for fixed non-zero  $\sigma$ , show the invertibility of  $\hat{P}(\sigma)$  for  $\sigma$  with  $0 < |\sigma| \leq \sigma_0$  by virtue of implying that the nullspace is trivial. *This proves the main theorem, Theorem 1.1, as well as Theorem 2.5!*

*Proof of Theorem 1.1 and of Theorem 2.5.* It only remains to prove (5.8). We proceed under the assumptions of, and using the notation of, Proposition 5.3, taking  $\tilde{r}'$  so that  $s > \tilde{r}' + 1$ ,  $\tilde{r} > \tilde{r}' + 3/2$ . If (5.8) is not true, there are sequences  $\sigma_j \rightarrow 0$  and  $u_j$ , for which one may assume that  $(x + \sigma_j)^\alpha u_j$  has unit norm in  $H_{\text{sc,b, res}}^{s,\tilde{r}+l-1/2,l}$ , and with  $(x + \sigma_j)^{\alpha-1} \hat{P}(\sigma_j) u_j \in H_{\text{sc,b, res}}^{s-2,\tilde{r}+l+1/2,l+1}$ , and such that  $(x + \sigma_j)^{\alpha-1} \hat{P}(\sigma_j) u_j \rightarrow 0$  in  $H_{\text{sc,b, res}}^{s-2,\tilde{r}+l+1/2,l+1}$ . By taking a subsequence (not shown in notation), using the sequential compactness of the unit ball in  $H_{\text{b}}^{\tilde{r}'+1+\delta',l}$  in the weak topology, and the continuity of the (family of) inclusion(s)  $H_{\text{sc,b, res}}^{s,\tilde{r}+l-1/2,l} \rightarrow H_{\text{b}}^{\tilde{r}'+1+\delta',l}$ , as well as the compactness of the inclusion  $H_{\text{b}}^{\tilde{r}'+1+\delta',l} \rightarrow H_{\text{b}}^{\tilde{r}'+1,l-\delta}$  for  $\delta' > 0$  sufficiently small, one may assume that there is  $v \in H_{\text{b}}^{\tilde{r}'+1+\delta',l}$  such that  $(x + \sigma_j)^\alpha u_j \rightarrow v$  weakly in  $H_{\text{b}}^{\tilde{r}'+1+\delta',l}$  and strongly in  $H_{\text{b}}^{\tilde{r}'+1,l-\delta}$ . By (5.5) then  $\liminf \|(x + \sigma_j)^\alpha u_j\|_{H_{\text{b}}^{\tilde{r}',l-\delta}} \geq C^{-1} > 0$ , so  $v \neq 0$  by the strong convergence. On the other hand,

$$(x + \sigma_j)^{\alpha-1} \hat{P}(\sigma_j) u_j \rightarrow x^{\alpha-1} \hat{P}(0) x^{-\alpha} v$$

in  $H_b^{\tilde{r}'-1, l-\delta}$  as

$$\begin{aligned} & (x + \sigma_j)^{\alpha-1} \hat{P}(\sigma_j) u_j - x^{\alpha-1} \hat{P}(0) x^{-\alpha} v \\ &= (x + \sigma_j)^{\alpha-1} (\hat{P}(\sigma_j) - \hat{P}(0)) (x + \sigma_j)^{-\alpha} ((x + \sigma_j)^\alpha u_j) \\ & \quad + (x + \sigma_j)^{\alpha-1} \hat{P}(0) (x + \sigma_j)^{-\alpha} ((x + \sigma_j)^\alpha u_j - v) \\ & \quad + (x + \sigma_j)^{\alpha-1} \hat{P}(0) ((x + \sigma_j)^{-\alpha} - x^{-\alpha}) v \\ & \quad + ((x + \sigma_j)^{\alpha-1} - x^{\alpha-1}) \hat{P}(0) x^{-\alpha} v \end{aligned}$$

since  $\hat{P}(\sigma_j) \rightarrow \hat{P}(0)$  as bounded operators in  $\mathcal{L}(H_b^{\tilde{r}'+1, l-\delta-\alpha}, H_b^{\tilde{r}'-1, l+1-\delta-\alpha})$ ,  $(x + \sigma_j)^{-\alpha}$ , resp.  $(x + \sigma_j)^{\alpha-1}$  are uniformly bounded between  $H_b$  spaces whose weight differs by  $-\alpha$ , resp.  $\alpha - 1$ , and  $(x + \sigma_j)^{-\alpha} u_j$  converges to  $v$  (thus is bounded) in  $H_b^{\tilde{r}'+1, l-\delta}$ , while finally  $(x + \sigma_j)^\alpha - x^\alpha = \sigma_j \int_0^1 \kappa(x + t\sigma_j)^{\alpha-1} dt$  shows that this difference goes to 0 (due to the  $\sigma_j$  factor) as an operator being weighted spaces whose order differs by  $\alpha - 1$ . Thus,  $\hat{P}(0) x^{-\alpha} v = 0$ , so  $u = x^{-\alpha} v$  is a non-trivial element of the nullspace of  $\hat{P}(0)$  on  $H_b^{\tilde{r}'-1/2, l-\alpha}$ , with  $l - \alpha \in (\nu_-^c, \nu_+^c) = (-1 - \frac{n-2}{2}, -1 + \frac{n-2}{2})$ , which contradicts our assumptions. This proves (5.8), as desired.  $\square$

We now return to the proof of Proposition 5.1.

The main claim is that (5.3) follows from the basic scattering estimate, namely the limiting absorption principle on corresponding spaces at energy 1, applied on a scattering manifold which also has a conic point in its interior; we now recall this.

We are thus working with b-Sobolev spaces at both ends, denoted by  $H_b^{\tilde{r}, l, \nu}$ ,  $\nu$  the weight at the conic point, using the scattering density at the sc-end, the conic density at the conic point, for the  $L^2$ -space (with trivial weight, i.e.  $\nu = l = 0$ ), or the second microlocal sc-b space at the scattering end, but the standard b-space at the conic point, denoting it by  $H_{sc, b}^{s, r, l, \nu}$ , with  $\nu$  the weight at the conic point. We recall that for conic points indicial roots are the poles of the Mellin transformed normal operator family in the b-pseudodifferential algebra (normalized to make the family dilation invariant), and we denote the ‘conic point central weight interval’, see the beginning of the section, by  $(\nu_-^c, \nu_+^c)$  which for the actual Laplacian is

$$(\nu_-^c, \nu_+^c) = \left(1 - \frac{n-2}{2}, 1 + \frac{n-2}{2}\right).$$

The latter corresponds to the indicial roots at the conic point. The conic density is  $s^{n-1} ds dy$  near  $s = 0$ , so  $s^{-n/2}$  is barely not in the weight 0 space, hence the allowed exponents are between those of  $s^{2-n}$  and  $s^0$ . We recall the limiting absorption principle in the present, conjugated setting:

**Proposition 5.4** (Limiting absorption principle, [18]). *For  $\tilde{P}(\varsigma)$  of conjugated spectral family type, with a scattering and a conic end,  $(\nu_-^c, \nu_+^c)$  the central weight interval for the conic point, and the unconjugated operator being formally self-adjoint, and for  $\varsigma \neq 0$ ,  $\text{Im } \varsigma \geq 0$ , we have*

$$\|v\|_{H_{sc, b}^{s, r, l, \nu}} \leq C \|\tilde{P}(\varsigma)v\|_{H_{sc, b}^{s-2, r+1, l+1, \nu-2}},$$

and its slightly lossy (in terms of differentiability, in other aspects its equivalent to the above version) version

$$(5.9) \quad \|v\|_{H_b^{\tilde{r}, l, \nu}} \leq C \|\tilde{P}(\varsigma)v\|_{H_b^{\tilde{r}, l+1, \nu-2}}.$$

These estimates are valid if, first of all,  $\nu \in (\nu_-^c, \nu_+^c)$  and, for the sc-b case either  $r > -1/2$  and  $l < -1/2$ , or  $r < -1/2$  and  $l > -1/2$ , while for the b case either  $\tilde{r} + l > -1/2$  and  $l < -1/2$ , or  $\tilde{r} + l < -1/2$  and  $l > -1/2$ .

If we drop the assumption of the unconjugated operator being formally self-adjoint, these are replaced by Fredholm estimates, with  $r > -1/2$  replaced by  $r > -1/2 + \frac{\text{Im}(\beta+\gamma)}{2}$ ,  $l < -1/2$  replaced by  $l < -1/2 + \frac{\text{Im}(\beta-\gamma)}{2}$ ,  $\tilde{r} + l > -1/2$  replaced by  $\tilde{r} + l > -1/2 + \frac{\text{Im}(\beta+\gamma)}{2}$ , and similarly with the inequalities reversed. If the skew-adjoint part is sufficiently small, the above invertibility estimates remain valid.

*Remark 5.5.* In [18] conic points were not considered explicitly. However, since the operator is elliptic near the conic point in the b-sense, near them one has standard b-estimates provided that the weight is not the negative of the imaginary part of an indicial root, see e.g. [11], cf. also [16] where the analytically similar cylindrical ends are discussed. Since [18] uses elliptic estimates away from the scattering end, this is a minor change that fits seamlessly into the framework. This gives Fredholm estimates directly. However, for formally self-adjoint operators, thus with  $\beta = 0$  (since it is skew-adjoint by hypothesis!), for the central weight interval for real  $\varsigma$  the boundary pairing formula goes through showing absence of kernel and cokernel, thus proving the stated version of the proposition. Indeed, (5.1) means that

$$(5.10) \quad (\nu_-^c, \nu_+^c) = \left(1 - \text{Re} \sqrt{\left(\frac{n-2}{2}\right)^2 + \beta'}, 1 + \text{Re} \sqrt{\left(\frac{n-2}{2}\right)^2 + \beta'}\right),$$

which includes a neighborhood of 1 as  $\text{Re} \beta' > -\left(\frac{n-2}{2}\right)^2$  by hypothesis, so elements of the nullspace are in weighted spaces of more than first order vanishing at the conic point (in terms of weighted b-spaces, relative to the metric  $L^2$ -weight), which suffices for the integration by parts arguments. Moreover, for  $\text{Im} \varsigma > 0$  the standard pairing formula for  $P(\varsigma)$  showing the absence of nullspace works as well: note that the undoing the conjugation that gave us  $\tilde{P}(\varsigma)$  means multiplication of the elements of the kernel by  $e^{i\varsigma/X}$  (with  $X$  denoting the conic variable), which are thus exponentially decaying at the scattering end (and no change at the conic point). As the family is uniformly Fredholm, in the sense discussed in [14], the index is constant 0 (as it is such at the real axis where we have invertibility), so this also shows the absence of cokernel for  $\text{Im} \varsigma > 0$ . Notice that it is at this point that the choice of the central weight interval (as opposed to another weight interval with negative free of indicial roots) plays a key role. Since the estimates are perturbation stable, the conclusions hold for sufficiently small skew-adjoint parts.

*Proof of Proposition 5.1.* We apply the scattering-conic estimate for our model operator which takes the form

$$N_0(\hat{P}(\sigma)) = \Delta_0 + \beta \left( x^3 D_x + i \frac{n-2}{2} x^2 \right) + \beta' x^2 - 2\sigma x \left( x D_x + i \frac{n-1}{2} + \frac{\beta-\gamma}{2} \right)$$

where  $\Delta_0$  is the Laplacian of the *exact* conic metric  $g_0 = \frac{dx^2}{x^4} + \frac{h_0}{x^2}$ ,  $h_0$  the metric on  $\partial X$ , after suitable rescaling via  $X = x/|\sigma|$ . Under this rescaling, with  $\hat{\sigma} = \sigma/|\sigma|$ ,  $N_0(\hat{P}(\sigma))$  becomes  $|\sigma|^2 \tilde{P}(\hat{\sigma})$ , where the rescaled operator is

$$\tilde{P}(\varsigma) = \Delta_0 + \beta \left( X^3 D_X + i \frac{n-2}{2} X^2 \right) + \beta' X^2 - 2\varsigma X \left( X D_X + i \frac{n-1}{2} + \frac{\beta-\gamma}{2} \right),$$

with  $\Delta_0$  still the Laplacian of the exact conic metric  $\frac{dX^2}{X^4} + \frac{h_0}{X^2}$ , and  $\tilde{P}(\varsigma)$  indeed enjoys the above limiting absorption principle estimate.

Now, the unweighted b-Sobolev spaces *relative to a b-density* are dilation invariant since the b-derivatives are such, and the b-density corresponds to weights  $-n/2$ , resp.  $n/2$  at the sc-end, resp. the conic point, relative to a scattering density in our normalization, so under the map  $\kappa_\sigma : (x, y) \rightarrow (x/\sigma, y)$ ,

$$\|\kappa_\sigma^* v\|_{H_b^{\tilde{r}, -n/2, n/2}} = \|v\|_{H_b^{\tilde{r}, -n/2, n/2}}.$$

(Note that there is no support condition on  $v$  here!) Since

$$\|v\|_{H_b^{\tilde{r}, l, \nu}} = \|(1+X)^{\nu-n/2}(1+X^{-1})^{l+n/2}v\|_{H_b^{\tilde{r}, -n/2, n/2}},$$

we have

$$\begin{aligned} \|v\|_{H_b^{\tilde{r}, l, \nu}} &= \|(1+X)^{\nu-n/2}(1+X^{-1})^{l+n/2}v\|_{H_b^{\tilde{r}, -n/2, n/2}} \\ &= \|(1+x/\sigma)^{\nu-n/2}(1+\sigma/x)^{l+n/2}\kappa_\sigma^* v\|_{H_b^{\tilde{r}, -n/2, n/2}} \\ &= \|(1+x/\sigma)^{\nu+l}x^{-l-n/2}\sigma^{l+n/2}\kappa_\sigma^* v\|_{H_b^{\tilde{r}, -n/2, n/2}} \\ &= |\sigma|^{l+n/2}\|(1+x/\sigma)^{\nu+l}\kappa_\sigma^* v\|_{H_b^{\tilde{r}, l, -l}}. \end{aligned}$$

Thus, (5.9) becomes

$$\begin{aligned} &|\sigma|^{l+n/2}\|(1+x/\sigma)^{\nu+l}\kappa_\sigma^* v\|_{H_b^{\tilde{r}, l, -l}} \\ &= \|v\|_{H_b^{\tilde{r}, l, \nu}} \\ &\leq C\|\tilde{P}(1)v\|_{H_b^{\tilde{r}, l+1, \nu-2}} \\ &= C|\sigma|^{l+1+n/2}\|(1+x/\sigma)^{\nu+l-1}\kappa_\sigma^* \tilde{P}(1)v\|_{H_b^{\tilde{r}, l+1, -l-1}} \\ &= C|\sigma|^{-2}|\sigma|^{l+1+n/2}\|(1+x/\sigma)^{\nu+l-1}N_0(\hat{P}(\sigma))\kappa_\sigma^* v\|_{H_b^{\tilde{r}, l+1, -l-1}} \\ &= C|\sigma|^{l+n/2}\|(1+x/\sigma)^{\nu+l}(x+\sigma)^{-1}N_0(\hat{P}(\sigma))\kappa_\sigma^* v\|_{H_b^{\tilde{r}, l+1, -l-1}}, \end{aligned}$$

which in summary gives

$$(5.11) \quad \|(1+x/\sigma)^{\nu+l}\kappa_\sigma^* v\|_{H_b^{\tilde{r}, l, -l}} \leq C\|(1+x/\sigma)^{\nu+l}(x+\sigma)^{-1}N_0(\hat{P}(\sigma))\kappa_\sigma^* v\|_{H_b^{\tilde{r}, l+1, -l-1}},$$

which is exactly (5.3), once we replace  $\kappa_\sigma^* v$  by  $v$ ,  $\nu+l$  by  $\alpha$  and keep in mind that due to the support condition, only the weight  $l+1$  on the sc-end is relevant.

The computation for the sc-b-resolved spaces is completely similar: the additional microlocal weights involve  $\tau = x\tau_b$  and  $\mu = x\mu_b$  which means that the corresponding weights using  $\tau_{\text{res}} = X\tau_b$  and  $\mu_{\text{res}} = X\mu_b$  yield  $\|\kappa_\sigma^* v\|_{H_{\text{sc}, b}^{s, r, -n/2, n/2}} = \|v\|_{H_{\text{sc}, b}^{s, r, -n/2, n/2}}$ , so following the above computation gives

$$(5.12) \quad \begin{aligned} &\|(1+x/\sigma)^{\nu+l}\kappa_\sigma^* v\|_{H_b^{s, r, l, -l}} \\ &\leq C\|(1+x/\sigma)^{\nu+l}(x+\sigma)^{-1}N_0(\hat{P}(\sigma))\kappa_\sigma^* v\|_{H_{\text{sc}, b}^{s-2, r, l+1, -l-1}}, \end{aligned}$$

completing the proof.  $\square$

## 6. ZERO ENERGY NULLSPACE

We now discuss, assuming  $\beta = 0$ ,  $\beta' = 0$  and  $\gamma = 0$ , what happens in the presence of non-trivial nullspace of  $\hat{P}(0)$ , i.e. of  $P(0)$ , on the relevant function space,  $H_b^{\infty, l-1/2}$ . As discussed in [17], one issue is that the domain of  $\hat{P}(\sigma)$  varies with  $\sigma$  in a serious way in that, depending on the dimension,  $\text{Ker}\hat{P}(0)$  need not lie in the domain of  $\hat{P}(\sigma)$ ; another issue is that the resolution we introduced here is not very easy to use for perturbation theory directly (since one needs a smooth, continuous, etc., *family* of operators). As in [17], and indeed following a long tradition in scattering theory, this can be remedied by letting  $\check{P}(\sigma)$  be a perturbation of  $\hat{P}(\sigma)$  in the same class but with  $\check{P}(0)$  invertible, where one can arrange this with

$$V(\sigma) = \check{P}(\sigma) - \hat{P}(\sigma)$$

even compactly supported in the interior of  $X$ , though this is not necessary. Then one considers

$$\hat{P}(\sigma)\check{P}(\sigma)^{-1} : \mathcal{Y} \rightarrow \mathcal{Y}, \quad \mathcal{Y} = H_b^{\bar{r}-1, l+3/2};$$

its invertibility at  $\sigma = 0$  is equivalent to that of  $\hat{P}(0)$  on the above discussed space. We have

$$\hat{P}(\sigma)\check{P}(\sigma)^{-1} = \text{Id} - V(\sigma)\check{P}(\sigma)^{-1}.$$

The nullspace of this is the image of that of  $\hat{P}(0)$  under  $\check{P}(0)$ , while the  $L^2$ -orthocomplement of the range is the nullspace of  $P(0)^*$  in  $\mathcal{Y}^*$ . One can decompose  $\mathcal{Y}$  into  $\check{P}(0)\text{Ker}\hat{P}(0)$  and its orthocomplement, and similarly on the target space side into  $\text{Ran}\hat{P}(0)$  and  $\text{Ker}\hat{P}(0)^*$ . Since  $\hat{P}(0)u = 0$  means  $\check{P}(0)u = V(0)u$ , we have  $\check{P}(0)|_{\text{Ker}\hat{P}(0)} = V(0)|_{\text{Ker}\hat{P}(0)}$ . Thus, the entries of the block matrix from  $\text{Ker}\hat{P}(0)\check{P}(0)^{-1} = \check{P}(0)\text{Ker}\hat{P}(0)$  are

(6.1)

$$\begin{aligned} \hat{P}(\sigma)\check{P}(\sigma)^{-1}\check{P}(0)|_{\text{Ker}\hat{P}(0)} &= \check{P}(0) - V(\sigma)\check{P}(\sigma)^{-1}\check{P}(0)|_{\text{Ker}\hat{P}(0)} \\ &= V(0) - V(\sigma)\check{P}(\sigma)^{-1}\check{P}(0)|_{\text{Ker}\hat{P}(0)} \\ &= V(0) - V(\sigma) + V(\sigma)(\check{P}(0)^{-1} - \check{P}(\sigma)^{-1})V(0)|_{\text{Ker}\hat{P}(0)}. \end{aligned}$$

Similarly, for  $\text{Ker}\hat{P}(0)^*$ , we have

$$\begin{aligned} (\check{P}(\sigma)^{-1})^*\hat{P}(\sigma)^*|_{\text{Ker}\hat{P}(0)^*} &= \text{Id} - (\check{P}(\sigma)^{-1})^*V(\sigma)^*|_{\text{Ker}\hat{P}(0)^*} \\ &= ((\check{P}(0)^{-1})^* - (\check{P}(\sigma)^{-1})^*)V(\sigma)^* \\ &\quad + (\check{P}(0)^{-1})^*(V(0) - V(\sigma))|_{\text{Ker}\hat{P}(0)^*}. \end{aligned}$$

Now, the argument would proceed by using that  $V(0) - V(\sigma)$  and  $\check{P}(0)^{-1} - \check{P}(\sigma)^{-1}$  (and also the difference of adjoints) are small as  $\sigma \rightarrow 0$ . This is clear in the case of  $V(0) - V(\sigma)$ , as it is  $O(\sigma)$  with suitably decaying coefficients, but it is much less so for  $(\check{P}(0)^{-1}) - (\check{P}(\sigma)^{-1})$ .

Formally (thus imprecisely),

$$\check{P}(0)^{-1} - \check{P}(\sigma)^{-1} = \check{P}(0)^{-1}(\check{P}(\sigma) - \check{P}(0))\check{P}(\sigma)^{-1} = \check{P}(\sigma)^{-1}(\check{P}(\sigma) - \check{P}(0))\check{P}(0)^{-1},$$

but the composition on the right does not make sense as  $\check{P}(0)^{-1}$  loses 2 orders of decay, and  $\check{P}(\sigma) - \check{P}(0)$  is  $O(\sigma)$  with a gain of one order of decay, which is not sufficient in general for the result to be in the domain of  $P(\sigma)^{-1}$ . *Notice that this is still better than the unconjugated case, where  $\check{P}(\sigma) - \check{P}(0)$  is  $O(\sigma^2)$  but gains no*

decay at all. However, when applied to  $V(0)|_{\text{Ker}\hat{P}(0)}$ ,  $\check{P}(0)^{-1}$  gives the element of  $\text{Ker}\hat{P}(0)$ . Now, the great advantage of the present conjugated setting is that in fact  $\check{P}(\sigma) - \check{P}(0)|_{\text{Ker}\hat{P}(0)}$  is  $O(\sigma)$  with 2 orders of decay gained. This follows from this difference, modulo terms with the claimed 2 orders of decay, being

$$-2\sigma x(xD_x + i(n-1)/2).$$

Now, if  $\lambda_j$  are the eigenvalues of  $\Delta_{\partial X}$ , then elements of  $\text{Ker}\hat{P}(0)$  have an expansion starting with  $\frac{n-2}{2} + \sqrt{(\frac{n-2}{2})^2 + \lambda_j}$ . Thus, if the  $\lambda_j$  are the spherical eigenvalues (thus non-negative integers), all terms, when an  $O(x^2)$  b-differential operator is applied to them, and all, except the  $\lambda_0 = 0$  term in case  $n = 3$ , when an  $O(x)$  b-differential operator is applied to them, are in the full range of decay orders for the domain of  $\hat{P}(\sigma)$ , as they are in  $x^2 H_b^{\infty, -1 + \frac{n-2}{2} - \epsilon} = H_b^{\infty, \frac{n}{2} - \epsilon}$ , resp.  $x H_b^{\infty, -1 + \frac{n-2}{2} + 1 - \epsilon} = H_b^{\infty, \frac{n}{2} - \epsilon}$ , and the acceptable range of weights for the domain start (on the most decaying end) at  $\frac{3}{2} - \epsilon$ , so  $\frac{n}{2} \leq \frac{3}{2}$  shows the claim. For the remaining  $\lambda_0$  case with an  $O(x)$  b-differential operator applied to it the result is in  $x H_b^{\infty, -1 + \frac{n-2}{2} - \epsilon} = H_b^{\infty, \frac{n}{2} - 1 - \epsilon}$ , which is in the acceptable range of weights for the domain which starts, on the least decaying end, at  $-\frac{n-4}{2} + \epsilon$ , so provided  $-\frac{n-4}{2} < \frac{n}{2} - 1$ , which holds for  $n > 3$ , shows that for  $n \geq 4$  in fact all terms make sense. It remains to consider the most singular case,  $n = 3$ , but in fact even then this holds for the  $\lambda_0$  term since  $(xD_x + i(n-1)/2)$  annihilates the leading order asymptotics as  $\frac{n-1}{2} = n-2$  in this case. Thus, assuming our formal computation can be justified, all of the terms, other than that of the big block (00 block) are  $O(|\sigma|)$ . Moreover, if the the 11 block is of the form  $\sigma$  times an invertible operator, Gaussian elimination shows that the whole operator is invertible for  $\sigma \neq 0$ , and the inverse has the form

$$(6.2) \quad \begin{pmatrix} O(1) & O(1) \\ O(1) & O(|\sigma|^{-1}) \end{pmatrix}.$$

Now, unlike in the unconjugated case, when for  $\sigma \neq 0$  the computation makes sense directly by wave front set considerations, we need to explicitly justify it here even then. Thus, we need to consider

$$\check{P}(0)^{-1} - \check{P}(\sigma)^{-1} = \check{P}(0)^{-1} - \check{P}(\sigma)^{-1}(\check{P}(0)\check{P}(0)^{-1})$$

and reparenthesize the last term, which we do by inserting a regularizer (in terms of decay)  $\chi_\epsilon = \chi(\cdot/\epsilon)$ ,  $\chi \geq 0$  smooth,  $\chi \equiv 1$  on  $[1, \infty)$ , supported in  $(0, \infty)$ . Thus, with the limits being strong operator limits,

$$\begin{aligned} \check{P}(0)^{-1} - \check{P}(\sigma)^{-1} &= \lim_{\epsilon \rightarrow 0} (\check{P}(\sigma)^{-1} \check{P}(\sigma)) \chi_\epsilon \check{P}(0)^{-1} - \check{P}(\sigma)^{-1} \chi_\epsilon (\check{P}(0)\check{P}(0)^{-1}) \\ &= \lim_{\epsilon \rightarrow 0} \left( (\check{P}(\sigma)^{-1} \check{P}(\sigma)) \chi_\epsilon \check{P}(0)^{-1} - (\check{P}(\sigma)^{-1} \chi_\epsilon \check{P}(0)) \check{P}(0)^{-1} \right) \\ &= \lim_{\epsilon \rightarrow 0} \left( \check{P}(\sigma)^{-1} \check{P}(\sigma) \chi_\epsilon - \check{P}(\sigma)^{-1} \chi_\epsilon \check{P}(0) \right) \check{P}(0)^{-1} \\ &= \lim_{\epsilon \rightarrow 0} \left( \check{P}(\sigma)^{-1} \check{P}(\sigma) \chi_\epsilon + \check{P}(\sigma)^{-1} [\check{P}(0), \chi_\epsilon] - \check{P}(\sigma)^{-1} \check{P}(0) \chi_\epsilon \right) \check{P}(0)^{-1} \\ &= \lim_{\epsilon \rightarrow 0} \check{P}(\sigma)^{-1} \left( \check{P}(\sigma) \chi_\epsilon + [\check{P}(0), \chi_\epsilon] - \check{P}(0) \chi_\epsilon \right) \check{P}(0)^{-1} \\ &= \check{P}(\sigma)^{-1} \lim_{\epsilon \rightarrow 0} \left( (\check{P}(\sigma) - \check{P}(0)) \chi_\epsilon + [\check{P}(0), \chi_\epsilon] \right) \check{P}(0)^{-1} \end{aligned}$$

which on  $\text{Ker}\hat{P}(0)$ , where we already saw that the first term indeed makes sense by direct domain considerations, is

$$\check{P}(\sigma)^{-1}(\check{P}(\sigma) - \check{P}(0))\check{P}(0)^{-1} + \check{P}(\sigma)^{-1} \lim_{\epsilon \rightarrow 0} [\check{P}(0), \chi_\epsilon] \check{P}(0)^{-1},$$

with similar considerations applying to  $\text{Ker}\hat{P}(0)^*$ . Now, the commutator is uniformly (in  $\epsilon$ ) in  $\Psi_b^{2,-2}(X)$ , and is tending to 0 in the strong topology

$$\mathcal{L}(H_b^{\tilde{r},l-1/2}, H_b^{\tilde{r}-2,l+3/2}),$$

so this term vanishes in the limit. This completes the justification of our computation, and thus that

$$\check{P}(0)^{-1} - \check{P}(\sigma)^{-1}$$

is  $O(|\sigma|)$ . This also shows that modulo  $O(|\sigma|^2)$ , the result is

$$\check{P}(0)^{-1}(\check{P}(\sigma) - \check{P}(0))\check{P}(0)^{-1}.$$

In summary then, mod  $O(|\sigma|^2)$ , (6.1) becomes

$$\begin{aligned} (6.3) \quad & V(0) - V(\sigma) + V(\sigma)(\check{P}(0)^{-1} - \check{P}(\sigma)^{-1})V(0)|_{\text{Ker}\hat{P}(0)} \\ & = V(0) - V(\sigma) + \check{P}(\sigma) - \check{P}(0)|_{\text{Ker}\hat{P}(0)} \\ & = \hat{P}(\sigma) - \hat{P}(0)|_{\text{Ker}\hat{P}(0)}. \end{aligned}$$

Thus, if this pairing between  $\text{Ker}P(0)$  and  $\text{Ker}P(0)^*$  is a non-degenerate multiple of  $\sigma$ , we have that the inverse satisfies bounds as in (6.2).

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