ASYMPTOTICS OF SCALAR WAVES ON LONG-RANGE ASYMPTOTICALLY MINKOWSKI SPACES

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Abstract. We show the existence of the full compound asymptotics of solutions to the scalar wave equation on long-range non-trapping Lorentzian manifolds modeled on the radial compactification of Minkowski space. In particular, we show that there is a joint asymptotic expansion at null and timelike infinity for forward solutions of the inhomogeneous equation. In two appendices we show how these results apply to certain spacetimes whose null infinity is modeled on that of the Kerr family. In these cases the leading order logarithmic term in our asymptotic expansions at null infinity is shown to be nonzero.

1. Introduction

In this paper we analyze the full compound asymptotics of solutions to the scalar wave equation on long-range non-trapping Lorentzian scattering manifolds. This class of Lorentzian scattering manifolds, introduced in [2], includes short-range perturbations of the Minkowski spacetimes as well as a broad class of rather different spacetimes that admit a compactification analogous to the spherical compactification of Minkowski space. In this paper we extend these results to the more physically meaningful setting of *long-range* perturbations of gravitational type: this entails adding a term to our metric that involves a constant Bondi mass. We analyze the compound asymptotics of scalar waves near the boundary at infinity. The most interesting region for this expansion is near the boundary of the light cone, where we obtain a full understanding of the asymptotics via an appropriately scaled blow-up; the *front face* of this blow-up, i.e., the new boundary face obtained by introduction of polar coordinates, is $\mathscr{I}^+$, the null infinity of our spacetime. We analyze the Friedlander radiation field, which is given by the restriction of the rescaled solution to $\mathscr{I}^+$; in particular we find as in [2] that the asymptotics of the radiation field in the "time-delay" parameter (given

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Date: February 15, 2016.

2000 *Mathematics Subject Classification.* Primary 35L05; Secondary 35P25, 58J45.

The authors acknowledge partial support from NSF grants DMS-1500646 (DB), DMS-1361432 (AV) and DMS-1001463 (JW), and the support of NSF Postdoctoral Fellowship DMS-1103436 (DB). The authors gratefully acknowledge the hospitality of the Erwin Schrödinger Institute program "Modern Theory of Wave Equations," at which some of this work was carried out in summer 2015. The first and third author also thank the Institut Henri Poincaré for support through its "Research in Paris" program in February 2016.

by $s = 2(t-r)$ in Minkowski space and subtler here owing to long-range effects) are determined by the resonance poles of an associated Laplace-like operator for an asymptotically hyperbolic metric on the "cap" in the sphere at infinity reached by forward limits of time-like geodesics. Among the main differences of the construction here and that used for the short-range case in [2] is the necessity of a change of $\mathcal{C}^\infty$ structure on the compactified spacetime, prior to the radiation field blow-up, in order to construct the correct $\mathscr{I}^+$.

In particular, in the following theorem, the variable s is analogous to the "lapse function" $2(t-r)$ in Euclidean space; in the long-range case it is given instead by

$$s = 2(t-r) + m \log r^{-1};$$

here the logarithmic correction has a coefficient, denoted m, related to the long-range asymptotics permitted in our spacetimes. The geometric hypotheses of the theorem are spelled out in detail in Section 3 below, and indeed we will restate the theorem in a more precise fashion there.

Theorem 1.1. *Let (M, g) be a non-trapping Lorentzian scattering manifold, and let*

$$\Box_g u = f$$

with $u \in \mathcal{C}^{-\infty}(M)$, $f \in \dot{\mathcal{C}}^\infty(M)$. Assume that u is a forward solution. Then u has a joint polyhomogeneous asymptotic expansion in $s \to \infty,\ r \to \infty$

$$u \sim r^{-(n-2)/2} \sum_j \sum_{\kappa \le m_j} \sum_{\ell=0}^{\infty} \sum_{\alpha \le 2\ell} a_{j\kappa\ell\alpha} s^{-\imath\sigma_j} (\log s)^\kappa (s/r)^\ell (\log(s/r))^\alpha. \tag{1.1}$$

If $m = 0$ then only $\alpha = 0$ terms appear.

The slightly eccentric-looking presentation of the terms in the sum is motivated by the fact that the variables s^{-1} and (s/r) should be viewed as defining functions of the two faces of a manifold with corners obtained by the blowup of the light cone at infinity (depicted on the right side of Figure 1). The exponents σ_j have an explicit description as resonance poles of a family of operators closely related to the spectral family of the Laplacian on asymptotically hyperbolic space. The radiation field, which is conventionally defined [7] by taking the s-derivative of the restriction to $r = \infty$ of $r^{(n-2)/2}u$, is thus well defined and enjoys an asymptotic expansion as $s \to \infty$ with terms (given by taking $\ell = \alpha = 0$ in (1.1)) of the form $s^{-\imath\sigma_j - 1}(\log s)^\kappa$. Note that because u is a forward solution, u is Schwartz for $s \ll 0$ when ρ is small, hence the regime $s \to +\infty$ is the only interesting one.

This theorem represents an improvement over the results of [2] even in the case $m = 0$ (which was all that was treated in that paper), as the appearance of log terms in the expansion is considerably clarified.

In practice, the variables in (1.1) are not well suited to the problem, as, for instance, the regime $r, s \to +\infty$ is the more interesting part of the

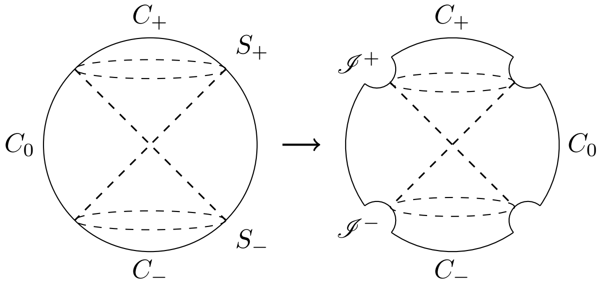

Figure 1. A schematic view of the blow-up. The lapse function s increases along $\mathscr{I}^+$ towards C_+. In the typical Penrose diagram of Minkowski space, $C_\pm$ are collapsed to $i_\pm$ and C_0 is collapsed to i_0.

parameter range. As in [2] we will in fact view our spacetime as the interior of a *compact* manifold with boundary M, analogous to the radial compactification of Minkowski space to the ball B^n. We will let $S_\pm$ denote the forward/backward light cones where they intersect the boundary at infinity, and let $C_\pm$ denote the interiors of ∂M interior to these light cones, i.e., the future/past timelike infinity, which here is a smooth manifold with boundary $S_\pm$. Then ρ denotes a boundary defining function (analogous to $(1+t^2+r^2)^{-1/2}$) while v will denote a function such that $v=0$ defines S_+, the future light cone at infinity. Near S_+, we can in fact simply change the ρ variable to $\rho = r^{-1}$ for simplicity. Then *in the short-range case*, $s = v/\rho$ and $r = \rho^{-1}$ are the variables used above, while $s^{-1} = \rho/v$ and $s/r = v$ will be defining functions for the boundary faces of the blow-up of $\rho = v = 0$, and the compound asymptotics (1.1) are thus expressed in these variables. Note that $\mathscr{I}^+ = \{v = 0\}$ while ρ/v tends to 0 as we move along $\mathscr{I}^+$ to forward timelike infinity. In the long-range case treated here, these definitions are seriously affected by the mass parameter, and the necessary changes are addressed extensively in Section 7 below.

1.1. **Notation.** We use the notation $O(f)$ to denote an element of $f \cdot \mathcal{C}^\infty$ and the notation $O(f_1, \dots, f_k)$ to denote an element of $f_1 \cdot \mathcal{C}^\infty + \dots + f_k \cdot \mathcal{C}^\infty$. We use $O_{\log}(f)$ and $O_{\log}(f_1, \dots, f_k)$ similarly (but with $\mathcal{C}^\infty_{\log}$ in place of $\mathcal{C}^\infty$).

Our convention is that the natural numbers include 0 :

$$\mathbb{N} \equiv \{0, 1, 2, 3, \dots\}.$$

1.2. **Sketch of proof.** As the method of proof is in certain respects somewhat round-about, we sketch it here. The strategy mimics that developed by the authors for the short-range case in [2] up to a certain point, where we prove conormality of the solution up to the boundary and to S_+. The subsequent treatment of the full asymptotic expansion is completely new, however, and much improves the earlier treatment even in the short-range case.

The main steps in the proof are as follows:

Set-up. As noted above, we will let ρ be a boundary defining function for the boundary (at infinity!) of M (e.g., $\rho = r^{-1}$ in the region of most interest) while v cuts out S_+, the future null cone (e.g. $t/r - 1$ in Minkowski space); let y denote the remaining variables (analogous to $\theta = x/|x| \in S^{n-2}$ when $(t,x) \in \mathbb{R}^{1,n-1}$).

We now consider the equation

$$\Box_g u = f$$

but then rescale and conjugate[1] to rewrite it as

$$Lw = g$$

where

$$L \equiv \rho^{-(n-2)/2-2} \Box_g \rho^{(n-2)/2}, \tag{1.2}$$

$$w = \rho^{-(n-2)/2} u \in \mathcal{C}^{-\infty}(M), \quad g = \rho^{-(n-2)/2-2} f \in \dot{\mathcal{C}}^\infty(M).$$

This is advantageous because L is then a "b-differential operator" in the sense of Melrose [17], and enables us to employ the b-pseudodifferential calculus to obtain microlocal estimates on w near $X = \partial M$.

Propagation of b-regularity. We first prove a propagation of b-regularity, which is to say (microlocalized) conormality with respect to the boundary, starting at the backward null cone, where by hypothesis the solution is trivial (zero near the boundary), along $X = \partial M$. This works easily until we reach S_+, where the relevant bicharacteristic flow has radial points, and we need to use subtler estimates. The idea is that instead of proving conormality with respect to X, which is to say, iterated regularity under $\rho\partial_\rho$, ∂_v, and ∂_y, we must settle for less: we only obtain regularity under vector fields that are additionally tangent to $S_+ \subset X$ as well as to X. This is the content of Proposition 5.4 below.

We also employ a refined version of the b-estimates described above that have a semiclassical parameter corresponding to the Mellin dual of the vector field $\rho\partial_\rho$.

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[1] The conjugation here rescales $\Box_g$ to a b-operator and makes it formally self-adjoint in the b-sense. It also removes the leading asymptotics of solutions of the wave equation. It is advantageous to work in the b-setting because the operator is degenerate in the sense of the scattering calculus, corresponding to its characteristic set over the boundary being singular at the zero section, the tip of the light cone in the fibers.

Fredholm estimates. The propagation estimates are the necessary ingredient, following the strategy developed by the second author in [25], in showing that we may set up a global Fredholm problem on X for the family of "reduced normal operators" P_σ. This is the family of operators given by an appropriate freezing of coefficients at $\rho = 0$ and conjugating L by the Mellin transform in the boundary defining function (which simply acts on L by replacing the vector field ρD_ρ by the parameter σ wherever it appears). Crucially, P_σ is taken to act on spaces with varying degrees of regularity, with more regularity mandated at the backward (in the sense of the time orientation) end of the flow lines than at the forward end. The non-trapping hypothesis is used here in a crucial way to show that the family P_σ^{-1} (which we show always exists as a meromorphic operator family) may moreover only have finitely many poles in a given horizontal strip in $\mathbb{C}$, and satisfies polynomial estimates as $|\mathrm{Re}\,\sigma| \to \infty$.

Global asymptotic expansion. We then begin the asymptotic development of w (and hence u) near the boundary as follows. Cutting off near ∂M and Mellin transforming, the equation

$$Lw = g$$

becomes a family of equations of the form

$$P_\sigma \tilde{w} = \tilde{g}.$$

A priori, all we know is that $\tilde{w}$ is analytic in a half-space $\mathrm{Im}\,\sigma \geq \varsigma_0$. However we may invert P_σ to obtain *meromorphy* of $\tilde{w}$, with poles arising from the poles of P_σ^{-1}. If L were in fact dilation-invariant near ∂M, we would immediately have global meromorphy, but as error terms need to be dealt with at each stage, we are only able to improve our domain of meromorphy a little at a time, increasing the half-space in which we know meromorphy by finite increments in an iterative argument. This iteration does eventually yield global meromorphy, but with the subtlety that poles may arise not merely from the poles of P_σ^{-1} itself but also from their shifts by $\imath j$ for $j \in \mathbb{N}$.

Applying the inverse Mellin transform has the effect of turning poles of $\tilde{w}$ into terms in an asymptotic expansion, with a pole at $\sigma = z$ of degree k becoming a term $\rho^{\imath z} \log \rho^{k-1}$. The coefficients of this expansion, however, are functions on $X = \partial M$ that *become worse* in their regularity at S_+ as $\mathrm{Im}\, z$ decreases, i.e. as we obtain more decay in ρ. This development thus suffices to get an asymptotic expansion valid as $\rho \downarrow 0$ for $v \neq 0$, and indeed to get this expansion uniformly as $\rho/v \downarrow 0$ near $v = 0$, which is in effect one of the two asymptotic expansions at intersecting boundary faces in (1.1) (where $s^{-1} = \rho/v$ and v should be regarded as the defining functions for two intersecting boundary faces of a manifold with corners, where we seek a joint asymptotic expansion).

Full asymptotics. It thus remains to obtain the full expansion (1.1), in both variables. It will suffice, via an argument discussed in Section 2, to show that in the short-range case w has improved asymptotics under applications of vector fields of the form

$$(R+\imath j)\dots(R+\imath)(R)$$

where $R = vD_v + \rho D_\rho$ is the scaling vector field about S_+. In the long-range case, we must employ instead

$$(R+\imath j)^{2j+1}\dots(R+\imath)^3(R),$$

with the increased multiplicity corresponding to the additional log terms appearing in the long-range case. These extra difficulties arise because of a change of coordinates necessary to obtain good estimates at S_+: if we (locally) replace the variable v that defines S_+ with $v + m\rho\log\rho$, this change of variables has the considerable virtue of making the leading order form of L near S_+ the same in the long- and short-range cases, but also the considerable defect of introducing additional log singularities into the coefficients of the remaining terms in L. It is these additional error terms that are responsible for the additional log singularities in our expansion. This change of variables and its consequences (in particular, a change of $\mathcal{C}^\infty$ structure on the manifold M) are discussed in Section 7 below.

2. BASICS OF B- AND SCATTERING-GEOMETRY

2.1. **b-geometry.**

The main microlocal tool that we employ is the *b-pseudodifferential calculus* of Melrose, together with refinements involving conormal regularity at submanifolds. We therefore begin by recalling notation and basic results about these objects.

Accordingly, the following preliminaries are essentially taken from [2]. For a more thorough discussion of b-pseudodifferential operators and b-geometry, we refer the reader to Chapter 4 of Melrose [17].

In this section and the following, we initially take M to be a manifold with boundary with coordinates $(\rho, y) \in [0,1) \times \mathbb{R}^{n-1}$ yielding a product decomposition $M \supset U \sim [0,1) \times \partial M$ of a collar neighborhood of ∂M. In particular, for now we lump the v variable in with the other boundary variables as it will not play a distinguished role.

The space of *b-vector fields*, denoted $\mathcal{V}_b(M)$, is the vector space of vector fields on M tangent to ∂M. In local coordinates (ρ, y) near ∂M, they are spanned over $C^\infty(M)$ by the vector fields $\rho\partial_\rho$ and ∂_y. We note that $\rho\partial_\rho$ is well-defined, independent of choices of coordinates, modulo $\rho\mathcal{V}_b(M)$; one may call this the *b-normal vector field* to the boundary. One easily verifies that $\mathcal{V}_b(M)$ forms a Lie algebra. The set of b-differential operators, $\mathrm{Diff}_{\mathrm{b}}^*(M)$, is the universal enveloping algebra of this Lie algebra: it is the filtered algebra consisting of operators of the form

$$A = \sum_{|\alpha|+j\le m} a_{j,\alpha}(\rho, y)(\rho D_\rho)^j D_y^\alpha \in \mathrm{Diff}_{\mathrm{b}}^m(M) \tag{2.1}$$

(locally near ∂M) with the coefficients $a_{j,\alpha} \in \mathcal{C}^\infty(M)$. We further define a bi-filtered algebra by setting

$$\operatorname{Diff}_{\mathrm{b}}^{m,l}(M) \equiv \rho^{-l} \operatorname{Diff}_{\mathrm{b}}^{m}(M).$$

The b-pseudodifferential operators $\Psi_{\mathrm{b}}^*(M)$ are the "quantization" of this Lie algebra, formally consisting of operators of the form

$$b(\rho, y, \rho D_\rho, D_y)$$

with $b(\rho, y, \xi, \eta)$ a Kohn–Nirenberg symbol (i.e., a symbol smooth in all variables with an asymptotic expansion in decreasing powers of $(\xi^2 + |\eta|^2)^{1/2}$); likewise we let

$$\Psi_{\mathrm{b}}^{m,l}(M) = \rho^{-l} \Psi_{\mathrm{b}}^{m}(M)$$

and obtain a bi-graded algebra.

The space $\mathcal{V}_b(M)$ is in fact the space of sections of a smooth vector bundle over M, the *b-tangent bundle*, denoted ${}^{\mathrm{b}}TM$. The sections of this bundle are of course locally spanned by the vector fields $\rho\partial_\rho, \partial_y$. The dual bundle to ${}^{\mathrm{b}}TM$ is denoted ${}^bT^*M$ and has sections locally spanned over $\mathcal{C}^\infty(M)$ by the one-forms $d\rho/\rho, dy$. We also employ the *fiber compactification* $\overline{{}^{\mathrm{b}}T^*M}$ of ${}^bT^*M$, in which we radially compactify each fiber. If we let

$$\xi \frac{d\rho}{\rho} + \eta \cdot dy$$

denote the canonical one-form on ${}^bT^*M$ then a defining function for the boundary "at infinity" of the fiber-compactification is

$$\nu = (\xi^2 + |\eta|^2)^{-1/2};$$

a (redundant) set of local coordinates on each fiber of the compactification near $\{v = \rho = 0\}$ is given by

$$\nu,\ \hat{\xi} = \nu\xi,\ \hat{\eta} = \nu\eta.$$

The symbols of operators in $\Psi_{\mathrm{b}}^*(M)$ are thus Kohn-Nirenberg symbols defined on ${}^bT^*M$. The principal symbol map, denoted σ_{b}, maps (the classical subalgebra of) $\Psi_{\mathrm{b}}^{m,l}(M)$ to ρ^{-l} times homogeneous functions of order m on ${}^bT^*M$. In the particular case of the subalgebra $\operatorname{Diff}_{\mathrm{b}}^{m,l}(M)$, if A is given by (2.1) we have

$$\sigma_{\mathrm{b},m,l}(\rho^{-l}A) = \rho^{-l} \sum_{|\alpha|+j\leq m} a_{j,\alpha}(\rho, y)\xi^j \eta^\alpha$$

where ξ, η are "canonical" fiber coordinates on ${}^bT^*M$ defined by specifying that the canonical one-form be

$$\xi \frac{d\rho}{\rho} + \eta \cdot dy.$$

There is a canonical symplectic structure of ${}^bT^*M^\circ$ given by the exterior derivative of the canonical one-form

$$\frac{1}{\rho}d\xi \wedge d\rho + d\eta \wedge dy.$$

The symbol of the commutator operators in $\Psi_{\mathrm{b}}^*(M)$ is one order lower than the product, with principal symbol given by the Poisson bracket of the principal symbols with respect to this structure. By contrast, the *weight* (second index) of the commutator is, in general, no better than that of the product, owing to noncommutativity of the normal operators introduced below.

Here and throughout this paper we fix a "b-density," which is to say a density which near the boundary is of the form

$$f(\rho, y)\left|\frac{d\rho}{\rho} \wedge dy_1 \wedge \cdots \wedge dy_{n-1}\right|$$

with $f > 0$ everywhere and smooth down to $\rho = 0$. Let $L^2_{\mathrm{b}}(M)$ denote the space of square integrable functions with respect to the b-density. We let $H^m_{\mathrm{b}}(M)$ denote the Sobolev space of order m relative to $L^2_{\mathrm{b}}(M)$ corresponding to the algebras $\mathrm{Diff}^m_{\mathrm{b}}(M)$ and $\Psi^m_{\mathrm{b}}(M)$. In other words, for $m \geq 0$, fixing $A \in \Psi^m_{\mathrm{b}}(M)$ elliptic, one has $w \in H^m_{\mathrm{b}}(M)$ if $w \in L^2_{\mathrm{b}}(M)$ and $Aw \in L^2_{\mathrm{b}}(M)$; this is independent of the choice of the elliptic A. For m negative, the space is defined by dualization. (For m a positive integer, one can alternatively give a characterization in terms of $\mathrm{Diff}^m_{\mathrm{b}}(M)$.) Let $H^{m,l}_{\mathrm{b}}(M) = \rho^l H^m_{\mathrm{b}}(M)$ denote the corresponding weighted spaces. The space $H^{\infty,l}_{\mathrm{b}}(M)$ are of special importance, as they are the spaces of *conormal distributions* with respect to the boundary (having different possible boundary weights). They can more be easily characterized without any microlocal methods by the iterated regularity condition

$$u \in H^{\infty,l}_{\mathrm{b}}(M) \Longleftrightarrow V_1, \dots V_N u \in \rho^l L^2(M)\ \forall N,\ \forall V_j \in \mathcal{V}_{\mathrm{b}}(M).$$

We recall also that associated to the algebra $\Psi^*_{\mathrm{b}}(M)$ is associated a notion of Sobolev wavefront set: $\mathrm{WF}^{m,l}_{\mathrm{b}}(w) \subset {}^{\mathrm{b}}S^*M$ is defined only for $w \in H^{-\infty,l}_{\mathrm{b}}$ (since $\Psi_{\mathrm{b}}(M)$ is *not* commutative to leading order in the decay filtration); the definition is then $\alpha \notin \mathrm{WF}^{m,l}_{\mathrm{b}}(w)$ if there is $Q \in \Psi^{0,0}_{\mathrm{b}}(M)$ elliptic at α such that $Qw \in H^{m,l}_{\mathrm{b}}(M)$, or equivalently if there is $Q' \in \Psi^{m,l}_{\mathrm{b}}(M)$ elliptic at α such that $Q'w \in L^2_{\mathrm{b}}(M)$. We refer to [13, Section 18.3] for a discussion of WF_{b} from a more classical perspective, and [19, Section 3] for a general description of the wave front set in the setting of various pseudodifferential algebras; [26, Sections 2 and 3] provide another discussion, including on the b-wave front set relative to spaces other than $L^2_{\mathrm{b}}(M)$.

In addition to the principal symbol, which specifies high-frequency asymptotics of an operator, we will employ the "normal operator" which measures the boundary asymptotics. For a b-differential operator given by (2.1), this is simply the dilation-invariant operator given by freezing the coefficients of ρD_ρ and D_y at $\rho = 0$, hence

$$N(A) \equiv \sum_{|\alpha|+j\leq m} a_{j,\alpha}(0,y)(\rho D_\rho)^j D_y^\alpha \in \mathrm{Diff}^m_{\mathrm{b}}([0,\infty)\times \partial M). \tag{2.2}$$

It is instructive in studying operators that are approximately dilation-invariant near ∂M to employ the Mellin transform. Thus we define the Mellin transform of u a distribution on M (suitably localized near the boundary) by setting

$$\mathcal{M}u(\sigma, y) = \int \chi(\rho)\rho^{-\imath\sigma-1}\, d\rho \tag{2.3}$$

where χ is compactly supported and equal to 1 near 0.

The Mellin conjugate of the operator $N(A)$ is known as the "reduced normal operator" and, if $N(A)$ is given by (2.2), the reduced normal operator is simply the family (in σ) of operators on ∂M given by

$$\widehat{N}(A) \equiv \sum_{|\alpha|+j\leq m} a_{j,\alpha}(0,y)\sigma^j D_y^\alpha. \tag{2.4}$$

This construction can be extended to b-pseudodifferential operators, but we will only require it in the differential setting here. Moreover, while the construction is more subtle if we extend our coefficient ring to $\mathcal{C}^\infty_{\log}$, as we would need to do to consider the d'Alembertian following the logarithmic coordinate change we will employ below, we will in practice only employ this construction in the setting of our original manifold with its smooth coordinates.

The Mellin transform is a useful tool in studying asymptotic expansions in powers of ρ (and $\log\rho$). In particular, we recall from Section 5.10 of [17] that if u is a distribution on our manifold with boundary M, we write

$$u \in \mathcal{A}^E_{\mathrm{phg}}(M)$$

iff u is conormal to ∂M with

$$u \sim \sum_{(z,k)\in E} \rho^{\imath z}(\log\rho)^k a_{z,k}$$

where $a_{z,k}$ are smooth coefficients on $y \in \partial M$. Here E is an *index set,* which is required to satisfy the following properties[2]:

- $E \subset \mathbb{C} \times \{0, 1, 2, \dots\}$.
- E is discrete.
- $(z_j, k_j) \in E$ and $|(z_j, k_j)| \to \infty \implies \operatorname{Im} z_j \to -\infty$.
- $(z, k) \in E \implies (z, \ell) \in E$ for $\ell = 0, \dots, k-1$ as well.
- $(z, k) \in E \implies (z - j\imath, k) \in E$ for $j \in \mathbb{N}$.

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[2] We have chosen to use the index set conventions of [21] rather than those in [17], which differ by a factor of $\imath$ in how the powers in the expansion relate to the z variable in the index set.

We refer the reader to [17] for an account of why these conditions are natural ones to impose. When $z \in \mathbb{C}$ denotes an index set, this means the smallest index set containing $(z, 0)$, i.e., $\{(z - j\imath, 0) : j \in \mathbb{N}\}$.

We now remark that we may characterize distributions in $\mathcal{A}^E_{\text{phg}}(M)$ in two different ways: by Mellin transform, or by applying scaling vector fields.

To see the former, we recall that by Proposition 5.27 of [17], we have $u \in \mathcal{A}^E_{\text{phg}}(M)$ iff its Mellin transform $\mathcal{M}u$ is meromorphic, with poles of order k only at points z such that $(z, k-1) \in E$, as well as satisfying appropriate decay estimates in σ. (We will state a quantitative L^2 version of this result below, hence will not discuss the estimates here.)

Alternatively, we recall that we may test for polyhomogeneity by use of *radial vector fields*: Let R denote the vector field ρD_ρ (recalling that D_ρ has a factor of $\imath^{-1}$ built into it). We can characterize $u \in \mathcal{A}^E_{\text{phg}}(M)$ for $E = \{(z_j, k_j)\}$ by the requirement that for all l there exists γ_l with $\gamma_l \to \infty$ as $l \to +\infty$ such that

$$\left(\prod_{(z,k) \in E,\ \operatorname{Im} z > -l} (R - z) \right) u \in H_{\mathrm{b}}^{\infty, \gamma_l}(M). \tag{2.5}$$

(Note that by our index set conventions, the product includes $k+1$ factors of $(R - z_j)$ if $(z_j, k) \in E$, since $(z_j, 0), \dots (z_j, k-1)$ are in E as well.)

Theorem 1.1 is about polyhomogeneity not just to one but to *two* boundary hypersurfaces of a manifold with codimension-two corners given by blow-up of our original spacetime M at S_+. We thus make a few remarks here on the generalization of the theory of polyhomogeneity to this context; it is covered in some detail in Section 5.10 of [17], but that treatment only deals with the case where all but one of the index sets are the set

$$0 \equiv \{(-j\imath, 0) : j \in \mathbb{N}\}$$

of indices for smooth functions. The more general case is treated in the unpublished [21], but follows similarly. Thus, here we have an index set at a codimension two corner with defining functions ρ_1, ρ_2 such that $E = (E_1, E_2)$ with E_j an index set at each of the boundary hypersurfaces individually. The idea is simply that u has an expansion at each boundary hypersurface with coefficients that are polyhomogeneous at the other:

$$u \in \mathcal{A}^E_{\text{phg}}(M)$$

iff for each $\ell = 1, 2$, we have

$$u \sim \sum_{(z,k) \in E_\ell} \phi_\ell(z, k) \rho_\ell^{\imath z} (\log \rho_\ell)^k \bmod H^{\infty, \gamma_\ell}(M),$$

where for each (z, k) we have coefficients

$$\phi_\ell(z, k) \in \mathcal{A}^{E(\ell)}_{\text{phg}}$$

with $E(\ell)$ given by $(0, E_2)$ resp. $(E_1, 0)$ for $\ell = 1, 2$ and where for $\ell = 1, 2$ $\gamma_\ell = (\infty, -A)$ resp. $(-A, \infty)$ with fixed $A > \sup\{\operatorname{Im} z : (z, k) \in E_\ell, \ell = 1, 2\}$.

In testing for polyhomogeneity at two boundary hypersurfaces by radial vector fields, it is of considerable importance that it suffices to test *individually* at each boundary hypersurface, with uniform estimates at the other; this is a consequence of a characterization by multiple Mellin transforms (see Chapter 4 of Melrose's book [21], or indeed the Appendix of the PhD thesis of Economakis [4], where a proof provided by Mazzeo is presented). Thus we will in particular use the following:

Proposition 2.1 (Mazzeo, Melrose). *Let R_ℓ denote $\rho_\ell D_{\rho_\ell}$, the scaling vector field at the ℓ'th hypersurface. Suppose that for each $\ell = 1, 2$, there exists a γ', and for all l there is a γ_l, with $\lim_{l\to+\infty}\gamma_l = \infty$ such that*

$$\left(\prod_{(z,k)\in E_\ell,\ \operatorname{Im} z>-l} (R_\ell - z)\right) u \in H_{\mathrm{b}}^{\infty,\gamma_l,\gamma'}, \tag{2.6}$$

where γ_l refers to the growth slot for the ℓ'th hypersurface, and (abusing notation) γ' to the growth at the other boundary hypersurfaces. Then $u \in \mathcal{A}^E_{phg}(M)$ where $E = (E_1, E_2)$.

Note that there is no requirement in (2.6) that the coefficients in the expansion (or indeed the remainder on the right hand side) be polyhomogeneous; this follows automatically when (2.6) is required for all boundary hypersurfaces H.

We let $\mathcal{E}_0$ denote the "index set"[3] of poles of the operator family P_σ^{-1} with imaginary part less than some fixed ς_0. Here we have

$$P_\sigma \equiv \widehat{N}(L)(\sigma)$$

with L the rescaled conjugate of $\Box_g$ given by (1.2) and $\widehat{N}(L)$ the reduced normal operator as defined in (2.4). (The spaces on which we should consider this operator to act will be defined below, in Section 6.) Thus $(\sigma_0, k) \in \mathcal{E}_0$ if σ_0 is a pole of P_σ^{-1} of order at least $k+1$. Though P_σ^{-1} may have poles σ_j with $\operatorname{Im}\sigma_j \to +\infty$, the presence of ς_0 in the definition restricts our attention to a lower half-plane. In practice, we fix ς_0 large enough to consider only the half-plane in which our function is not a priori holomorphic.

To account for accidental multiplicities arising from multiplication by $\mathcal{C}^\infty$ (or $\mathcal{C}^\infty_{\log}$) functions, we must also include in the resonance set the shifts of $\mathcal{E}_0$ corresponding to the index sets of $\mathcal{C}^\infty$ (or $\mathcal{C}^\infty_{\log}$) functions. Namely, for each $j = 1, 2, \dots$, we set

$$\mathcal{E}_j = \{(\sigma - \imath j, k) : (\sigma, k) \in \mathcal{E}_0(\varsigma_0)\}$$

We define the massless resonance set as the extended union of $\mathcal{E}_j$:

$$\mathcal{E}^0_{\mathrm{res}} \equiv \mathcal{E}_0 \overline{\cup} \mathcal{E}_1 \overline{\cup} \mathcal{E}_2 \overline{\cup} \dots, \tag{2.7}$$

---

[3] Note that this is not technically an index set as defined, as it is not closed under shifts by $-\imath$.

where $\overline{\cup}$ denotes the extended union of index sets as in [17, Section 5.18]:

$$E\overline{\cup}F \equiv E \cup F \cup \{(z,k) : (z,\ell_1) \in E,\ (z,\ell_2) \in F,\ k = \ell_1 + \ell_2 + 1\};$$

this corresponds to the increase in order of the poles of a product of meromorphic function in the case when poles of the two functions coincide. Finally we define the resonance set that we obtain on the "logified" space—when $m \neq 0$—by transformation of $\mathcal{E}^0_{\mathrm{res}}$ (see Proposition 7.8):

$$\mathcal{E}_{\mathrm{res}} \equiv \begin{cases} \mathcal{E}^0_{\mathrm{res}} & m = 0 \\ \{(\sigma - j\imath, k + \ell) : (\sigma, k) \in \mathcal{E}^0_{\mathrm{res}},\ 0 \leq \ell \leq j\} & m \neq 0 \end{cases}$$

We also set

$$\mathcal{E}_{\mathscr{I}} = \begin{cases} 0, & m = 0 \\ \{(-j\imath, \ell),\ \ell \in \{0, \dots, 2j\}, & m \neq 0. \end{cases}$$

Thus, the latter is the index set describing an expansion in $\rho^j(\log\rho)^\ell$ for $\ell = 0, \dots, 2j$.

Finally, write the "total index set" as

$$\mathcal{E}_{\mathrm{tot}} = (\mathcal{E}_{\mathrm{res}}, \mathcal{E}_{\mathscr{I}})$$

where the two index sets on the RHS are for the lift of C_+ and $\mathscr{I}^+$ respectively in the radiation field blowup. This is the index set that we will show occurs in the asymptotic expansion.

In an intermediate step of our construction, we will need to consider slightly different kinds of asymptotic expansions: those that are global expansions in the ρ variable of M but with coefficients that are not smooth: they will have conormal singularities of increasing orders at S_+. Index sets and manipulations for these expansions are defined analogously to those of the smooth expansions described above, but we need to slightly clarify the testing definition: Suppose that

$$u \sim \sum a_j(v,y)\rho^{\imath\sigma_j}(\log\rho)^{k_j} \tag{2.8}$$

where now we assume that overall, u is a conormal distribution with respect to $\rho = v = 0$, and that for some fixed q_0, s_0, L

$$a_j \in I^{(q_0 - \mathrm{Re}(\imath\sigma_j))}$$

are also conormal, and where the asymptotic sum now means that

$$u - \sum_{\mathrm{Im}\,\sigma_j \geq -A} a_j(v,y)\rho^{\imath\sigma_j}(\log\rho)^{k_j} \in \rho^{L+A-0}H_{\mathrm{b}}^{s_0-A}(M).$$

Thus the the remainder has better decay at the cost of conormal regularity (and since u is a priori conormal w.r.t. N^*S_+, this loss of regularity is only there).

Proposition 2.2. *A distribution u conormal with respect to N^*S_+ enjoys the expansion* (2.8) *(interpreted as described above) if and only if we have*

$$\prod_{\operatorname{Im}\sigma_j>-A}(\rho D_\rho-\sigma_j)^{k_j}u\in\rho^{L+A-0}H_{\mathrm{b}}^{q_0-A}(M).$$

2.2. **Scattering geometry.** In addition to the notion of b-geometry, we also need to study a different set of metric and operator structures on a manifold with boundary. If we radially compactify Euclidean space, we remark that linear vector fields become the b-vector fields described above, while *constant coefficient* vector fields become elements of

$$\mathcal{V}_{\mathrm{sc}}(M)\equiv\rho\mathcal{V}_{\mathrm{b}}(M).$$

These "scattering" vector fields are thus spanned over $\mathcal{C}^\infty(M)$ by $\rho^2\partial_\rho$ and $\rho\partial_y$ in the coordinates employed above, and as with b-vector fields, they are sections of a bundle, denoted ${}^{\mathrm{sc}}TM$. The dual bundle, ${}^{\mathrm{sc}}T^*M$, has sections spanned by the one-forms

$$\frac{d\rho}{\rho^2},\ \frac{dy}{\rho}.$$

The Euclidean and Minkowski metrics, under radial compactification of Euclidean resp. Minkowski spaces to a ball, are quadratic in these one-forms, and hence (non-degenerate) quadratic forms on ${}^{\mathrm{sc}}TM$.

We may build "scattering differential operators" out of scattering vector fields by setting

$$A=\sum_{|\alpha|+j\le m}a_{j,\alpha}(\rho,y)(\rho^2D_\rho)^j(\rho D_y)^\alpha\in\operatorname{Diff}^m_{\mathrm{sc}}(M)$$

There is a well defined "scattering principal symbol" $\sigma^k_{\mathrm{sc}}(P)$ which replaces ρ^2D_ρ resp. ρD_y by ξ_{sc} resp. η_{sc}, their canonical dual variables in the fibers of ${}^{\mathrm{sc}}T^*M$. See [18] for details, as well as the construction of the associated pseudodifferential calculus.

3. LONG-RANGE SCATTERING GEOMETRY

In this section, we specify our geometric hypotheses in detail.

Let (M,g) be an n-dimensional manifold with boundary $X=\partial M$ equipped with a Lorentzian metric g over M° such that g extends to be a nondegenerate quadratic form on ${}^{\mathrm{sc}}TM$ of signature $(+,-,\dots,-)$.

We motivate our definition of Lorentzian scattering metrics by recalling that if we radially compactify Minkowski space by setting $t=\rho^{-1}\cos\theta$, $x_j=\rho^{-1}\omega_j\sin\theta$ with $\omega\in S^{n-2}$ and then set $v=\cos 2\theta$, the metric becomes:

$$g=v\frac{d\rho^2}{\rho^4}-\frac{v}{4(1-v^2)}\frac{dv^2}{\rho^2}-\frac{1}{2}\Big(\frac{d\rho}{\rho^2}\otimes\frac{dv}{\rho}+\frac{dv}{\rho}\otimes\frac{d\rho}{\rho^2}\Big)-\frac{1-v}{2}\frac{d\omega^2}{\rho^2}. \tag{3.1}$$

This motivates the form of the following definition when $m=0$. The more general case is of course motivated by the need to include the case of nontrivial solution to the Einstein vacuum equations, and in particular we show below in Appendix A that the Kerr solution of the Einstein equations is of this form *sufficiently far from the event horizon and away from timelike infinity*: for any $\epsilon > 0$ and C_ϵ sufficiently large, the region described in Boyer-Lindquist coordinates by $r > C_\epsilon + \epsilon t$ will have the desired metric form.

Definition 3.1. We say that g is a long-range Lorentzian scattering metric if g is a smooth, Lorentzian signature, symmetric bilinear form on ${}^{\mathrm{sc}}TM$, and there exist a boundary defining function ρ for M, a function $v \in \mathcal{C}^\infty(M)$, and a constant m so that

(1) When V is a scattering normal vector field, $g(V,V)$ has the same sign as v at $\rho = 0$,
(2) in a neighborhood of $\{v = 0, \rho = 0\}$ we have

$$g = (v - m\rho)\frac{d\rho^2}{\rho^4} - \left(\frac{d\rho}{\rho^2} \otimes \frac{\vartheta}{\rho} + \frac{\vartheta}{\rho} \otimes \frac{d\rho}{\rho^2}\right) - \frac{\tilde{g}}{\rho^2}$$

with ϑ a smooth 1-form on M and $\tilde{g}$ a smooth symmetric 2-cotensor on M so that

$$\tilde{g}|_{\mathrm{Ann}(d\rho, dv)} \text{ is positive definite.}$$

We further require that

$$\vartheta = \frac{1}{2}dv + O(v) + O(\rho) \text{ near } \rho = v = 0.$$

We make two additional global assumptions on the structure of our spacetime:

Definition 3.2. A Lorentzian scattering metric is *non-trapping* if

(1) The set $S = \{v = 0, \rho = 0\} \subset X$ splits into S_+ and S_-, each a disjoint union of connected components; we further assume that $\{v > 0\} \subset X$ splits into components $C_\pm$ with $S_\pm = \partial C_\pm$. We denote by C_0 the subset of X where $v < 0$.
(2) The projections of all null bicharacteristics on ${}^{\mathrm{sc}}T^*M \setminus o$ tend to $S_\pm$ as their parameter tends to $\pm\infty$ (or vice versa).

In particular, non-trapping guarantees the time-orientability of (M, g) by specifying the future light cone as the one from which the forward (in the sense of the Hamilton flow) null bicharacteristics tend to S_+.

The final definition needed to make sense of the statement of Theorem 1.1 is the following.

Definition 3.3. Let $\Box_g u = f$ on (M, g) a Lorentzian scattering manifold. We say that u is a *forward solution* if u is smooth near $\overline{C_-}$ and vanishes to infinite order there.

We now analyze the inverse metric. Our metric, as a metric on the fibers of ${}^{\mathrm{sc}}T_X M$, i.e., in the frame

$$\rho^2\partial_\rho,\ \rho\partial_v,\ \rho\partial_y$$

has the block form

$$G_0 = \begin{pmatrix} v & -\frac{1}{2} + a_0 v & a_1 v & \dots & a_{n-2} v \\ -\frac{1}{2} + a_0 v & b & c_1 & \dots & c_{n-2} \\ a_1 v & c_1 & -h_{1,1} & \dots & -h_{n-2,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-2} v & c_{n-2} & -h_{1,n-2} & \dots & -h_{n-2,n-2} \end{pmatrix}, \tag{3.2}$$

with the lower $(n-2) \times (n-2)$ block negative definite, hence h_{ij} is positive definite. Blockwise inversion shows that in the frame

$$\frac{d\rho}{\rho^2}, \frac{dv}{\rho}, \frac{dy}{\rho},$$

the inverse metric when restricted to the boundary has the block form

$$G_0^{-1} = \begin{pmatrix} \omega & -2 + \alpha v & -\frac{1}{2}\mu^{\mathrm{T}} + O(v) \\ -2 + \alpha v & -4v + \beta v^2 & -v\Upsilon^{\mathrm{T}} + O(v^2) \\ -\frac{1}{2}\mu + O(v) & -v\Upsilon + O(v^2) & -h^{-1} + O(v) \end{pmatrix}.$$

In the above, $h^{-1} = h^{ij}$ is the inverse matrix of h_{ij}, while $\omega, \alpha, \beta, \mu_j$, and Υ_j are smooth near $\rho = v = 0$, and A^{T} denotes the transpose of a matrix A.

In a neighborhood of the boundary, i.e., at $\rho \neq 0$, there are further correction terms in the inverse metric as the actual metric is given by

$$\begin{aligned} G &= G_0 + H, \\ H &= \begin{pmatrix} -m\rho + O(\rho^2) & O(\rho) & O(\rho) \\ O(\rho) & O(\rho) & O(\rho) \\ O(\rho) & O(\rho) & O(\rho) \end{pmatrix}. \end{aligned}$$

Thus in the inverse frame above,

$$G^{-1} = G_0^{-1} + \begin{pmatrix} O(\rho) & O(\rho) & O(\rho) \\ O(\rho) & 4m\rho + O(\rho^2) + O(\rho v) & O(\rho) \\ O(\rho) & O(\rho) & O(\rho) \end{pmatrix}. \tag{3.3}$$

Thus in the coordinate frame ∂_ρ, ∂_v, ∂_y, the dual metric becomes

$$\begin{pmatrix} g^{\rho\rho}\rho^4 + O(\rho^5) & g^{\rho v}\rho^3 + O(\rho^4) & g^{\rho y}\rho^3 + O(\rho^4) \\ g^{\rho v}\rho^3 + O(\rho^4) & g^{vv}\rho^2 + O(\rho^4) + O(\rho^3 v) & g^{vy}\rho^2 + O(\rho^3) \\ g^{\rho y}\rho^3 + O(\rho^4) & g^{vy}\rho^2 + O(\rho^3) & g^{yy}\rho^2 + O(\rho^3) \end{pmatrix}, \tag{3.4}$$

where $g^{\bullet\bullet}$ are given by

$$\begin{aligned} g^{\rho\rho} &= \omega & g^{\rho v} &= -2 + \alpha v & g^{\rho y} &= -\frac{1}{2}\mu + O(v) \\ g^{vv} &= -4v + 4m\rho + \beta v^2 & g^{vy} &= -v\Upsilon + O(v^2) & g^{yy} &= -h^{-1} + O(v) \end{aligned} \tag{3.5}$$

Again all terms are smooth. We remark at this juncture that the appearance of m only at level of $O(\rho)$ terms means that the normal operator of rescaled $\Box$ will be independent of m, and arguments involving only the inversion of this normal operator will thus be identical to those in [2]. Arguments involving the detailed structure of $\Box$ near S_+, however, require serious modifications.

From (3.4) it is easy to read off the scattering principal symbol of $\Box_g$: if the canonical one-form on ${}^{\mathrm{sc}}T^*M$ is given by

$$\xi_{\mathrm{sc}}\frac{d\rho}{\rho^2}+\gamma_{\mathrm{sc}}\frac{dv}{\rho}+\eta_{\mathrm{sc}}\frac{dy}{\rho},$$

then

$$\begin{gathered}\sigma^2_{\mathrm{sc}}(\Box_g)=(\omega-m\rho+O(\rho^2))\xi_{\mathrm{sc}}^2+(-4+2\alpha v+O(\rho))\xi_{\mathrm{sc}}\gamma_{\mathrm{sc}}+(-4v+\beta v^2)\gamma^2_{\mathrm{sc}}\\ -\big(h^{ij}+O(v)+O(\rho)\big)(\eta_{\mathrm{sc}})_i(\eta_{\mathrm{sc}})_j+(-2v\Upsilon+O(v^2)+O(\rho))\gamma_{\mathrm{sc}}\eta_{\mathrm{sc}}+(-\mu+O(v)+O(\rho))\xi_{\mathrm{sc}}\eta_{\mathrm{sc}}.\end{gathered}\tag{3.6}$$

The transition to the b-principal symbol is likewise quite simple, since dividing by ρ^2 simply converts all sc-vector fields into corresponding b-vector fields, with commutator terms contributing only at lower order. Hence we simply obtain

$$\begin{gathered}\sigma^2_{\mathrm{b}}(\rho^{-2}\Box_g)=(\omega-m\rho+O(\rho^2))\xi^2+(-4+2\alpha v+O(\rho))\xi\gamma+(-4v+\beta v^2)\gamma^2\\ -\big(h^{ij}+O(v)+O(\rho)\big)\eta_i\eta_j+(-2v\Upsilon+O(v^2)+O(\rho))\gamma\eta+(-\mu+O(v)+O(\rho))\xi\eta.\end{gathered}\tag{3.7}$$

4. The Hamilton vector field and its radial set

We record the form of the b-Hamilton vector field of the conjugated operator. If λ is the b-principal symbol of the conjugated and rescaled operator

$$L=\rho^{-(n-2)/2-2}\Box_g\rho^{(n-2)/2}\tag{4.1}$$

then, since conjugation does not affect the principal symbol, we still have, by (3.7)

$$\begin{aligned}\lambda\equiv\sigma_{\mathrm{b}}(L)=g^{\rho\rho}\xi^2-2(2-\alpha v+O(\rho))\xi\gamma-(4v-4m\rho-\beta v^2+O(\rho v)+O(\rho^2))\gamma^2\\ +2g^{\rho y}\cdot\eta\xi-2(v\Upsilon+O(\rho))\cdot\eta\gamma+g^{y_iy_j}\eta_i\eta_j,\end{aligned}\tag{4.2}$$

which yields the Hamilton vector field

$$\begin{aligned}\mathsf{H}_\lambda=&\,(2g^{\rho\rho}\xi-2(2-\alpha v+O(\rho))\gamma+2g^{\rho y}\cdot\eta)\,\rho\partial_\rho\\ &-2\left[\big(4v-\beta v^2-4m\rho+O(\rho v)+O(\rho^2)\big)\gamma+(2-\alpha v+O(\rho))\xi+(v\Upsilon+O(\rho))\cdot\eta\right]\partial_v\\ &+2\left(g^{\rho y}\xi-(v\Upsilon+O(\rho))\gamma+g^{yy}\eta\right)\cdot\partial_y-(\rho\partial_\rho\lambda)\partial_\xi-(\partial_v\lambda)\partial_\gamma-(\partial_y\lambda)\cdot\partial_\eta.\end{aligned}$$

We now analyze the *radial set* $\mathcal{R}$ of the Hamilton vector field within the characteristic set of L. This is defined as the conic set

$$\mathcal{R}=\{p\in{}^bT^*M:\lambda(p)=0,\mathsf{H}_\lambda|_p\in\mathbb{R}R\},$$

where R denotes the scaling vector field in the fibers of ${}^bT^*M$. In order for $\mathsf{H}_\lambda|_p\in\mathbb{R}R$, the projection $\pi\mathsf{H}_\lambda$ of H_λ to the base must vanish as a smooth vector field. We recall that λ is a nondegenerate Lorentzian metric on the fibers of ${}^bT^*M$ and denote the induced b-metric on ${}^{\mathrm{b}}TM$ by g_b. For a point $p = (x^i, \zeta_i) \in {}^bT^*M$ not in the zero section, the projection $\pi\mathsf{H}_\lambda$ is given by

$$\pi\mathsf{H}_\lambda = 2\left(g^{\rho j}\zeta_j\rho\partial_\rho + g^{vj}\zeta_j\partial_v + g^{yj}\zeta_j\partial y\right).$$

In other words, at a point $p = (x, \zeta) \in {}^bT^*M$, the projection $\pi\mathsf{H}_\lambda$ is the vector at x associated to ζ by regarding g_b as a linear map ${}^bT^*_xM \to {}^{\mathrm{b}}T_xM$. Thus $\pi\mathsf{H}_\lambda$ must be a non-vanishing b-vector field. In particular, for it to vanish as a smooth vector field, it must be a nonzero multiple of $\rho\partial_\rho$. We further have that

$$g_b\left(\pi\mathsf{H}_\lambda, \pi\mathsf{H}_\lambda\right) = 4g_{ij}(g^{ik}\zeta_k)(g^{j\ell}\zeta_\ell) = 4(g^{j\ell}\zeta_j\zeta_\ell) = 4\lambda(p),$$

and so $\rho\partial_\rho$ must be a null vector field at $\rho = 0$ and thus $v = 0$. An examination of the coefficients of the spatial vector fields then shows that the radial set $\mathcal{R}$ within $\rho = 0$ is exactly $v = 0$, $\eta = 0$, $\xi = 0$. Equivalently (and this will be used below—cf. (5.3)), we can take it to be defined by λ, ρ, η, ξ, substituting λ for v as a defining function.

On the fiber compactification of ${}^bT^*M$ near $\mathcal{R}$, we use local coordinates

$$\nu = \frac{1}{\gamma}, \hat{\xi} = \frac{\xi}{\gamma}, \hat{\eta} = \frac{\eta}{\gamma},$$

and compute the linearization of H_λ at $\mathcal{R}$. Modulo terms vanishing quadratically at $\partial\mathcal{R}$, we have

$$\begin{aligned}\nu\mathsf{H}_\lambda = &-4\rho\partial_\rho + (-8v - 4\hat{\xi} + 8m\rho)\partial_v + 2\left(g^{\rho y}\hat{\xi} - v\Upsilon + c\rho + g^{yy}\hat{\eta}\right)\partial_y \\ &- 4m\rho\partial_{\hat{\xi}} - 4\left(\nu\partial_\nu + \hat{\xi}\partial_{\hat{\xi}} + \hat{\eta}\partial_{\hat{\eta}}\right) + \mathcal{I}^2\mathcal{V}(\overline{{}^bT^*}M),\end{aligned}$$

with c smooth.

In particular, the linearization of $\nu\mathsf{H}_\lambda$ has eigenvectors and eigenvalues given by

$dv + d\hat{\xi} - m\,d\rho$ with eigenvalue -8,

$d\rho, d\nu, d\hat{\eta},$ with eigenvalue -4,

$4dy + 2g^{yy}d\hat{\eta} + (2c - 3m\Upsilon - 2mg^{\rho y})d\rho - \Upsilon dv + (2g^{\rho y} + \Upsilon)d\hat{\xi},$ with eigenvalue 0.

For $m \neq 0$, this leaves one dimension unaccounted for, and in a notable difference with the short-range case, for $m \neq 0$, there is in fact a nontrivial Jordan block in the generalized eigenspace -4, spanned by $d\rho$ and $d\hat{\xi}$.

Consequently, we must revisit the proof of propagation of b-regularity to radial points (Proposition 4.4 of [2]) in this context. We undertake this in the following section.

5. Propagation of b-regularity and module regularity

Definition 5.1. Let $\mathcal{M} \subset \Psi^1_{\mathrm{b}}(M)$ denote the $\Psi^0_{\mathrm{b}}(M)$-module of pseudodifferential operators with principal symbol vanishing on the radial set $\mathcal{R} = \{\rho = 0,\ v = 0,\ \xi = 0,\ \eta = 0\}$. We also let $\mathcal{M}_{\mathrm{D}} \subset \mathrm{Diff}^1(M)$ denote the module of differential operators with principal symbol vanishing on the radial set $\mathcal{R}$.

Note that a set of generators for $\mathcal{M}$ over $\Psi_{\mathrm{b}}^{0}(M)$ is given by the vector fields $\rho\partial_\rho$, $\rho\partial_v$, $v\partial_v$, ∂_y, and I. The differential module $\mathcal{M}_{\mathrm{D}}$ is generated by the same vector fields over $\mathcal{C}^\infty(M)$.

We recall from [2] that the module $\mathcal{M}$ is closed under commutators.

If we disregard factors in $\mathcal{M}^2$ we note that the operator L defined by equation (4.1) takes a particularly simple form:

Lemma 5.2.

$$L = 4\partial_v\left(\rho\partial_\rho + v\partial_v\right) - 4m\rho\partial_v^2 + \mathcal{M}^2. \tag{5.1}$$

Proof. As in the previous section, we let g_b denote the induced b-metric given by λ, so that $g_b = \rho^2 g$. We observe that L and $\Box_{g_b}$ have the same principal symbol and are both self-adjoint with respect to the volume $\rho^n\sqrt{g}$, hence these operators agree up to a smooth zero-th order term (which is automatically in $\mathcal{M}^2$). We must thus show that $\Box_{g_b}$ has the desired form.

To see this we start by noting that $\Box_{g_b}$ is an element of $\mathrm{Diff}_{\mathrm{b}}^2(M)$ and so the only terms of $\Box_{g_b}$ not lying in $\mathcal{M}^2$ are those terms containing a ∂_v (because $\rho\partial_\rho$ and ∂_y lie in $\mathcal{M}$). We then observe that

$$\begin{aligned} g_b^{vy} &= O(v), \quad g_b^{\rho v} = -2\rho + O(\rho v),\\ g_b^{vv} &= -4v + 4m\rho + O(v^2) + O(\rho v) + O(\rho^2). \end{aligned}$$

Because $\sqrt{g_b} = \rho^{-2}A$, where A is smooth and non-vanishing, it follows that $\Box_{g_b}$ (and hence L) has the desired form. $\Box$

We begin by recalling, just as in [2], that regularity/singularities of solutions to $Lw = f \in \dot{\mathcal{C}}^\infty(M)$ propagates along maximally extended integral curves of the Hamilton vector field for a wide class of operators L: let $L \in \Psi_{\mathrm{b}}^{s,r}(M)$ be arbitrary, and let $\Sigma \subset {}^{\mathrm{b}}S^*M$ denote the characteristic set of L, λ denote the principal symbol of L in $\Psi_{\mathrm{b}}^{s,r}(M)$.

Proposition 5.3. *Suppose $w \in H_{\mathrm{b}}^{-\infty,l}(M)$. Then*

1. *Elliptic regularity holds away from Σ, i.e.,*

$$\mathrm{WF}_{\mathrm{b}}^{m,l}(w) \subset \mathrm{WF}_{\mathrm{b}}^{m-s,l-r}(Lw) \cup \Sigma,$$

2. *In Σ, $\mathrm{WF}_{\mathrm{b}}^{m,l}(w) \setminus \mathrm{WF}_{\mathrm{b}}^{m-s+1,r-l}(Lw)$ is a union of maximally extended bicharacteristics, i.e., integral curves of H_λ.*

Note that the order in $\mathrm{WF}_{\mathrm{b}}^{m-s+1,r-l}(Lw)$ is shifted by 1 relative to the elliptic estimates, corresponding to the usual hyperbolic loss. This arises naturally in the positive commutator estimates used to prove such hyperbolic estimates: commutators in $\Psi_{\mathrm{b}}(M)$ are one order lower than products in the differentiability sense (the first index), but not in the decay order (the second index); hence the change in the first order relative to elliptic estimates but not in the second. We refer the reader to, e.g., [26], for a proof; the idea is essentially a version of the usual real-principal type propagation argument by positive commutators.

Proposition 5.3 by itself fails to give any useful information exactly at $\mathcal{R}$, the radial set of L. To analyze the solutions at $\mathcal{R}$ we require a considerably subtler result that yields propagation into and out of the radial set but which is sensitive to the order of Sobolev regularity under study. The statement below is thus *only about the particular operator* L under study here, as it depends in detail on the behavior near $\mathcal{R}$. Our result here has the same statement as Proposition 4.4 of [2] but as noted above is complicated by the existence of a nontrivial Jordan block in the linearization of the Hamilton vector field about $\mathcal{R}$ in the long-range case considered here.

Proposition 5.4. *Let* $L = \rho^{-(n-2)/2-2}\Box_g \rho^{(n-2)/2}$. *If* $w \in H_{\mathrm{b}}^{-\infty,l}(M)$ *for some* l, $Lw \in H_{\mathrm{b}}^{m-1,l}$, *and* $w \in H_{\mathrm{b}}^{m,l}$ *on a punctured neighborhood* $U \setminus \partial\mathcal{R}$ *of* $\partial\mathcal{R}$ *in* ${}^{\mathrm{b}}S^*M$ *(i.e.,* $\mathrm{WF}_{\mathrm{b}}^{m,l}(w) \cap (U \setminus \partial\mathcal{R}) = \emptyset$*) then for* $m' \leq m$ *with* $m' + l < 1/2$, $w \in H_{\mathrm{b}}^{m',l}(M)$ *at* $\partial\mathcal{R}$ *(i.e.,* $\mathrm{WF}_{\mathrm{b}}^{m',l}(w) \cap \partial\mathcal{R} = \emptyset$*) and for* $N \in \mathbb{N}$ *with* $m' + N \leq m$ *and for* $A \in \mathcal{M}^N$, Aw *is in* $H_{\mathrm{b}}^{m',l}(M)$ *at* $\partial\mathcal{R}$ *(i.e.,* $\mathrm{WF}_{\mathrm{b}}^{m',l}(Aw) \cap \partial\mathcal{R} = \emptyset$*).*

We sketch the proof, focusing on the differences from [2].

Proof. First we show propagation of ordinary b-regularity up to the threshold regularity m'. We inductively show that $\mathrm{WF}_{\mathrm{b}}^{\tilde{m},l}(w) \cap \partial\mathcal{R} = \emptyset$ assuming that we already have shown $\mathrm{WF}_{\mathrm{b}}^{m'',l}(w) \cap \partial\mathcal{R} = \emptyset$ with $m'' = \tilde{m} - 1/2$. As $w \in H_{\mathrm{b}}^{m_0,l}(M)$ for some m_0, we start with $\tilde{m} = \min(m_0 + 1/2, m')$ and then, increasing $\tilde{m}$ by an amount $\leq 1/2$ at each step, we may reach $\tilde{m} = m'$ in finitely many steps.

To do this, we set

$$a = \rho^{-r}\nu^{-s}\phi^2,$$

where $\phi \geq 0$, $\phi \equiv 1$ near $\mathcal{R}$ and $\operatorname{supp}\phi \subset U$. Taking $r + s < 0$ and constraining the support of ϕ appropriately gives

$$\nu \mathsf{H}_\lambda a = -b^2 + e,$$

with b elliptic near $\mathcal{R}$ and e supported on $\operatorname{supp} d\phi$, which we choose to be away from $\mathcal{R}$. Choosing $A \in \Psi_{\mathrm{b}}^{s,r}(M)$ with symbol a then gives

$$\imath[L, A] = -B^*B + E + F,$$

with $E \in \Psi_{\mathrm{b}}^{s+1,r}(M)$ microsupported away from $\mathcal{R}$, $B \in \Psi_{\mathrm{b}}^{(s+1)/2,r/2}(M)$, and $F \in \Psi_{\mathrm{b}}^{s,r}(M)$. Hence we have an estimate

$$\|Bw\|^2 \leq |\langle Ew, w\rangle| + |\langle Fw, w\rangle| + 2\,|\langle Lw, Aw\rangle| \tag{5.2}$$

when w is a priori sufficiently regular. Given $\tilde{m}, l$, we take $s = 2\tilde{m} - 1$ and $r = 2l$ so that $s + r < 0$ is satisfied. As F has order $\leq 2m''$, the inductive assumption gives a bound on $\langle Fw, w\rangle$. A standard regularization argument to justify the pairing then proves the proposition for $N = 0$.

Now we turn to the general case, following the methods of Hassell, Melrose, and Vasy [10, 11] (cf. also the appendix of [19]). In particular, we follow the treatment of Section 6 of [10], which covers the propagation of regularity under test modules into and out of radial points.

Thus as generators of the module we may take quantizations of $\nu^{-1}g$, where g runs over the set

$$\{\hat{\eta}, \rho, \hat{\xi}, \nu^2\lambda\}. \tag{5.3}$$

Recall that $d\hat{\eta},\ d\rho$ are eigenvectors of the linearization of $\nu\mathsf{H}_\lambda$ with eigenvalue -4 while $d\hat{\xi}$ lies in the same generalized eigenspace.

Now let $G_0 = I$ and let $G_1, \dots, G_{n-1}$ be given by quantizing $\nu^{-1}\hat{\eta}$ and $\nu^{-1}\rho$; let G_n be the quantization of $\nu^{-1}\hat{\xi}$ and let $G_{n+1} = \Lambda L$. Here $\Lambda \in \Psi_{\mathrm{b}}^{-1}$ has symbol ν near $\mathcal{R}$. We employ the obvious multi-index notation for G^α.

Since $d\nu,\ d\hat{\eta},\ d\rho$ have equal eigenvalues for $1 \leq j \leq n-1$, we have

$$\imath\Lambda[G_i, L] = \sum_{j=1}^{n+1} C_{ij} G_j + E_i,$$

where $E_i \in \Psi_{\mathrm{b}}^{-\infty}(M)$ and for $i \leq n-1$,

$$\sigma_{\mathrm{b},0,0}(C_{ij})|_{\mathcal{R}} = 0. \tag{5.4}$$

By contrast,

$$\imath\Lambda[G_n, L] = \sum_j C_{n,j} G_j + E_n,$$

with $E_n \in \Psi_{\mathrm{b}}^{-\infty}(M)$ and

$$\sigma_{\mathrm{b},0,0}(C_{n,j})|_{\mathcal{R}} = 0,\ j \neq n-1; \tag{5.5}$$

the term $C_{n,(n-1)}$ *will not enjoy this vanishing property, however.*

We now inductively control regularity under $\mathcal{M}^N$, with $N = 0$ being the case established above. In proving regularity of w under $\mathcal{M}^N$ given regularity under $\mathcal{M}^{N-1}$, we recall that it suffices to consider the application of elements G^α with $|\alpha| = N$ and with $\alpha_{n+1} = 0$, since the presence of a single factor of G_{n+1} in the correct slot renders w residual. (We can arrange that factors of G_{n+1} are always in the correct slot as the induction hypothesis allows us to bound the commutators.)

We thus consider the *system* of commutators

$$\imath[L, W_\alpha],$$

with

$$W_\alpha = \epsilon^{-\alpha_{n-1}} \operatorname{Op}(\sqrt{a})^*(G^\alpha)^*(G^\alpha)\operatorname{Op}(\sqrt{a}),$$

where a is chosen as above, $\epsilon > 0$ is small (to be fixed later), and where we let α run over all values with

$$|\alpha| = N, \alpha_{n+1} = 0.$$

As before, since $s+r<0$ (and if the support of ϕ is sufficiently small), we have

$$\begin{aligned}\imath[L, W_\alpha] &= -\epsilon^{-\alpha_{n-1}} B^*(G^\alpha)^*(G^\alpha)B\\ &+\sum_\beta \epsilon^{-\alpha_{n-1}} \operatorname{Op}(\sqrt{a})^* \Big((G^\beta)^* C_{\alpha\beta}(G^\alpha) + (G^\alpha)^* C'_{\alpha\beta}(G^\beta)\Big) \operatorname{Op}(\sqrt{a})\\ &+ E_\alpha + F_\alpha\end{aligned} \tag{5.6}$$

Here the terms involving the G^β (and adjoint) arise from the commutators of L with the G^α (and adjoint) factors; B is elliptic near $\mathcal{R}$, as before. E_α is microsupported away from $\mathcal{R}$, and F_α has lower order. Crucially, the vanishing of the symbols of C_{ij} on $\mathcal{R}$ imply that

$$\sigma_{\mathrm{b},0,0}(C_{\alpha\beta}) = 0,\ \sigma_{\mathrm{b},0,0}(C'_{\alpha\beta}) = 0,\ \text{on } \mathcal{R} \text{ unless } \beta = \alpha + \delta^{n-1} - \delta^n,$$

where δ^j is the multi-index with $\delta^j_i = 0$ for $i \neq j$, $\delta^j_j = 1$. Now on pairing equation (5.6) with w, we note that:

- Terms with $\beta_{n+1} \neq 0$ are trivially bounded because Lw is residual.
- Terms with $|\beta| < N$ can be absorbed in the positive terms by the inductive hypothesis and Cauchy–Schwarz.
- Terms with $|\beta| = |\alpha|$ can likewise be absorbed in the main positive terms unless $\beta = \alpha + \delta^{n-1} - \delta^n$ by the vanishing of the symbol (shrinking supports if necessary).
- Terms with $\beta = \alpha + \delta^{n-1} - \delta^n$ can be likewise handled by Cauchy–Schwarz, as they come with a coefficient $\epsilon^{-\alpha_{n-1}}$, while the corresponding positive term has coefficient $\epsilon^{-\beta_{n-1}} = \epsilon^{-\alpha_{n-1}-1}$. Hence for ϵ sufficiently small, these terms, too, may be controlled by the main commutator terms.

□

6. Fredholm properties

We now turn to the Fredholm properties of the operator family P_σ on variable-order Sobolev spaces, which we can deduce from the propagation theorems above. This argument is identical to that used in [2], again following the strategy first used by the second author in [25].

Definition 6.1. Let $\mathbb{C}_\nu$ denote the halfspace $\operatorname{Im}\sigma > -\nu$ and let $\mathcal{H}(\mathbb{C}_\nu)$ denote holomorphic functions on this space. For a Fréchet space $\mathcal{F}$, let

$$\mathcal{H}(\mathbb{C}_\nu) \cap \langle\sigma\rangle^{-k} L^\infty L^2(\mathbb{R}; \mathcal{F})$$

denote the space of g_σ holomorphic in $\sigma \in \mathbb{C}_\nu$ taking values in $\mathcal{F}$ such that each seminorm

$$\int_{-\infty}^{\infty} \big\|g_{\mu+\imath\nu'}\big\|_\bullet^2 \langle\mu\rangle^{2k}\, d\mu$$

is uniformly bounded in $\nu' > -\nu$.

Note the choice of signs: as ν increases, the halfspace gets larger.

We will further allow elements of $\mathcal{H}(\mathbb{C}_\nu)$ to take values in *σ-dependent* Sobolev spaces, or rather Sobolev spaces with σ-dependent norms. In particular, we allow values in the standard semiclassical Sobolev spaces H_h^m on a compact manifold (without boundary), with semiclassical parameter $h = \langle\sigma\rangle^{-1}$. Recall (see [30, Section 8.3]) that these are the standard Sobolev spaces and up to the equivalence of norms, for h in compact subsets of $(0, \infty)$, the norm is just the standard H^m norm, but the norm is h-dependent: for non-negative integers m, in coordinates y_j, locally the norm $\|g\|_{H_h^m}$ is equivalent to $\sqrt{\sum_{|\alpha|\leq m} \|(hD_{y_j})^\alpha g\|_{L^2}^2}$.

We let P_σ be the normal operator for the conjugated operator $L = \rho^{-(n-2)/2}\rho^{-2}\Box_g\rho^{(n-2)/2}$. Recall that under our global assumptions, the characteristic set of P_σ in S^*X has two parts $\Sigma_\pm$ such that the integral curves of the Hamilton flow in $\Sigma_\pm$ tend to $S_\pm$ as the parameter tends to $+\infty$.

We recall now from [2] that the radial points of the Hamilton vector field of P_σ (an operator on $X = \partial M$) occur when $v = 0$. Indeed, they occur at

$$\Lambda_{\epsilon_2}^{\epsilon_1} = \{v = 0, \eta = 0, \epsilon_2\gamma > 0\} \cap \Sigma_{\epsilon_1} \subset T^*X, \quad \epsilon_i = \pm, \tag{6.1}$$

so that the $\pm$ in the superscript distinguishes "past" from "future" null infinity, while that in the subscript separates the intersections with the two components of the characteristic set. The past and future radial sets are denoted $\Lambda^\pm = \Lambda_+^\pm \cup \Lambda_-^\pm$.

The operator family P_σ is Fredholm on appropriate variable-order Sobolev spaces, which we now recall. Let $\bar{s}^\pm(\sigma)$ denote the threshold Sobolev exponents at $\Lambda^\pm$, i.e., at the future and past radial sets. From [2], we recall that in fact

$$\bar{s}^\pm(\sigma) = \frac{1}{2} + \operatorname{Im}\sigma.$$

Now let s_{ftr} be a function on S^*X so that

1. s_{ftr} is constant near $\Lambda^\pm$,
2. s_{ftr} is decreasing along the H_p-flow on Σ_+ and increasing on Σ_-,
3. s_{ftr} is less than the threshold exponents at Λ^+, towards which we propagate our estimates, i.e., $s_{\mathrm{ftr}}|_{\Lambda^+} < \bar{s}^+(\sigma)$, and
4. s_{ftr} is greater than the threshold value at Λ^-, away from which we propagate our estimates.

We also require a function s_{past^*} on S^*X satisfying the above assumptions for P_σ^* with $\bar{s}^{\pm,*}(\sigma) = -\bar{s}^\pm(\sigma) + 1$, and may thus take $s_{\mathrm{past}}^* = -s_{\mathrm{ftr}} + 1$ so that

$$(H^{s_{\mathrm{ftr}}})^* = H^{s_{\mathrm{past}^*}-1}, \quad (H^{s_{\mathrm{ftr}}-1})^* = H^{s_{\mathrm{past}}^*}.$$

As $\operatorname{Im}\sigma$ decreases, the constant value $s(S_+)$ assumed by s_{ftr} near S_+ must satisfy $s(S_+) < \frac{1}{2} + \operatorname{Im}\sigma$. Because we are ultimately interested in functions that are identically zero near S_-, we may typically choose s_{ftr} and s_{past} so that they are constant on the support of our functions.

For $U \in H^{s_{\mathrm{ftr}}}$ near Λ^-, propagation of regularity from Λ^- to Λ^+ yields estimates of the form

$$\|U\|_{H^{s_{\mathrm{ftr}}}} \leq C\left(\|P_\sigma U\|_{H^{s_{\mathrm{ftr}}-1}} + \|U\|_{H^{-N}}\right),$$

with similar estimates holding for P_σ. We may thus obtain Fredholm properties for P_σ and P_σ^* by changing the spaces slightly. We set

$$\mathcal{Y}^{s_{\mathrm{ftr}}-1} = H^{s_{\mathrm{ftr}}-1}, \quad \mathcal{X}^{s_{\mathrm{ftr}}} = \{U \in H^{s_{\mathrm{ftr}}} : P_\sigma U \in \mathcal{Y}^{s_{\mathrm{ftr}}-1}\}.$$

(Recall that the last statement in the definition of $\mathcal{X}^{s_{\mathrm{ftr}}}$ depends only on the principal symbol of P_σ, which is independent of σ.)

The following proposition then holds for P_σ:

Proposition 6.2 ([2], Proposition 5.1)**.** *The family of maps P_σ enjoys the following properties:*

(1) *$P_\sigma : \mathcal{X}^{s_{\mathrm{ftr}}} \to \mathcal{Y}^{s_{\mathrm{ftr}}-1}$ and $P_\sigma^* : \mathcal{X}^{s^*_{\mathrm{past}}} \to \mathcal{Y}^{s^*_{\mathrm{past}}-1}$ are Fredholm maps.*

(2) *P_σ is a holomorphic Fredholm family on these spaces in*

$$\mathbb{C}_{s_+,s_-} = \{\sigma \in \mathbb{C} \mid s_+ < \bar{s}^+(\sigma), s_- > \bar{s}^-(\sigma)\},$$

with $s_{\mathrm{ftr}}|_{\Lambda^\pm} = s_\pm$. P_σ^* is antiholomorphic in the same region.

Non-trapping versions of the above estimates yield the following proposition as well:

Proposition 6.3 ([2], Proposition 5.2)**.** *If the non-trapping hypothesis holds, then*

(1) *P_σ^{-1} has finitely many poles in each strip $a < \operatorname{Im}\sigma < b$.*

(2) *For all a, b there exists V such that*

$$\left\|P_\sigma^{-1}\right\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}_{|\sigma|^{-1}} \to \mathcal{X}^{s_{\mathrm{ftr}}}_{|\sigma|^{-1}}} \leq C\langle \operatorname{Re}\sigma\rangle^{-1}$$

for $a < \operatorname{Im}\sigma < b$ and $|\operatorname{Re}\sigma| > C$.

Here the spaces with $|\sigma|^{-1}$ subscripts refer to the variable order versions of the semiclassical Sobolev spaces.

An inductive argument about the Jordan block structure of P_σ^{-1} and the Cauchy integral formula establish the following lemma as well:

Lemma 6.4 ([2], Lemma 8.3)**.** *Let σ_0 be a pole of order k of the operator family*

$$P_\sigma^{-1} : \mathcal{Y}^{s_{\mathrm{ftr}}-1} \to \mathcal{X}^{s_{\mathrm{ftr}}}$$

and let

$$(\sigma - \sigma_0)^{-k} A_k + (\sigma - \sigma_0)^{-k+1} A_{k-1} + \cdots + (\sigma - \sigma_0)^{-1} A_1 + A_0$$

denote the Laurent expansion near σ_0, with A_0 locally holomorphic. If a function f vanishes in a neighborhood of $\overline{C_-}$, then $A_\ell f$ is supported in $\overline{C_+}$ for $\ell = 1, \ldots, k$.

7. Logification

The long-range term in the metric induces a logarithmic divergence of the light cones near infinity when compared to the short-range setting. There are several ways to compensate for this fact: for example, one could introduce a logarithmic correction when blowing up $S_\pm$. This method, however, causes problems, as the resulting manifold is no longer a smooth manifold with corners. We adopt a different strategy here: we *change the smooth structure* on M to obtain a new smooth manifold with boundary M before the blow-up. This process removes the ambiguity surrounding what sort of object the blown-up manifold becomes, but at the cost of introducing logarithmic singularities in the metric coefficients. All methods require fixing a product structure in X near $S_\pm$, but the results do not depend on the choice of product structure. We emphasize that this change of smooth structure will be employed only in the final stage of our arguments (denoted "Full Asymptotics" in the sketch from the introduction), when we perform the radiation field blowup and deduce our asymptotic expansion at $\mathscr{I}^+$; in the intervening stage, at which we iteratively invert the reduced normal operator of L globally on ∂M, we are using the original smooth structure.

In what follows, the coordinates on the new, "logified" space are denoted by the same letters but in different fonts. We typically distinguish the logified function spaces with a subscript "log".

Our assumptions on the metric g imply that dv is non-degenerate in a neighborhood of $S_\pm$. In particular, we now consider an atlas of coordinate charts $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n_+$ of this neighborhood so that ρ and v are always two of the coordinates. (We denote the remaining coordinates by φ^y_α.) Note that restricting our attention to such charts fixes a product decomposition near $S_\pm$.

Because $S_\pm$ are compact, there is some constant C so that $\{\rho = 0, |v| \leq C\}$ is covered by the union of the U_α. Fix now a function $\chi \in C^\infty_c(\mathbb{R})$ so that $\chi \equiv 1$ near 0 and $\chi(v) \equiv 0$ for $|v| \geq C$.

We now introduce the functions $\varrho = \rho$ and $\mathrm{v} = v + \chi(v) m\rho \log\rho$ and observe that the restriction of v to X agrees with v. We change the smooth structure of the manifold by defining a new atlas in the neighborhoods U_α. Indeed, we define charts $\tilde{\varphi}_\alpha : U_\alpha \to \tilde{V}_\alpha \subset \mathbb{R}^n_+$ on M by

$$\tilde{\varphi}_\alpha = (\varrho, \mathrm{v}, \varphi^y_\alpha).$$

In other words, we change the smooth structure of M by asking that the function v (rather than v) be smooth. We also use the notation $\mathrm{y} = y$ in these coordinates.

Because $v = \mathrm{v} - \chi m \varrho \log\varrho$, smooth functions on M (i.e., those admitting expansions in (ρ, v, y) with nonnegative integer exponents) no longer are smooth on M but instead admit expansions in ϱ, $\varrho\log\varrho$, v, and y with nonnegative integer exponents.

We now define the algebras of functions and of differential operators with mildly singular coefficients that we employ.

Definition 7.1. We let $\mathcal{C}^\infty(\mathrm{M})$ denote the coefficient ring of functions smooth on M (i.e., M equipped with the new smooth structure), while we let $\mathcal{C}^\infty_{\log}(\mathrm{M})$ denote the coefficient ring consisting of smooth functions of ϱ, $\varrho\log\varrho$, v, and y.

Observe that $\mathcal{C}^\infty_{\log}$ is also the set of distributions conormal to $X = \partial\mathrm{M}$ that are polyhomogeneous with index set

$$\mathcal{E}_{\mathcal{C}^\infty_{\log}} = \{(k,j) : k = 0,1,2,\dots, j = 0,1,\dots,k\}.$$

To clarify which manifold we are working on, we introduce the notation

$$\iota : M \to \mathrm{M}$$

for the tautological map between these two manifolds. We let

$$\iota^* : \mathcal{C}^\infty(\mathrm{M}) \to \mathcal{C}^\infty_{\log}(M)$$

denote the natural pullback map, and also, by modest abuse of notation, let

$$\iota_* : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty_{\log}(\mathrm{M}),$$

denote pullback under ι^{-1}. We will employ the analogous notation for push-forward (and pullback!) of vector fields as well.

Definition 7.2. We say that $P \in \mathrm{Diff}^k_{\mathrm{b},\log}(\mathrm{M})$ if $P \in \mathcal{C}^\infty_{\log}(\mathrm{M}) \otimes \mathrm{Diff}^k_{\mathrm{b}}(\mathrm{M})$. In other words, $P \in \mathrm{Diff}^k_{\mathrm{b},\log}(\mathrm{M})$ if there are coefficients $a_\alpha \in \mathcal{C}^\infty_{\log}(\mathrm{M})$ so that

$$P = \sum_{|\alpha|\leq k} a_\alpha D^\alpha,$$

where D^α are monomials in the vector fields ϱD_ϱ, D_{v}, and D_{y_j}.

Even though the lift of $\iota_* L$ of L to M lives in this space, we include here a more general class of operators for future work. In particular, we also allow terms of the form $\varrho\log\varrho D_\varrho$, which are not in $\mathrm{Diff}_{\mathrm{b},\log}$. We consider the slightly larger space $\widetilde{\mathrm{Diff}}^k_{\mathrm{b},\log}(\mathrm{M})$. Elements of this space have the form

$$\sum_{|\alpha|\leq k} a_\alpha D^\alpha,$$

where $a_\alpha \in \mathcal{C}^\infty_{\log}$ and D^α are monomials in the vector fields ϱD_ϱ, $\varrho\log\varrho D_\varrho$, D_{v}, and D_{y_j}. Observe that $\mathrm{Diff}^k_{\mathrm{b},\log} \subset \widetilde{\mathrm{Diff}}^k_{\mathrm{b},\log}$.[4]

Let $\mathcal{I} \subset \mathcal{C}^\infty$ denote the ideal of smooth functions vanishing at S_+, while $\mathcal{I}_{\log} \subset \mathcal{C}^\infty_{\log}$ is the ideal of $\mathcal{C}^\infty_{\log}$ functions vanishing at S_+. In other words, $f \in \mathcal{I}$ if there are smooth functions a_1 and a_2 so that $f = \varrho a_1 + \mathrm{v} a_2$, while $f \in \mathcal{I}_{\log}$ if there are $\mathcal{C}^\infty_{\log}$ functions a_1, a_2, and a_3 so that $f = \varrho a_1 + \mathrm{v} a_2 + (\varrho\log\varrho) a_3$.

---

[4] Unfortunately, this larger space is not a graded algebra (and $\widetilde{\mathrm{Diff}}^1_{\mathrm{b},\log}/\widetilde{\mathrm{Diff}}^0_{\mathrm{b},\log}$ is not a Lie algebra), but we avoid these problems by working with $\mathrm{Diff}_{\mathrm{b},\log}$ when possible.

We now define the module $\mathcal{M}_{\mathrm{D,log}} \subset \mathrm{Diff}^1_{\mathrm{b,log}}$ to be the module of vector fields logarithmically tangent to $\rho = 0$ and S_+, i.e. that map ϱ, v to $O(\varrho) + O(\varrho \log \varrho) + O(\mathrm{v})$. Over $\mathcal{C}^\infty_{\log}$, this module is generated by ϱD_ϱ, ϱD_{v}, $\mathrm{v} D_{\mathrm{v}}$, and D_{y}.

Finally, we define the "bad module" $\mathcal{N} \subset \widetilde{\mathrm{Diff}}^1_{\mathrm{b,log}}$ as the corresponding module in the larger space, so it is generated over $\mathcal{C}^\infty_{\log}$ by ϱD_ϱ, $\varrho \log \varrho D_\varrho$, ϱD_{v}, $\varrho \log \varrho D_{\mathrm{v}}$, $\mathrm{v} D_{\mathrm{v}}$, and D_{y_j}.

Observe that just as the module $\mathcal{M}$ maps $\mathcal{I}$ to itself, $\mathcal{M}_{\mathrm{D,log}}$ preserves $\mathcal{I}_{\log}$. The "bad module" $\mathcal{N}$ maps $\mathcal{I}$ to $\mathcal{I}_{\log}$.

Lemma 7.3. *We may characterize $\mathcal{M}_{\mathrm{D,log}}$ as*

$$\mathcal{M}_{\mathrm{D,log}} = \mathcal{C}^\infty_{\log} \otimes \mathcal{M}_{\mathrm{D}}(\mathrm{M})$$

and the following inclusion holds for $\mathcal{N}$:

$$\mathcal{N} \subset \mathcal{M}_{\mathrm{D,log}} + (\log \varrho)\mathcal{M}_{\mathrm{D,log}}.$$

Moreover,

$$\iota_* \mathcal{M}_{\mathrm{D}} \subset \mathcal{N},$$

while

$$\mathcal{N} \subset \iota_* \mathcal{M}_{\mathrm{D}}(M) + \iota_*(\log \rho)\mathcal{M}_{\mathrm{D}}(M).$$

Proof. The first statement follows because $\mathcal{M}_{\mathrm{D}}$ and $\mathcal{M}_{\mathrm{D,log}}$ are generated by the same vector fields but over different rings. The second statement follows by examination of the generators of $\mathcal{N}$.

For the final statement, we must only calculate the lifts of the generators. For instance,

$$\iota_* \rho D_\rho = \varrho D_\varrho + \varrho \left(\frac{\partial \mathrm{v}}{\partial \rho} D_{\mathrm{v}} \right) = \varrho D_\varrho + \varrho m (\log \varrho + 1) D_{\mathrm{v}} + O(\varrho^2 \log \varrho) D_{\mathrm{v}}.$$

□

A consequence of Lemma 7.3 is that the passage from M to M does not materially change the b-Sobolev spaces. In particular, we have the following equalities for all s and γ:

$$\iota^* H^{s,\gamma-0}_{\mathrm{b}}(\mathrm{M}) = H^{s,\gamma-0}_{\mathrm{b}}(M). \tag{7.1}$$

This means that, provided we are willing to lose a small amount of regularity and decay, neither the Sobolev nor conormal spaces change. A further consequence of this fact is that Proposition 5.4 implies that module regularity under $\mathcal{M}(M)$ immediately implies module regularity under $\mathcal{M}_{\mathrm{D,log}}(\mathrm{M})$ and $\mathcal{N}$. In particular, the bad module loses just an epsilon relative to the good one owing to log terms:

Proposition 7.4. *For each $k \in \mathbb{N}$, we have*

$$\mathcal{M}^N_{\mathrm{D}} u \in H^{s,\gamma}_{\mathrm{b}}(M)\ \forall N \Longrightarrow \mathcal{M}^N_{\mathrm{D}} \mathcal{N}^k \iota_* u \in H^{s,\gamma-0}_{\mathrm{b}}(\mathrm{M})\ \forall N. \tag{7.2}$$

We now easily verify the following:

Proposition 7.5. *The space* $\mathrm{Diff}^1_{\mathrm{b,log}}(\mathrm{M})/\mathrm{Diff}^0_{\mathrm{b,log}}(\mathrm{M})$ *consisting of b-vector fields with coefficients in* $\mathcal{C}^\infty_{\log}$ *is a Lie algebra;* $\mathrm{Diff}^*_{\mathrm{b,log}}(\mathrm{M})$ *is a graded algebra.*

Proof. The only new ingredient compared to the usual, smooth, case is the fact that

$$[\varrho, D_\varrho, \varrho\log\varrho] = \imath^{-1}(\varrho\log\varrho + \varrho) \in \mathcal{C}^\infty_{\log}.$$

□

An essential ingredient in our iterative argument will be the following refinement of Lemma 5.2; this is essentially the main point of our change of variables from v to v, which makes the $-4m\rho D_v^2$ term in the operator disappear.

Lemma 7.6. *We have*

$$\iota_* L = 4\partial_{\mathrm{v}}\left(\varrho\partial_\varrho + \mathrm{v}\partial_{\mathrm{v}}\right) + \mathcal{N}^2 \tag{7.3}$$

Proof. We note that in the coordinate change from v to v, we have

$$\iota_*\partial_v = (1+\chi'(v)m\varrho\log\varrho)\partial_{\mathrm{v}},\ \iota_*\partial_\rho = \partial_\varrho + \chi(v)m(1+\log\varrho)\partial_{\mathrm{v}}$$

and hence

$$\begin{aligned}\iota_* v\partial_v &= (\mathrm{v} - \chi m\varrho\log\varrho)\left(1+\chi' m\varrho\log\varrho\right)\partial_{\mathrm{v}}\\ \iota_*\rho\partial_\rho &= \varrho\partial_\varrho + \chi m\varrho(1+\log\varrho)\partial_{\mathrm{v}}\end{aligned}$$

Applying Lemma 7.3 yields (7.3). □

The following lemma is useful in Section 9; it shows that additional vanishing at S_+ in fact improves regularity.

Lemma 7.7. $\mathcal{I} \subset \Psi_{\mathrm{b}}^{-1}\mathcal{M}_{\mathrm{D}}$ *and* $\mathcal{I}_{\log} \subset \Psi_{\mathrm{b}}^{-1}\mathcal{M}_{\mathrm{D,log}}$.

Proof. We prove the lemma in the first case; the proof is nearly identical in the logified setting.

It suffices to show that $\rho, v \in \Psi_{\mathrm{b}}^{-1}\mathcal{M}_{\mathrm{D}}$. To do this, note that as a composition of operators,

$$\begin{pmatrix}\rho\partial_\rho\\ \partial_v\\ \partial_y\end{pmatrix} \circ v \in \begin{pmatrix}\mathcal{M}_{\mathrm{D}}\\ \vdots\\ \mathcal{M}_{\mathrm{D}}\end{pmatrix}.$$

The vector-valued b-operator on the left has a left-invertible symbol and thus has a left inverse in $(\Psi_{\mathrm{b}}^{-1},\dots,\Psi_{\mathrm{b}}^{-1})$ modulo $\Psi_{\mathrm{b}}^{-\infty}$, hence (since certainly $\Psi_{\mathrm{b}}^{-\infty} \subset \Psi_{\mathrm{b}}^{-1}\mathcal{M}_{\mathrm{D}}$) we have

$$v \in \Psi_{\mathrm{b}}^{-1}\mathcal{M}_{\mathrm{D}}.$$

The proof for ρ proceeds in the same way. □

We now discuss how asymptotic expansions are transformed by the logification process.

Proposition 7.8. *Let* $u \in \mathcal{A}^{E}_{phg}(M)$*. For each* $j \in \mathbb{N}$ *let*

$$E(j) \equiv \{(z - j\imath, \ell) : (z,k) \in E, 0 \le \ell \le k + j\}.$$

Let

$$E' \equiv E(0) \cup E(1) \cup E(2) \cup \dots$$

Then

$$\iota_* u \in \mathcal{A}^{E'}_{phg}(\mathrm{M}).$$

We note that an alternative definition of E', since E is an index set and therefore closed under $(z,k) \to (z - \imath, k)$, is in terms of extended unions as the set

$$E' = E \overline{\cup} E_1 \overline{\cup} E_2 \overline{\cup} \dots$$

with $E_j = \{(z - j\imath, k) : (z,k) \in E\}$.

Proof. We employ the method of testing by radial vector fields. Note that

$$\mathrm{R} \equiv \varrho D_\varrho = \iota_* \left(\rho D_\rho - \chi m \rho (1 + \log \rho) D_v\right) + \chi' m \varrho^2 \log \varrho (1 + \log \varrho) D_{\mathrm{v}}.$$

Thus,

$$\iota^* (\mathrm{R} - z)^{k+1} = (\rho D_\rho - z)^{k+1} + F,$$

where

$$F \in \rho \operatorname{Diff}^{k+1}_{\mathrm{b,log}}(M) + \rho \log \rho \operatorname{Diff}^{k+1}_{\mathrm{b,log}}(M),$$

and more precisely, F is a sum of products of smooth b-vector fields times coefficients containing powers of ρ and $\rho \log \rho$ between 1 and $k+2$ (though we do not need this characterization).

Now for any index set G let S denote the shift operation with increase of multiplicity:

$$S(G) \equiv \{(z - \imath, k + 1) : (z,k) \in G\}.$$

Hence $E(j+1) = S(E(j))$ and E' is closed under S.

Now, since application of b-vector fields preserves index sets while multiplication by ρ and $\rho \log \rho$ shifts them according to the map S, we find in general that if w has index set G on M and if $(z,k) \in G$, then

$$\iota^* (\mathrm{R} - z)^{k+1} w \in \mathcal{A}^{G_z}_{\mathrm{phg}}(M),$$

where

$$G_z = (G \setminus (z,k)) \cup S(G) \cup S^2(G) \cup \dots.$$

Letting z be the value of in G with largest imaginary part, we then see that this process yields an index set with strictly smaller imaginary parts. (If there are several with same imaginary part, we must of course repeat the process finitely many times.)

Now we apply this argument iteratively to u: if u has index set E, i.e., improved decay under application of

$$\prod_{(z,k) \in E} (\rho D_\rho - z)^{k+1},$$

then we pick (z_0, k_0) with largest imaginary part (again iterating if this is not unique) and note that

$$\iota^*(\mathrm{R} - z_0)^{k_0+1} u \in \mathcal{A}^{E_z}_{\mathrm{phg}}(M),$$

where now E_z is an index set with smaller imaginary part, and is contained in E'. Continuing inductively (and remembering at every stage that E' is conserved by S) we see that u has improved decay under application of

$$\iota^* \prod_{(z,k)\in E'} (\mathrm{R} - z)^{k+1}.$$

Pushing forward (and recalling that the scale of weighted Sobolev spaces is essentially unchanged by ι_*) we see that $\iota^* u$ has improved decay under

$$\prod_{(z,k)\in E'} (\mathrm{R} - z)^{k+1},$$

as desired. □

We will in practice need the version of this result that deals with the rougher expansions with coefficients conormal at S_+. To this end, we say that u lies in the L^2-based conormal space $I^{(s)}(\Lambda^+)$ if $u \in H^s(X)$ and $A_1 \dots A_k u \in H^s(X)$ for all $k \in \mathbb{N}$ and $A_j \in \mathcal{M}_{\mathrm{D}}$. We then have the following

Proposition 7.9. *If a distribution u on M conormal with respect to $N^* S_+$ enjoys an expansion*

$$u \sim \sum_E a_j(v, y) \rho^{\imath\sigma_j} (\log \rho)^{k_j} \tag{7.4}$$

with index set E and with

$$a_j \in I^{(q_0 - \mathrm{Re}(\imath\sigma_j))}$$

then on M, *$\iota_* u$ has an expansion*

$$\iota_* u \sim \sum_{E'} b_j(v, y) \rho^{\imath\sigma_j} (\log \rho)^{k_j}$$

where

$$b_j \in I^{(q_0 - \mathrm{Re}(\imath\sigma_j) - 0)}.$$

and where the index set

$$E' \equiv E(0) \cup E(1) \cup E(2) \cup \dots$$

with

$$E(j) \equiv \{(z - j\imath, k + \ell) : (z, k) \in E, 0 \leq \ell \leq j\},$$

Proof. The proof is just as in Proposition 7.8, using the oscillatory testing characterization (Proposition 2.2) but with the additional feature that we note that the logarithmic change of variables shifts conormal orders by ϵ for any $\epsilon > 0$. □

8. THE RADIATION FIELD BLOW-UP

In this section we recall from [2] the construction of the manifold $[\mathrm{M}; S]$ on which the radiation field lives.

We now *blow up* $S = \{\mathrm{v} = \varrho = 0\}$ in M by replacing it with its inward pointing spherical normal bundle.[5] This process replaces M with a new manifold with corners $[\mathrm{M}; S]$ on which polar coordinates in ϱ, v are smooth, and depends only on S and the smooth structure of M. The blow-up comes equipped with a natural blow-down map $[\mathrm{M}; S] \to \mathrm{M}$ which is a diffeomorphism on the interior. $[\mathrm{M}; S]$ is a manifold with corners with several boundary hypersurfaces: the closure of the lifts of C_0 and $C_\pm$ to $[\mathrm{M}; S]$, which we still denote C_0 and $C_\pm$, and $\mathscr{I}$, which we define as the lift of S to $[\mathrm{M}; S]$. Further, the fibers of $\mathscr{I}$ over the base, S, are diffeomorphic to intervals, and indeed, the interior of a fiber is naturally an affine space (i.e., these interiors have $\mathbb{R}$ acting by translations, but there is no natural origin).

Given v and ϱ, the fibers of the interior of $\mathscr{I}$ in $[\mathrm{M}; S]$ can be identified with $\mathbb{R}$ via the coordinate $s = \mathrm{v}/\varrho$. In particular, ∂_s is a well-defined vector field on the fibers.

In what follows, we note that $s = \mathrm{v}/\varrho$ is a smooth coordinate along $\mathscr{I}^+$, and s^{-1}, ϱ are respectively the defining functions of (the lift of) C_+ and $\mathscr{I}^+$. If we are interested in studying forward solutions, this corner and the two faces meeting at it are the only places where u has nontrivial asymptotics.

With the notation of the previous sections in hand, we finally restate our main theorem in more detail:

Theorem 1.1. *Let (M, g) be a non-trapping Lorentzian scattering manifold, and let*

$$\Box_g u = f$$

with $u \in \mathcal{C}^{-\infty}(M)$, $f \in \dot{\mathcal{C}}^\infty(M)$. Assume that u is a forward solution. Then u lifts to $[\mathrm{M}; S]$ to have a joint polyhomogeneous expansion at all boundary faces, vanishing except at the face C_+ and the front face $\mathscr{I}^+$ of the blowup of S_+. At that pair of faces the powers in the polyhomogeneous expansion are given by $\mathcal{E}_{tot}$ described above in Section 2.1, hence with terms that are powers of a defining function at C_+ described in terms of poles of the family of P_σ and at $\mathscr{I}^+$ given by terms $\rho^j (\log \rho)^\ell$ for $\ell = 0, \dots, 2j$ for $m \neq 0$ and simply by ρ^j for $m = 0$.

9. ASYMPTOTIC EXPANSIONS

We are now ready to derive the asymptotic expansion of solutions to the wave equation on M, thereby proving Theorem 1.1. In the case $m = 0$, such an asymptotic expansion was derived in [2], but adapting the argument given there would be rather cumbersome. Instead, we proceed with a different, and (we hope) more transparent, argument which in fact yields more.

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[5] The reader may wish to consult [17] for more details on the blow-up construction than we give here.

The proof will proceed in two steps, one for each boundary face of the radiation field blowup. By Proposition 2.1, it will suffice to obtain the asymptotics at each boundary face with uniform control of error terms at the other face. To begin, we work at C_+; while this argument will initially appear to be a global one near ∂M, the worsening error terms at S_+ will mean that this first step will yield only the asymptotics at C_+, uniformly up to $\mathscr{I}^+$ after the radiation field blowup.

As we use the following spaces many times, it is convenient to introduce a compact notation:

Definition 9.1. For $\varsigma, s \in \mathbb{R}$, we let

$$\mathfrak{B}(\varsigma, s) = \mathcal{H}(\mathbb{C}_\varsigma) \cap \langle \sigma \rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(s)}(\Lambda^+)).$$

9.1. Asymptotics at C_+.

We start by recalling a portion of the argument of [2] yielding asymptotic expansions at C_+.

As in Lemma 7.6, we write the operator $L = N(L) + E$, where $E \in \varrho \operatorname{Diff}^2_{\mathrm{b}}(M)$. We let R_σ be the family of operators intertwining E with the Mellin transform, i.e., satisfying

$$\mathcal{M} \circ E = R_\sigma \circ \mathcal{M}.$$

R_σ is thus an operator on meromorphic families in σ in which ϱD_ϱ is replaced by σ and multiplication by ϱ translates the imaginary part.

Note that since the mass term only appears with an $O(\rho)$ relative to the main terms in L when written as a b-operator, $N(L)$ is independent of m, hence agrees with the expression found in [2].

By Lemma 7.6 we have the following result on the mapping properties of R_σ. The mapping properties of R_σ are slightly worse here than in our previous work owing to the presence of a term of the form ρD_v^2 in L in the long-range setting.

Lemma 9.2 ([2], Lemma 9.1)**.** *For each ν, k, ℓ, s, the operator family R_σ enjoys the following mapping properties:*

1. *R_σ enlarges the region of holomorphy at the cost of regularity at Λ^+:*
$$\begin{aligned} R_\sigma : &\mathcal{H}(\mathbb{C}_\nu) \cap \langle \sigma \rangle^{-k} L^\infty L^2(\mathbb{R}; I^{(s)}(\Lambda^+)) \\ &\to \mathcal{H}(\mathbb{C}_{\nu+1}) \cap \langle \sigma \rangle^{-k+2} L^\infty L^2(\mathbb{R}; I^{(s-2)}(\Lambda^+)) \end{aligned} \tag{9.1}$$
2. *If f_σ vanishes near $\overline{C_-}$ for $\operatorname{Im} \sigma \geq -\nu$, then $R_\sigma f_\sigma$ also vanishes near $\overline{C_-}$ for $\operatorname{Im} \sigma \geq -\nu - 1$.*

As discussed above, we transform the equation

$$\Box_g u = f$$

by rescaling and conjugation to rewrite it as

$$Lw = g$$

where

$$L \equiv \rho^{-(n-2)/2-2} \Box_g \rho^{(n-2)/2}, \tag{9.2}$$

$$w = \rho^{-(n-2)/2} u \in \mathcal{C}^{-\infty}(M), \quad g = \rho^{-(n-2)/2-2} f \in \dot{\mathcal{C}}^{\infty}(M).$$

Thus, suppose $Lw = g$, where $g \in \dot{\mathcal{C}}^{\infty}(M)$ and u vanishes in a neighborhood of $\overline{C_-}$. Taking the Mellin transform, we obtain

$$P_\sigma \tilde{w}_\sigma = \tilde{g}_\sigma - R_\sigma \tilde{w}_\sigma. \tag{9.3}$$

As $g \in \dot{\mathcal{C}}^{\infty}(M)$, we have

$$\tilde{g}_\sigma \in \mathfrak{B}(C, s') \text{ for all } C, s'.$$

Because $\rho^{(n-2)/2} w$ lies in some $H_{\mathrm{b}}^{s,\gamma}(M)$, we have

$$\tilde{w}_\sigma \in \mathcal{H}(\mathbb{C}_{\varsigma_0}) \cap \langle\sigma\rangle^{\max(0,-s)} L^\infty L^2(\mathbb{R}; H^s), \tag{9.4}$$

where $\varsigma_0 = \gamma - (n-2)/2$. By reducing s, we may assume that $s + \gamma < 1/2$ so as to be able to apply the module regularity results of Proposition 5.4. We may also arrange that $\tilde{w}_\sigma$ vanishes in a neighborhood of $\overline{C_-}$ in X because, by hypothesis, w vanishes near $\overline{C_-}$ in M.

Because the metric is non-trapping, we know that w has module regularity with respect to $\mathcal{M}$, and so

$$\tilde{w}_\sigma \in \mathfrak{B}(\varsigma_0, -\infty),$$

and thus, by interpolation with (9.4),

$$\tilde{w}_\sigma \in \mathfrak{B}(\varsigma_0, s - 0).$$

In particular, $R_\sigma \tilde{w}_\sigma$ (and hence $P_\sigma \tilde{w}_\sigma$) lies in

$$\mathfrak{B}(\varsigma_0 + 1, s - 2 - 0)$$

Because $P_\sigma \tilde{w}_\sigma$ is known to be holomorphic in a larger half-plane, we can now invert P_σ to obtain meromorphy of $\tilde{w}_\sigma$ on this larger space: by Propositions 6.2 and 6.3, P_σ is Fredholm as a map

$$\mathcal{X}^{s_{\mathrm{ftr}}} \to \mathcal{Y}^{s_{\mathrm{ftr}}-1},$$

and P_σ^{-1} has finitely many poles in any horizontal strip $\operatorname{Im}\sigma \in [a, b]$. Moreover, P_σ^{-1} satisfies polynomial growth estimates as $|\operatorname{Re}\sigma| \to \infty$. Here we recall from Section 6 that given any ς', in order for P_σ to be Fredholm for $\sigma \in \mathbb{C}_{\varsigma'}$, the (constant) value $s(S_+)$ assumed by the variable Sobolev order s_{ftr} near S_+ must satisfy $s(S_+) < 1/2 - \varsigma'$; thus as one enlarges the domain of meromorphy for $\tilde{w}_\sigma$, one needs to relax the control of the derivatives. Thus $\tilde{w}_\sigma$ is obtained by applying P_σ^{-1} to the right hand side of (9.1); this term is meromorphic in $\mathbb{C}_{\varsigma_0+1}$ with values in

$$\langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; H^{\min(s-1-0, 1/2-\varsigma_0-1-0)}) \tag{9.5}$$

with (finitely many) poles in this strip, arising from the poles of P_σ^{-1}. Here (and below) we are ignoring the distinction between $\mathcal{X}^{s_{\mathrm{ftr}}}$ and H^s as $\tilde{w}_\sigma$ is trivial by hypothesis on the set where the regularity in the variable-order Sobolev space differs from H^s.

Now we can improve our description of the remainder terms (going back to Lemma 9.2 for the description of $R_\sigma \tilde{w}_\sigma$.) since P_σ maps the expression in question to $\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(s-2-0)})$. Thus the term (9.5) must in fact be meromorphic with values in the *conormal* space

$$\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{\min(s-1-0,1/2-\varsigma_0-1-0)}),$$

by propagation of singularities away from radial points (Proposition 4.1 of [2]) and the first case of Theorem 6.3 of [9], which deals with propagation of Lagrangian regularity into conic Lagrangian submanifolds of radial points.[6]

Thus we have now shown that

$$\tilde{w}_\sigma \in \mathfrak{B}(\varsigma_0+1, \min(s-1-0, 1/2-\varsigma_0-1-0)) \tag{9.6}$$

$$+ \sum_{\substack{(\sigma_j,m_j)\in\mathcal{E}_0\\ \operatorname{Im}\sigma_j>-\varsigma_0-1}} (\sigma-\sigma_j)^{-m_j} a_j, \tag{9.7}$$

where

$$a_j \in \mathfrak{B}(\varsigma_0+1, \operatorname{Im}\sigma_j + 1/2 - 0).$$

Here the conormal regularity of the coefficients of the polar part follows from the Cauchy integral formula.

We now iterate this argument as follows. (The argument is simpler than the analogous argument in [2], as we will allow derivative losses in our conormal spaces that we will recoup later.)

Assume inductively that

$$\tilde{w}_\sigma \in \mathfrak{B}(\varsigma_0+N, \min(s-N-0, 1/2-\varsigma_0-N-0)) + \dots + \sum_{\substack{(\sigma_j,m_j)\in\mathcal{E}_0\tilde{\cup}\dots\tilde{\cup}\mathcal{E}_N,\\ \operatorname{Im}\sigma_j>-\varsigma_0-N}} (\sigma-\sigma_j)^{-m_j} a_j, \tag{9.8}$$

with

$$a_j \in \mathfrak{B}(\varsigma_0+2N, 1/2+\operatorname{Im}\sigma_j - 0).$$

By Lemma 9.2,

(9.9)

$$R_\sigma\tilde{w}_\sigma \in \mathfrak{B}(\varsigma_0+N+1, \min(s-N-2-0, 1/2-\varsigma_0-N-1-0)) + \dots + \sum_{\substack{(\sigma_j,m_j)\in\mathcal{E}_0\tilde{\cup}\dots\tilde{\cup}\mathcal{E}_N\\ \operatorname{Im}\sigma_j>-\varsigma_0-N}} (\sigma-(\sigma_j-\imath))^{-m_{j,1}} a_j'$$

(9.10)

where

$$a_j' \in \mathfrak{B}(\varsigma_0+N, 1/2+\operatorname{Im}(\sigma_j-\imath)-0),$$

---

[6] Here Theorem 6.3 is applied pointwise in σ; the result there is not stated in terms of bounds (just as a membership in the claimed set), but just as in the case of Proposition 5.4 here, estimates can be recovered from the statement of Theorem 6.3 by the closed graph theorem or alternatively recovered from examination of the proof, which proceeds via such estimates.

We remark that all the shifted poles in the above expressions lie in the new index set

$$\mathcal{E}_1 \overline{\cup} \dots \overline{\cup} \mathcal{E}_{N+1}.$$

Now we may apply P_σ^{-1} as above to solve for $\tilde{w}_\sigma$ on the left hand side of (9.3) and find that (9.8) holds for all N, since the new poles introduced by the operator family are given by the extended union with $\mathcal{E}_0$. Inverse Mellin transforming this result then yields the following asymptotic expansion. We thus have the following:

Proposition 9.3. *Let $\mathcal{E}^0_{\mathrm{res}}$ be the massless resonance index set (with ς_0 chosen to ignore those resonances where $\tilde{w}_\sigma$ is a priori holomorphic). Then on M,*

$$w = \sum_{\substack{(\sigma_j,k)\in\mathcal{E}^0_{\mathrm{res}}\\ \operatorname{Im}\sigma_j>-l}} \varrho^{\imath\sigma_j}(\log\varrho)^k a_{jk} + w',$$

where, for $C = s+\varsigma_0$,

$$w' \in \varrho^l H_{\mathrm{b}}^{\min(C-l-0,1/2-\varsigma_0-l-0),\gamma}(M).$$

The coefficients a_{jk} are $\mathcal{C}^\infty$ functions of ϱ taking values in $I^{(1/2-\mathrm{Re}(\imath\sigma_j)-0)}$ and are supported in $\overline{C_+}$.

Moreover on the logified manifold M *we have*

$$\iota_* w = \sum_{\substack{(\sigma_j,k)\in\mathcal{E}_{\mathrm{res}}\\ \operatorname{Im}\sigma_j>-l}} \varrho^{\imath\sigma_j}(\log\varrho)^k b_{jk} + w',$$

where the coefficients b_{jk} have the same properties as the a_{jk} and

$$w' \in \varrho^l H_{\mathrm{b}}^{\min(C-l-0,1/2-\varsigma_0-l-0),\gamma}(\mathrm{M}).$$

Proof. The proposition follows by taking the inverse Mellin transform of P_σ^{-1} applied to (9.9): the polar terms yield the terms in the sum, while the "remainder" term arises from the first $2N$ terms in (9.9). This yields the expansion with index set $\mathcal{E}^0_{\mathrm{res}}$ on M. Now applying Proposition 7.9 gives us the corresponding expansion with index set $\mathcal{E}_{\mathrm{res}}$ on M. □

Remark 9.4. Note that the expansion appears somewhat unsatisfactory as the conormal regularity declines as the power of ϱ increases, but we will cope with this inconvenience later.

We now continue our discussion of asymptotic expansions working exclusively on M. In what follows, though the exact value of the constant C is irrelevant, it may be taken to be $s+\varsigma_0$.

As a consequence of Proposition 9.3, we have

$$w' = \left(\prod_{(\sigma_j,k)\in\mathcal{E}_{\mathrm{res}}(\varsigma_0),\operatorname{Im}\sigma_j>-l} (\varrho D_\varrho - \sigma_j)\right) w \in \varrho^l H_{\mathrm{b}}^{\min(C-l-0,1/2-\varsigma_0-l-0),\gamma}(M).$$

Now by Proposition 5.4, w enjoys module regularity with respect to $\varrho^l H_{\mathrm{b}}^{s',\gamma}(\mathrm{M})$ for some s'. Thus for all N',

$$\mathcal{M}_{\log}^{N'}\left(\prod_{(\sigma_j,k)\in\mathcal{E}_{\mathrm{res}}(\varsigma_0),\operatorname{Im}\sigma_j>-l}(\varrho D_\varrho-\sigma_j)\right)w\in\varrho^l H_{\mathrm{b}}^{s',\gamma}(\mathrm{M});$$

here we have of course used the fact that all the factors $(\varrho D_\varrho-\sigma_j)$ lie in $\mathcal{M}_{\log}$. Interpolation now yields for all N

$$\mathcal{M}_{\log}^{N}\left(\prod_{(\sigma_j,k)\in\mathcal{E}_{\mathrm{res}}(\varsigma_0),\operatorname{Im}\sigma_j>-l}(\varrho D_\varrho-\sigma_j)\right)w\in\varrho^l H_{\mathrm{b}}^{\min(C-l-0,1/2-\varsigma_0-l-0),\gamma}(\mathrm{M}).$$

Now $\mathcal{M}$ includes a basis of vector fields in $\mathcal{V}_b(\mathrm{M})$ with the exception of ∂_{v}, but $\mathrm{v}\partial_{\mathrm{v}}$ is in $\mathcal{M}$. This leads to:

Lemma 9.5. *If* $\mathcal{M}_{\log}^{\ell}w\in H_{\mathrm{b}}^{p,q}(\mathrm{M})$*, then* $\mathrm{v}^\ell w\in H_{\mathrm{b}}^{p+\ell,q}(\mathrm{M})$*. More generally, if* $\mathcal{M}_{\log}^{N+\ell}w\in H_{\mathrm{b}}^{p,q}(\mathrm{M})$*, then* $\mathcal{M}_{\log}^{N}\mathrm{v}^\ell w\in H_{\mathrm{b}}^{p+\ell,q}(\mathrm{M})$*.*

Proof. Since $D_{\mathrm{v}}\mathrm{v}\in\mathcal{M}_{\log}$, we have

$$(\varrho D_\varrho)^\alpha D_{\mathrm{y}}^\beta D_{\mathrm{v}}^\gamma \mathrm{v}^\ell\in\mathcal{M}_{\log},$$

provided $\gamma\le\ell$, hence by our assumed module regularity,

$$(\varrho D_\varrho)^\alpha D_{\mathrm{y}}^\beta D_{\mathrm{v}}^\gamma v^\ell w\in H_{\mathrm{b}}^{p,q}(M),$$

provided $\alpha+|\beta|+\gamma\le\ell$. □

Applying Lemma 9.5 now yields, for all N,

$$\mathcal{M}_{\log}^{N}\mathrm{v}^l\left(\prod_{(\sigma_j,k)\in\mathcal{E}_{\mathrm{res}}(\varsigma_0),\operatorname{Im}\sigma_j>-l}(\varrho D_\varrho-\sigma_j)\right)w\in\varrho^l H_{\mathrm{b}}^{\min(C-0,1/2-\varsigma_0-0),\gamma}(\mathrm{M}).$$

Since ϱ^{-l} commutes with all generators of $\mathcal{M}_{\log}$ except ϱD_ϱ and since $\varrho D_\varrho\varrho^{-l}=\varrho^{-l}\varrho D_\varrho+\imath l\varrho^{-l}$, induction on N shows that we may commute ρ^{-l} through the module factors to obtain

$$\mathcal{M}_{\log}^{N}\varrho^{-l}\mathrm{v}^l\left(\prod_{(\sigma_j,k)\in\mathcal{E}_{\mathrm{res}}(\varsigma_0),\operatorname{Im}\sigma_j>-l}(\varrho D_\varrho-\sigma_j)\right)w\in H_{\mathrm{b}}^{\min(C-0,1/2-\varsigma_0-0),\gamma}(\mathrm{M}).$$

In other words, if we set

$$\varpi=\varrho/\mathrm{v}$$

(ignoring $|\mathrm{v}/\varrho|<1$ for notational convenience) is the defining function of the side faces C_+ and C_0 in the blow-up $[\mathrm{M};S]$,

$$\mathcal{M}_{\log}^{N}\varpi^{-l}\left(\prod_{(\sigma_j,k)\in\mathcal{E}_{\mathrm{res}}(\varsigma_0),\operatorname{Im}\sigma_j>-l}(\varrho D_\varrho-\sigma_j)\right)w\in H_{\mathrm{b}}^{\min(C-0,1/2-\varsigma_0-0),\gamma}(\mathrm{M}),$$

which is to say, switching over entirely to coordinates $\varpi = \varrho/\mathrm{v}$, v, and y valid in a neighborhood of C_+ including the corner $C_+ \cap \mathscr{I}$, we finally have the following:

Proposition 9.6. *On* C_+, *uniformly up to the corner* $C_+ \cap \mathscr{I}^+$ *in* $[\mathrm{M}; S_+]$, *w enjoys an asymptotic expansion with powers given by the resonance index set:*

$$\left(\prod_{(\sigma_k, k) \in \mathcal{E}_{\mathrm{res}}(\varsigma_0), \operatorname{Im} \sigma_j > -l} (\varpi D_\varpi - \sigma_j) \right) w \in \varpi^l H_{\mathrm{b}}^{\infty,*,*}([\mathrm{M}; S]),$$

where the $*$*'s represent fixed (i.e., independent of* l*) growth orders.*

By Proposition 2.1, this is now one of the two ingredients required to prove Theorem 1.1 *in the short-range case*, giving us the expansion at C_+ uniformly up to $\mathscr{I}^+$. To complete the proof of Theorem 1.1 for the short-range case, it thus suffices to obtain the expansion at $\mathscr{I}^+$, with uniform control at C_+.

9.2. **Expansion at $\mathscr{I}^+$: the short-range case.**

In describing asymptotics at $\mathscr{I}^+$, *we now specialize to the short-range case* for the sake of clarity of exposition, before returning to the more general long-range case in the following section.

Throughout this section we use R to denote the vector field that lifts to be the radial vector field at $\mathscr{I}$, i.e.,

$$R = \rho D_\rho + v D_v.$$

We further set $\mathbf{R}_k$ to be the appropriate product of shifted radial vector fields to test for the $\mathcal{C}^\infty$ index set $0 \equiv \{(-j\imath, 0),\ j \in \mathbb{N}\}$. In other words, we have

$$\mathbf{R}_k = \prod_{j=0}^{k} (R + \imath j).$$

Here $\mathbf{R}_{-1}$ denotes the empty product, i.e., the identity operator.

We begin by treating the short range case, i.e., assuming that $m = 0$; what we will want to prove is that one has a polyhomogeneous expansion on $[M; S]$. This means that at the lift of C_+, which is still denoted by C_+, one has an expansion given by the resonances (this is Proposition 9.6 above, while at $\mathscr{I}$ one has smooth behavior (i.e., the standard expansion), and at the lift of C_0 there is rapid decay. These expansions at boundary hypersurfaces are supposed to fit together smoothly at the corners; we recall that by Proposition 2.1, the apparent challenge of verifying matching conditions is moot.

We must thus prove such an expansion at C_+ (with the resonance index set), C_0 (with the empty index set), and $\mathscr{I}$ (with the smooth index set). The C_+ expansion we have already obtained in both the short- and long-range cases: this is Proposition 9.6 above.

At C_0, the same argument applies, but, due to the support property of the resonant states (Lemma 6.4), the coefficients all vanish to infinite order. In particular, this means we need not apply the radial factors to obtain the vanishing. In other words, for all l

$$\mathcal{M}^N w \in \varpi^l H_{\mathrm{b}}^{s,\gamma}(U), \tag{9.11}$$

where U is a neighborhood of C_0 in $[M;S]$ on which ϖ is bounded above[7] (say, $\varpi < 1$). Put differently,

$$w \in \varpi^l H^{\infty,*,*}(U).$$

We now turn to $\mathscr{I}$. In dealing with the expansion near $\mathscr{I}$ we consider the $\mathcal{C}^\infty(M)$-submodule $\mathcal{M}_{\mathrm{D}}$ of $\mathcal{M}$ consisting of the first order differential operators in $\mathcal{M}$. Because $\mathcal{M}$ is generated as a module over Ψ^0_{b} by differential operators, regularity with respect to the module $\mathcal{M}_{\mathrm{D}}$ is equivalent to regularity with respect to $\mathcal{M}$.

Lemma 9.7. *With $\mathcal{I}$ the ideal of $\mathcal{C}^\infty$ functions vanishing at S, one has*

$$(R + \imath k)\mathcal{I} \subset \mathcal{I}(R + \imath(k-1)) + \mathcal{I}^2$$

and

$$[R, \mathcal{M}_{\mathrm{D}}] \subset \rho\mathcal{M}_{\mathrm{D}} + v\mathcal{M}_{\mathrm{D}} = \mathcal{I}\mathcal{M}_{\mathrm{D}}.$$

The second part of the lemma is related to the statement that $\mathcal{M}$ lifts to b-pseudodifferential operators on $[M;S]$, while $\rho D_\rho + vD_v$ is the radial vector field associated with the front face, which has a commutative normal operator. Thus, its commutator with anything has an extra order of vanishing (i.e., in ρ or v) at the front face.

Proof. First if $a \in \mathcal{C}^\infty(M)$, then

$$[R, a] = \rho D_\rho a + v D_v a \in \mathcal{I}. \tag{9.12}$$

Since elements of $\mathcal{I}$ are of the form $\rho a_1 + v a_2$ with $a_j \in \mathcal{C}^\infty(M)$, and since

$$(R+\imath)\rho = \rho R,\ (R+\imath)v = vR,$$

we have

$$\begin{aligned}(R+\imath k)(\rho a_1 + v a_2) &= \rho(R+\imath(k-1))a_1 + v(R+\imath(k-1))a_2 \\ &= \rho[R,a_1] + \rho a_1(R+\imath(k-1)) + v[R,a_2] + v a_2(R+\imath(k-1)),\end{aligned}$$

with the commutators on the right hand side in $\mathcal{I}$ as remarked at the outset, so the membership of the right hand side in $\mathcal{I}(R+\imath(k-1)) + \mathcal{I}^2$ follows.

Turning to $[R, \mathcal{M}_{\mathrm{D}}]$, using (9.12) again, it suffices to show for a set of generators V_j of $\mathcal{M}_{\mathrm{D}}$ that $[R, V_j] \in \mathcal{I}\mathcal{M}_{\mathrm{D}}$. Using ρD_ρ, vD_v, ρD_v, and D_{y_j} in local coordinates as generators, all commute with R since they are homogeneous of degree zero under the action of dilations $(\rho, v, y) \to (t\rho, tv, y)$, $t > 0$, in the first two variables. □

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[7] Although U is a subset of the blow-up, we abuse notation by treating it as a subset of $\mathcal{M}$ in defining this weighted Sobolev space with a single weight.

In fact, more generally one has

Lemma 9.8. *With $\mathcal{I}$ as above:*

$$\begin{aligned}[\mathbf{R}_k, \mathcal{M}_{\mathrm{D}}^l] &\subset \sum_{j=0}^{k} \mathcal{I}^{j+1} \mathcal{M}_{\mathrm{D}}^l \mathbf{R}_{k-1-j} \\ &\subset \sum_{j=0}^{k+1} \Psi_{\mathrm{b}}^{-j-1} \mathcal{M}_{\mathrm{D}}^{l+j+1} \mathbf{R}_{k-1-j}\end{aligned}$$

Here the vanishing factor of powers of $\mathcal{I}$ arises from the classicality of the coefficients, so if one has logarithmic coefficients, one needs additional factors of the radial vector field plus appropriate constants.

Proof. The second inclusion in the statement of the lemma follows from $\mathcal{I} \subset \Psi_{\mathrm{b}}^{-1}\mathcal{M}_{\mathrm{D}}$, which we prove in Lemma 7.7.

First consider $k = 0$, i.e., $[R, V_1 \dots V_l]$ with $V_j \in \mathcal{M}_{\mathrm{D}}$. This is of the form

$$[R, V_1]V_2 \dots V_l + V_1[R, V_2]V_3 \dots V_l + \dots + V_1 \dots V_{l-1}[R, V_l],$$

and the commutators are in $\mathcal{I}\mathcal{M}_{\mathrm{D}}$ by the second half of Lemma 9.7. Now, as $[\mathcal{M}_{\mathrm{D}}, \mathcal{I}] \subset \mathcal{I}$, one can commute the $\mathcal{I}$ factors to the front iteratively. This proves the $k = 0$ case, namely that $[R, V_1 \dots V_l] \subset \mathcal{I}\mathcal{M}_{\mathrm{D}}^l$.

Now suppose $k \geq 1$, and that the lemma has been proved with k replaced by $k - 1$. Then

$$[\mathbf{R}_k, \mathcal{M}_{\mathrm{D}}^l] \subset (R + \imath k)[\mathbf{R}_{k-1}, \mathcal{M}_{\mathrm{D}}^l] + [R, \mathcal{M}_{\mathrm{D}}^l]\mathbf{R}_{k-1} \tag{9.13}$$

By the inductive hypothesis the first term on the right hand side is in

$$(R + \imath k) \sum_{j=0}^{k-1} \mathcal{I}^{j+1} \mathcal{M}_{\mathrm{D}}^l \mathbf{R}_{k-2-j}$$

Commuting $R+\imath k$ through the ideal factors using the first half of Lemma 9.7 iteratively, this itself lies in

$$\sum_{j=0}^{k-1} \Big(\mathcal{I}^{j+1} (R + \imath(k - j - 1)) \mathcal{M}_{\mathrm{D}}^l \mathbf{R}_{k-2-j} + \mathcal{I}^{j+2} \mathcal{M}_{\mathrm{D}}^l \mathbf{R}_{k-2-j} \Big)$$

By the $k = 0$ case, commuting $(R + \imath j)$ factors on the left of $\mathcal{M}_{\mathrm{D}}^l$ to the right gives commutators in $\mathcal{I}\mathcal{M}_{\mathrm{D}}^l$, so this expression is in

$$\sum_{j=0}^{k-1} \Big(\mathcal{I}^{j+1} \mathcal{M}_{\mathrm{D}}^l \mathbf{R}_{k-1-j} + \mathcal{I}^{j+2} \mathcal{M}_{\mathrm{D}}^l \mathbf{R}_{k-2-j} \Big) \subset \sum_{j=1}^{k} \mathcal{I}^{j+1} \mathcal{M}_{\mathrm{D}}^l \mathbf{R}_{k-1-j}.$$

which is of the form in the statement of the lemma. On the other hand, the second term in (9.13) is

$$[R, \mathcal{M}_{\mathrm{D}}^l]\mathbf{R}_{k-1},$$

so by the $k = 0$ case we get

$$\mathcal{I}\mathcal{M}_{\mathrm{D}}^l \mathbf{R}_{k-1}$$

for this term, which is of the form given in the last term in the statement of the lemma. ☐

The main claim is:

Proposition 9.9. *If* $w \in H_{\mathrm{b}}^{s,\gamma}(M)$ *with* $Lw \in \dot{\mathcal{C}}^\infty(M)$, *we have*

$$\mathcal{M}_{\mathrm{D}}^N \mathbf{R}_k w \in H_{\mathrm{b}}^{s+(k+1),\gamma}(M).$$

Notice that this proposition improves the b-regularity, but not the decay; in particular, this does *not* involve normal operators. However, once we have this, we can use the infinite order vanishing at C_0 to establish vanishing at the front face, as we show below.

Proof. The result follows from Proposition 5.4 if there are no radial vector factors (so $k = -1$). If $k = 0$, notice that

$$L + 4D_v(vD_v + \rho D_\rho) \in \mathcal{M}_{\mathrm{D}}^2,$$

so $Lw \in \dot{\mathcal{C}}^\infty$ and $\mathcal{M}_{\mathrm{D}}^N w \in H_{\mathrm{b}}^{s,\gamma}$ for all N implies that $D_v R w \in H_{\mathrm{b}}^{s,\gamma}$ by (5.1). Because D_v is elliptic on $\mathrm{WF}_{\mathrm{b}}(w)$, this yields $\mathbf{R}_0 w \in H_{\mathrm{b}}^{s+1,\gamma}(M)$. To finish the $k = 0$ case, we now rewrite $D_v \mathcal{M}_{\mathrm{D}}^N \mathbf{R}_0 w$ by commuting D_v with $\mathcal{M}_{\mathrm{D}}$. In particular, it suffices to consider the usual set of generators for $\mathcal{M}_{\mathrm{D}}$; the only one not commuting with D_v is vD_v, but $D_v(vD_v) = (D_v v)D_v \in \mathcal{M}_{\mathrm{D}} D_v$, so $D_v \mathcal{M}_{\mathrm{D}} \subset \mathcal{M}_{\mathrm{D}} D_v + \mathcal{M}_{\mathrm{D}}$. Thus, iterating we find that

$$D_v \mathcal{M}_{\mathrm{D}}^N \subset \mathcal{M}_{\mathrm{D}}^N D_v + \mathcal{M}_{\mathrm{D}}^N.$$

Consequently, we obtain

$$\begin{aligned} D_v \mathcal{M}_{\mathrm{D}}^N R w &\subset \mathcal{M}_{\mathrm{D}}^N D_v R w + \mathcal{M}_{\mathrm{D}}^N R w \\ &\subset \mathcal{M}_{\mathrm{D}}^N L w + \mathcal{M}_{\mathrm{D}}^{N+2} w \subset H_{\mathrm{b}}^{s,\gamma}(M). \end{aligned}$$

The ellipticity of D_v on $\mathrm{WF}_{\mathrm{b}}(u)$ now proves the $k = 0$ case of the proposition.

Now suppose $k \geq 1$, and that the proposition has been proved with k replaced by $k - 1$. We use then that

$$D_v \mathbf{R}_k = \mathbf{R}_{k-1} D_v R,$$

so

$$D_v \mathbf{R}_k \in \mathbf{R}_{k-1} L + \mathbf{R}_{k-1} \mathcal{M}_{\mathrm{D}}^2 \subset \mathbf{R}_{k-1} L + \sum_{j=0}^{k} \Psi_{\mathrm{b}}^{-j} \mathcal{M}_{\mathrm{D}}^{j+2} \mathbf{R}_{k-1-j},$$

where we applied Lemma 9.8 for the last inclusion. Thus, using the inductive hypothesis,

$$D_v \mathbf{R}_k w \in H_{\mathrm{b}}^{s+k,\gamma}.$$

Again, as D_v is elliptic in the microlocally relevant region,

$$\mathbf{R}_k w \in H_{\mathrm{b}}^{s+(k+1),\gamma}.$$

A similar result holds even with a factor $\mathcal{M}_{\mathrm{D}}^N$ added, by the same argument as in the $k = 0$ case, which completes the proof of the proposition. ☐

We now use the proposition, which as pointed out gives additional regularity without additional decay, to prove vanishing at the front face using the infinite order vanishing at C_0. First, fixing $v_0 < 0$, we already have $O(\rho^\infty)$ bounds for w near v_0. Further, we have the following estimate near C_0:

Lemma 9.10. *Let U be a neighborhood of $\overline{C_0}$ in $[M;S]$ as above. Then for any $\epsilon > 0$ and $N, N' \in \mathbb{N}$,*

$$D_v^{k+1}\mathcal{M}_{\mathrm{D}}^N \mathbf{R}_k w \in (\rho/v)^{N'} v^{-\epsilon} H_{\mathrm{b}}^{s,\gamma}(U). \tag{9.14}$$

Proof. Without the $(\rho/v)^{N'}$ or $v^{-\epsilon}$ factors, the desired estimate is just the regularity statement of Proposition 9.9. On the other hand, since $\mathbf{R}_k \in \mathcal{M}_{\mathrm{D}}^{k+1}$, the decay statement (9.11) yields the growth/decay statement

$$\mathcal{M}_{\mathrm{D}}^{\tilde{N}} \mathbf{R}_k w \in (\rho/v)^{N'} H_{\mathrm{b}}^{s,\gamma}(U).$$

As $v^{k+1}D_v^{k+1} \in \mathcal{M}_{\mathrm{D}}^{k+1}$, we then have

$$D_v^{k+1}\mathcal{M}_{\mathrm{D}}^N \mathbf{R}_k w \in v^{-k-1}(\rho/v)^{N'} H_{\mathrm{b}}^{s,\gamma}(U).$$

Fixing k, taking N' large, and interpolating with Proposition 9.9 completes the proof. ☐

Now integrating (9.14) $k+1$ times in v from v_0 gives

$$\mathcal{M}_{\mathrm{D}}^N \mathbf{R}_k w \in v^{-\epsilon}(\rho^{N'} + \rho^{N'} v^{-N'+k+1}) H_{\mathrm{b}}^{s,\gamma}(U),$$

which is, with $N' > k+1$, an order $k+1-\epsilon$ vanishing statement at the front face in the region U where $|\rho/v|$ is bounded.

Having this decay in U, we can proceed further into the front face. Since w has no b-wavefront set except at $\rho = v = 0$, it is in particular smooth in v, and we can rewrite $\mathcal{M}\mathbf{R}_k w$ as an iterated integral of its $(k+1)$-st derivative in v. Integrating from, say, $v = -\rho/2$, this gives an estimate

$$\mathcal{M}_{\mathrm{D}}^N \mathbf{R}_k w \in (v + C\rho)^{k+1} H_{\mathrm{b}}^{s,\gamma}(M),$$

with $v + C\rho$ being the length of the integration curve in the coordinates v, y, $\varpi \equiv \rho/v$ valid in a neighborhood of the interior of $\mathscr{I}^+$. Now we lift the module regularity statement on M to the blowup: since the generators of the module span a basis of b-vector fields on $[M;S]$, the module regularity lifts to give H_{b}^∞ regularity on the blowup (as the generators of the module lift to nondegenerate b-vector fields on the blow-up), i.e., the module regularity means that

$$\mathbf{R}_k w \in (v + C\rho)^{k+1} H_{\mathrm{b}}^{N,*,*}([M;S])$$

for suitable fixed (i.e., k-independent) weights $*$. As $v + C\rho$ defines the front face in the relevant region, this is exactly the desired polyhomogeneity statement at $\mathscr{I}$.

This finishes the proof of Theorem 1.1 in the short-range case.

9.3. **Expansion at $\mathscr{I}^+$: the long-range case.** In this section we return to the general setting $m \neq 0$. We proceed in much the same way as in the previous section, though significant modifications arise from the presence of $\varrho \log \varrho$ terms. In particular, the main difference is that, while in the short range case, we showed that w was polyhomogeneous at $\mathscr{I}$ with index set

$$\mathcal{E}_{\text{smooth}} = \{(-\imath k, 0) : k = 0, 1, 2, \dots\},$$

in the long-range setting we show that w is polyhomogeneous at $\mathscr{I}$ with index set

$$\mathcal{E}_{\log} = \{(-\imath k, j) : k = 0, 1, 2, \dots, j = 0, 1, \dots, 2k\}.$$

In what follows, we will abuse notation by letting w denote $\iota_* w$, its push-forward from M to M.

Let $\mathbf{R}_k$ be given by the following product of radial vector fields:

$$\mathbf{R}_k = \prod_{j=0}^{k} (\varrho D_\varrho + \mathrm{v} D_{\mathrm{v}} + \imath j)^{2j+1}$$

On $[\mathrm{M}; S]$ with coordinates ϱ, v, ϖ, observe that $\mathbf{R}_k$ has the following form:

$$\mathbf{R}_k = \prod_{j=0}^{k} (\varrho D_\varrho + \imath j)^{2j+1}$$

In other words, $\mathbf{R}_k$ is the appropriate product of radial vector fields at $\mathscr{I}$ to test for polyhomogeneity with index set $\mathcal{E}_{\log}$. For convenience with our bookkeeping, we also define the k-th triangular number as follows:

$$t_{-1} = 0, \quad t_k = t_{k-1} + k$$

As in the short range case, the support property of the resonant states means that all coefficients vanish to infinite order at C_0, so that for all ℓ, we have

$$w \in \varpi^\ell H_{\mathrm{b}}^{\infty,*,*}(U),$$

where U is a neighborhood of C_0 in $[\mathrm{M}; S]$ on which ϖ is bounded above.

The main difference in the proof concerns the behavior at $\mathscr{I}$. In the previous section, the crux of the proof was Proposition 9.9. The replacement for this proposition is the following:

Proposition 9.11. *If $w \in H_{\mathrm{b}}^{s,\gamma}(\mathrm{M})$ with $Lw \in \dot{\mathcal{C}}^\infty(M)$, we have*

$$\mathcal{M}_{\mathrm{D},\log}^{N} \mathbf{R}_k w \in H_{\mathrm{b}}^{s+(k+1),\gamma-0}(\mathrm{M}).$$

We defer for now a discussion of the proof of Proposition 9.11 and note that the following analogue of Lemma 9.10 immediately follows (with the same proof):

Lemma 9.12. *Let U be a neighborhood of C_0 in $[\mathrm{M}; S]$ as in the discussion immediately preceding equation (9.11). Then for any $\epsilon > 0$ and $N' \in \mathbb{N}$,*

$$D_{\mathrm{v}}^{k+1} \mathcal{M}_{\mathrm{D},\log} \mathbf{R}_k w \in (\varrho/\mathrm{v})^{N'} \mathrm{v}^{-\epsilon} H_{\mathrm{b}}^{s,\gamma-0}(U). \tag{9.15}$$

As in the previous section, we can then integrate $k+1$ times in v to obtain the desired vanishing (and hence polyhomogeneity) statements at $\mathscr{I}$. This completes the proof of Theorem 1.1 in the long-range case.

We now turn our attention to the proof of Proposition 9.11. Suppose we are able to prove the following lemma (which is the analogue of Lemma 9.8):

Lemma 9.13. *If* $w \in H_{\mathrm{b}}^{s,\gamma}$ *is as above and* $Lw \in \dot{\mathcal{C}}^\infty(\mathrm{M})$*, then*

$$D_{\mathrm{v}} \mathbf{R}_k w \in \sum_{j=0}^{k} \mathcal{I}_{\log}^{j} \mathcal{M}_{\mathrm{D,log}}^{1+2(t_k - t_{k-1-j})} \mathcal{N}^2 \mathbf{R}_{k-1-j} w + \dot{\mathcal{C}}^\infty(\mathrm{M}).$$

Proof of Proposition 9.11. The proposition holds if there are no factors of the radial vector field (i.e., if $k=-1$) by propagation of singularities (Proposition 5.4). If $k=0$, we notice that

$$L + 4 D_{\mathrm{v}} \mathbf{R}_0 \in \mathcal{N}^2,$$

so because $Lw \in \dot{\mathcal{C}}^\infty$ and $\mathcal{N}^2 w \in H_{\mathrm{b}}^{s,\gamma-0}$ (by (7.2)), we have $D_{\mathrm{v}} \mathbf{R}_0 \in H_{\mathrm{b}}^{s,\gamma-0}$. Because D_{v} is elliptic on $\mathrm{WF}_{\mathrm{b}}(w)$, this yields $\mathbf{R}_0 w \in H_{\mathrm{b}}^{s+1,\gamma-0}$.

To finish the $k=0$ case, we now rewrite $D_{\mathrm{v}} \mathcal{M}_{\mathrm{D,log}}^{N} \mathbf{R}_0 w$ by commuting D_{v} with $\mathcal{M}_{\mathrm{D,log}}$. In particular, $D_{\mathrm{v}} \mathcal{M}_{\mathrm{D,log}} \subset \mathcal{M}_{\mathrm{D,log}} D_{\mathrm{v}} + \mathcal{M}_{\mathrm{D,log}}$ and so

$$D_{\mathrm{v}} \mathcal{M}_{\mathrm{D,log}}^{N} \subset \mathcal{M}_{\mathrm{D,log}}^{N} D_{\mathrm{v}} + \mathcal{M}_{\mathrm{D,log}}^{N}.$$

We thus obtain

$$\begin{aligned} D_v \mathcal{M}_{\mathrm{D,log}}^{N} \mathbf{R}_0 w &\subset \mathcal{M}_{\mathrm{D,log}}^{N} D_{\mathrm{v}} \mathbf{R}_0 w + \mathcal{M}_{\mathrm{D,log}}^{N} \mathbf{R}_0 w \\ &\subset \mathcal{M}_{\mathrm{D,log}}^{N} L w + \mathcal{M}_{\mathrm{D,log}}^{N} \mathcal{N}^2 w \subset H_{\mathrm{b}}^{s,\gamma-0}(\mathrm{M}). \end{aligned}$$

The ellipticity of D_{v} on $\mathrm{WF}_{\mathrm{b}}(w)$ now proves the $k=0$ case of the proposition.

Now suppose $k \geq 1$ and that the proposition has been proved with k replaced by $k-1$. We then use Lemma 9.13 to see that

$$\begin{aligned} D_{\mathrm{v}} \mathbf{R}_k w &\in \sum_{j=0}^{k} \mathcal{I}_{\log}^{j} \mathcal{M}_{\mathrm{D,log}}^{1+2(t_k - t_{k-1-j})} \mathcal{N}^2 \mathbf{R}_{k-1-j} w \\ &\subset \sum_{j=0}^{k} \Psi_{\mathrm{b}}^{-j} \mathcal{M}_{\mathrm{D,log}}^{1+j+2(t_k - t_{k-1-j})} \mathcal{N}^2 \mathbf{R}_{k-1-j} w \in H_{\mathrm{b}}^{s+k,\gamma-0}(\mathrm{M}) \end{aligned}$$

by the induction hypothesis. Because D_{v} is elliptic in the microlocally relevant region, we see that $\mathbf{R}_k w \in H_{\mathrm{b}}^{s+(k+1),\gamma-0}(\mathrm{M})$.

A similar result holds even with a factor $\mathcal{M}_{\mathrm{D,log}}^{N}$ added, by the same argument as in the $k=0$ case (and using the fact that $\mathcal{M}_{\mathrm{D,log}}$ preserves $\mathcal{C}_{\log}^{\infty}$); this completes the proof of the proposition. □

We now turn our attention to the proof of Lemma 9.13. The intuitive idea behind the proof is as before, namely that commuting the radial vector field through the various factors yields an improvement. Unfortunately, it is a bit more complicated than in the short-range case:

Lemma 9.14. *Let* $R = \varrho D_\varrho + \mathrm{v} D_{\mathrm{v}}$*. The following relations hold:*

1. $(R + \imath k)\mathcal{N} \subset \mathcal{N}(R + \imath k) + \mathcal{M}_{\mathrm{D,log}}$
2. $\mathcal{N}\mathcal{M}_{\mathrm{D,log}} \subset \mathcal{M}_{\mathrm{D,log}}\mathcal{N}$
3. $(R + \imath k)\mathcal{M}_{\mathrm{D,log}} \subset \mathcal{M}_{\mathrm{D,log}}(R + \imath k) + \mathcal{I}_{\mathrm{log}}\mathcal{M}_{\mathrm{D,log}}$
4. $\mathcal{M}_{\mathrm{D,log}}\mathcal{I}_{\mathrm{log}} \subset \mathcal{I}_{\mathrm{log}}\mathcal{M}_{\mathrm{D,log}}$
5. $(R + \imath k)\mathcal{I}_{\mathrm{log}} \subset \mathcal{I}_{\mathrm{log}}(R + \imath(k-1)) + \varrho\mathcal{C}^\infty_{\mathrm{log}} + \mathcal{I}^2_{\mathrm{log}}$
6. $(R + \imath k)\varrho\mathcal{C}^\infty_{\mathrm{log}} \subset \varrho\mathcal{C}^\infty_{\mathrm{log}}(R + \imath(k-1)) + \varrho\mathcal{I}_{\mathrm{log}}$

Proof. We first observe that if $a \in \mathcal{I}_{\mathrm{log}}$, then $[R, a] = \varrho D_\varrho a + \mathrm{v} D_{\mathrm{v}} a \in \mathcal{I}_{\mathrm{log}}$.

Now we observe that $[R, \varrho \log \varrho D_\varrho] = \frac{1}{\imath}\varrho D_\varrho \in \mathcal{M}_{\mathrm{D,log}}$ and $[R, \varrho \log \varrho D_{\mathrm{v}}] = \frac{1}{\imath}\varrho D_{\mathrm{v}} \in \mathcal{M}_{\mathrm{D,log}}$. Any element of $\mathcal{N}$ can be written as $V + a_1 \varrho \log \varrho D_\varrho + a_2 \varrho \log \varrho D_{\mathrm{v}}$, where $V \in \mathcal{M}_{\mathrm{D,log}}$ and $a_i \in \mathcal{C}^\infty_{\mathrm{log}}$, proving the first statement.

The second statement follows from the observation that $[\mathcal{M}_{\mathrm{D,log}}, \mathcal{N}] \subset \mathcal{N}$.

The proof of Lemma 9.7, together with the observation that $[R, a] \in \mathcal{I}_{\mathrm{log}}$ shows that the third statement holds.

The fourth statement follows from the observation that $[\mathcal{M}_{\mathrm{D,log}}, \mathcal{I}_{\mathrm{log}}] \subset \mathcal{I}_{\mathrm{log}}$.

For the fifth statement, we compute. The proof of Lemma 9.7 shows that the statement is true for elements of $\mathcal{I}_{\mathrm{log}}$ of the form $a_1\varrho + a_2\mathrm{v}$, so we must only show it for elements of the form $a\varrho \log \varrho$, where $a \in \mathcal{C}^\infty_{\mathrm{log}}$. We then compute

$$(R + \imath k)a\varrho \log \varrho = a\varrho \log \varrho(R + \imath(k-1)) + \frac{1}{\imath}\rho a + \varrho \log \varrho (Ra),$$

which lies in the desired space.

The final statement is similar. Suppose $a \in \mathcal{C}^\infty_{\mathrm{log}}$, then

$$(R + \imath k)\rho a = \rho a(R + \imath(k-1)) + \rho(Ra).$$

□

By repeatedly applying Lemma 9.14, we obtain the following iterative version of the lemma:

Lemma 9.15. *Suppose* α*,* β*,* γ*,* δ*, and* ϵ *are integers, and that* $\gamma \geq 1$*. Let* $R = \varrho D_\varrho + \mathrm{v} D_{\mathrm{v}}$ *and let* $\tilde{R}^j$ *denote any product of* j *shifts of the radial vector field* R*. We then have that*

- $(R + \imath k)^\epsilon \rho^\alpha \subset \rho^\alpha (R + \imath(k - \alpha))^\epsilon + \sum_{i=1}^{\epsilon} \sum_{a=0}^{\min(i, \epsilon - i)} \rho^{\alpha + a} \mathcal{I}^{i-a}_{\mathrm{log}} \tilde{R}^{\epsilon - i - a}$
- $$(R + \imath k)^\epsilon \mathcal{I}^\beta_{\mathrm{log}} \subset \sum_{a=0}^{\min(\epsilon, \beta)} \rho^a \mathcal{I}^{\beta - a}_{\mathrm{log}} (R + \imath(k - \beta))^{\epsilon - a} + \sum_{i=1}^{\epsilon} \sum_{a=0}^{\min(\epsilon - i, \beta + i)} \rho^a \mathcal{I}^{\beta + i - a}_{\mathrm{log}} \tilde{R}^{\epsilon - i - a}$$

- $$(R+\imath k)^{\epsilon}\mathcal{M}^{\gamma}_{\mathrm{D,log}}\mathcal{N}^{\delta} \subset \sum_{d=0}^{\min(\delta,\epsilon)} \mathcal{M}^{\gamma+d}_{\mathrm{D,log}}\mathcal{N}^{\delta-d}(R+\imath k)^{\epsilon-d} + \sum_{d=0}^{\min(\delta,\epsilon)}\sum_{i=1}^{\epsilon-d}\sum_{a=0}^{\min(i,\epsilon-d-i)} \rho^{a}\mathcal{I}^{i-a}_{\log}\mathcal{M}^{\gamma+d}_{\mathrm{D,log}}\mathcal{N}^{\delta-d}\tilde{R}^{\epsilon-d-i-a}$$

Remark 9.16. Lemma 9.14 implies that $(R+\imath k)^{\epsilon}\varrho^{\alpha}\mathcal{I}^{\beta}_{\log}\mathcal{M}^{\gamma}_{\mathrm{D,log}}\mathcal{N}^{\delta}$ is contained in a sum of terms of the form

$$\varrho^{a}\mathcal{I}^{b}_{\log}\mathcal{M}^{c}_{\mathrm{D,log}}\mathcal{N}^{d}\tilde{R}^{e},$$

where all exponents are nonnegative, $\gamma+\delta=c+d$, and $2\alpha+\beta+2\gamma+\delta+\epsilon=2a+b+2c+d+e$. The leading terms are those with $a+b=\alpha+\beta$.

Proof. The main idea that one can "spend" a power of $(R+\imath k)$ to do one of the following:

- Turn a factor of $\mathcal{N}$ into a factor of $\mathcal{M}_{\mathrm{D,log}}$,
- Turn a factor of $\mathcal{M}_{\mathrm{D,log}}$ into a factor of $\mathcal{I}_{\log}$,
- Turn a factor of $\mathcal{I}_{\log}$ into a factor of $\rho\mathcal{C}^{\infty}_{\log}+\mathcal{I}^{2}_{\log}$, or
- Turn a factor of ρ into a factor of $\rho\mathcal{I}_{\log}$.

Moreover, commuting the radial vector field through a power of ρ or $\mathcal{I}_{\log}$ shifts it by $\imath$.

We show only the easiest of the three cases to indicate the method of proof.

By applying Lemma 9.14 repeatedly, we see that

$$(R+\imath k)\rho^{\alpha} \subset \rho^{\alpha}(R+\imath(k-\alpha)) + \rho^{\alpha}\mathcal{I}_{\log}.$$

Now suppose that we have shown the first statement for ϵ. We have

$$\begin{aligned}
(R+\imath k)^{\epsilon+1}\rho^{\alpha} &\subset (R+\imath k)\rho^{\alpha}(R+\imath(k-\alpha))^{\epsilon} \\
&\quad + (R+\imath k)\sum_{i=1}^{\epsilon}\sum_{a=0}^{\min(i,\epsilon-i)} \rho^{\alpha+a}\mathcal{I}^{i-a}_{\log}\tilde{R}^{\epsilon-i-a} \\
&\subset \rho^{\alpha}(R+\imath(k-\alpha))^{\epsilon+1} + \rho^{\alpha}\mathcal{I}_{\log}(R+\imath(k-\alpha))^{\epsilon} \\
&\quad + \sum_{i=1}^{\epsilon}\sum_{a=0}^{\min(i,\epsilon-i)} \rho^{\alpha+a}\mathcal{I}^{i-a}_{\log}\tilde{R}^{\epsilon+1-i-a} \\
&\quad + \sum_{i=1}^{\epsilon}\sum_{a=0}^{\min(i,\epsilon-i)} \left(\rho^{\alpha+a+1}\mathcal{I}^{i-a-1}_{\log} + \rho^{\alpha+a}\mathcal{I}^{i+1}_{\log}\right)\tilde{R}^{\epsilon-i-a} \\
&\subset \rho^{\alpha}(R+\imath(k-\alpha))^{\epsilon+1} + \sum_{i=1}^{\epsilon+1}\sum_{a=0}^{\min(i,\epsilon+1-i)} \rho^{\alpha+a}\mathcal{I}^{i-a}_{\log}\tilde{R}^{\epsilon+1-i-a},
\end{aligned}$$

as desired. $\square$

Putting Lemma 9.15 together, we have the following:

Lemma 9.17. *Again suppose that α, β, γ, δ, and ϵ are natural numbers with $\gamma \geq 1$. Then*

$$\begin{aligned}(R+\imath k)^{\epsilon}\rho^{\alpha}\mathcal{I}_{\log}^{\beta}\mathcal{M}_{\mathrm{D},\log}^{\gamma}\mathcal{N}^{\delta} \subset \sum_{d=0}^{\min(\delta,\epsilon)}\sum_{a=0}^{\min(\beta,\epsilon-d)} &\rho^{\alpha+a}\mathcal{I}_{\log}^{\beta-a}\mathcal{M}_{\mathrm{D},\log}^{\gamma+d}\mathcal{N}^{\delta-d}(R+\imath(k-\alpha-\beta))^{\epsilon-d-a} \\ &+ \sum_{d=0}^{\min(\delta,\epsilon)}\sum_{i=1}^{\epsilon-d}\sum_{a=0}^{\min(\beta+i,\epsilon-d-i)}\rho^{\alpha+a}\mathcal{I}_{\log}^{\beta+i-a}\mathcal{M}_{\mathrm{D},\log}^{\gamma+d}\mathcal{N}^{\delta-d}\tilde{R}^{\epsilon-d-i-a}.\end{aligned}$$

In particular, one has, for $\epsilon \geq \beta+2$,

$$\begin{aligned}(R+\imath k)^{\epsilon}\mathcal{M}_{\mathrm{D},\log}^{\gamma}\mathcal{N}^{2} &\subset \mathcal{M}_{\mathrm{D},\log}^{\gamma+2}\mathcal{N}^{2}(R+\imath k)^{\epsilon-2} \\ &\quad + \mathcal{I}_{\log}\mathcal{M}_{\mathrm{D},\log}^{\gamma+\epsilon-1}\mathcal{N}^{2} \\ (R+\imath k)^{\epsilon}\mathcal{I}_{\log}^{\beta}\mathcal{M}_{\mathrm{D},\log}^{\gamma}\mathcal{N}^{2} &\subset \mathcal{I}_{\log}^{\beta}\mathcal{M}_{\mathrm{D},\log}^{\gamma+\beta+2}\mathcal{N}^{2}(R+\imath(k-\beta))^{\epsilon-2-\beta} \\ &\quad + \mathcal{I}_{\log}^{\beta+1}\mathcal{M}_{\mathrm{D},\log}^{\gamma+\epsilon-1}\mathcal{N}^{2}\end{aligned}$$

Proof. The proof of the first statement merely combines the three statements of Lemma 9.15, while the second statement follows from the observations that $\rho \in \mathcal{I}_{\log}$ and $\tilde{R} \in \mathcal{M}_{\mathrm{D},\log}$. ☐

Proof of Lemma 9.13. We proceed via induction on k. Lemma 7.6 establishes that $L + 4D_{\mathrm{v}}\mathbf{R}_0 \in \mathcal{N}^2 \subset \mathcal{M}_{\mathrm{D},\log}\mathcal{N}^2$, finishing the $k=0$ case of the lemma.

We now suppose the lemma is true with k replaced by $k-1$. For convenience, we let

$$s(k,j) = 1 + 2(t_k - t_{k-1-j})$$

and observe that

$$\begin{aligned}D_{\mathrm{v}}\mathbf{R}_k w &= (R+\imath(k-1))^{2k+1}D_{\mathrm{v}}\mathbf{R}_{k-1}w \\ &\in (R+\imath(k-1))^{2k+1}\sum_{j=0}^{k-1}\mathcal{I}_{\log}^{j}\mathcal{M}_{\mathrm{D},\log}^{s(k-1,j)}\mathcal{N}^{2}\mathbf{R}_{k-2-j}w \\ &\subset \sum_{j=0}^{k-1}\mathcal{I}_{\log}^{j}\mathcal{M}_{\mathrm{D},\log}^{s(k-1,j)+j+2}\mathcal{N}^{2}(R+\imath(k-1-j))^{2k-1-j}\mathbf{R}_{k-2-j}w \\ &\quad + \sum_{j=0}^{k-1}\mathcal{I}_{\log}^{j+1}\mathcal{M}_{\mathrm{D},\log}^{s(k-1,j)+2k}\mathcal{N}^{2}\mathbf{R}_{k-2-j}w,\end{aligned}$$

where for the second inclusion we applied Lemma 9.17. Because $R \in \mathcal{M}_{\mathrm{D,log}}$, we then have

$$\begin{aligned} D_{\mathrm{v}} \mathbf{R}_k w \in \sum_{j=0}^{k-1} &\mathcal{I}_{\log}^{j} \mathcal{M}_{\mathrm{D,log}}^{s(k-1,j)+2j+2} \mathcal{N}^2 \mathbf{R}_{k-1-j} w \\ &+ \sum_{j=1}^{k} \mathcal{I}_{\log}^{j} \mathcal{M}_{\mathrm{D,log}}^{s(k-1,j-1)+2k} \mathcal{N}^2 \mathbf{R}_{k-1-j} w. \end{aligned}$$

We finally note that $s(k-1,j)+2j+2 = 1+2t_{k-1}-2t_{k-2-j}+2j+2 = 1+2(t_k - t_{k-1-j})$ and $s(k-1,j-1)+2k = 1+2(t_k - t_{k-1-j})$, finishing the proof. ☐

Appendix A. The Kerr metric

In this appendix, we discuss the Kerr metric near null infinity as an example of a Lorentzian scattering metric.

The Kerr metric (with our "mostly-minus" sign convention) can be written

$$\left(1-\frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar\sin^2\theta}{\Sigma}\, dt\, d\varphi - \frac{\Sigma}{\Delta}\, dr^2 - \Sigma\, d\theta^2 - \left(r^2+a^2+\frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\varphi^2,$$

$$\begin{aligned} \Sigma &= r^2 + a^2\cos^2\theta, \\ \Delta &= r^2 - 2Mr + a^2. \end{aligned}$$

We now introduce the new variables

$$\rho = \frac{1}{t},\ v_0 = 2\left(1 - \frac{r}{t}\right)$$

so that that the cone $r = t$ now becomes $v_0 = 0$. We easily compute

$$\frac{\Sigma}{\Delta} \sim 1 + 2M\rho + M\rho v_0, \tag{A.1}$$

$$\frac{r}{\Sigma} \sim \rho + \frac{\rho v_0}{2} \tag{A.2}$$

where we use the notation $f \sim g$ if $f - g = O(\rho^2) + O(\rho v_0^2)$ near $\rho = v_0 = 0$ and we will write $\mathcal{O}$ for terms that are $O(\rho^2) + O(\rho v_0^2)$ below. Thus we may write

(A.3)

$$\begin{aligned} g = \big(1-2M(\rho+\rho v_0/2+\mathcal{O})\big)\frac{d\rho^2}{\rho^4} - 2Mar\sin^2\theta(\rho+\rho v_0/2+\mathcal{O})\big(\frac{d\rho}{\rho^2}d\varphi + d\varphi\frac{d\rho}{\rho^2}\big) \\ - \big(1+2M\rho+M\rho v_0\big)\big(-(1-v_0/2)\frac{d\rho}{\rho^2} - \frac{1}{2}\frac{dv_0}{\rho}\big)^2 - \Sigma d\theta^2 - (*)d\varphi^2. \end{aligned}$$

We then compute the coefficient of $(d\rho/\rho^2)^2$ as

$$g_{\rho\rho} = v_0 - 4M\rho - v_0^2/4 + \mathcal{O}.$$

Meanwhile, the coefficient of $(d\rho dv + dvd\rho)/\rho^3$ is given by

$$\frac{1}{2}\big(1-\frac{v}{2}\big)\big(1+2M\rho+M\rho v+\mathcal{O}\big)=\frac{1}{2}+O(\rho)+O(v),$$

while all other cross terms with $d\rho$ are of the form $O(\rho^{-1})d\rho\, d\bullet$, with $\bullet = \theta, \varphi$, or v. Thus, setting $m = 4M$, and changing coordinates to

$$v \equiv v_0 - v_0^2/4$$

near $v_0 = 0$ brings the metric to the desired form.

Meanwhile, we continue to compute in the variables ρ, v_0 for the moment. The dual Kerr metric has the form

(A.4)

$$\frac{1}{\Delta}\big(r^2+a^2+\frac{2Ma^2r}{\Sigma}\sin^2\theta\big)\partial_t^2+\frac{2Mr}{\Sigma}\frac{a}{\Delta}(\partial_\varphi\partial_t+\partial_t\partial_\varphi)-\frac{1}{\Delta\sin^2\theta}\big(1-\frac{2Mr}{\Sigma}\big)\partial_\varphi^2-\frac{\Delta}{\Sigma}\partial_r^2-\frac{1}{\Sigma}\partial_\theta^2.$$

Changing coordinates from r, t to ρ, v_0 gives,

$$\partial_t \rightsquigarrow -\rho^2\partial_\rho + 2(1-v_0/2)\rho\partial_{v_0}, \quad \partial_r \rightsquigarrow -2\rho\partial_{v_0}.$$

Thus, using (A.1), (A.2), the r, t block of the metric can be rewritten in coordinates ρ, v, θ, φ as

$$\begin{aligned}&(1+O(\rho))\partial_t^2-(1+O(\rho))\partial_r^2\\&\quad=(1+O(\rho))(-\rho^2\partial_\rho+2(1-v_0/2)\rho\partial_{v_0})^2-(1+O(\rho))(-2\rho\partial_{v_0})^2.\end{aligned}$$

In other words, the scattering principal symbol associated to these terms (with canonical dual variables ξ, γ_0 to $d\rho/\rho^2,\ dv_0/\rho$) is

$$(1+O(\rho))\xi^2-4\xi\gamma_0(1-v_0/2)+4((1-v_0/2)^2-1)\gamma_0^2+O(\rho) \tag{A.5}$$

Now we change coordinates to the “correct” system of $\rho, v = v_0 - v_0^2/4$, in which the metric assumes the normal form. We have

$$v_0 = 2(1-\sqrt{1-v}),$$

hence in particular,

$$(1-v_0/2)=\sqrt{1-v},$$

while the vector fields are transformed by

$$\partial_{v_0} \rightsquigarrow \sqrt{1-v}\partial_v, \quad \partial_\rho \rightsquigarrow \partial_\rho,$$

so that

$$\gamma_0 \rightsquigarrow \sqrt{1-v}\gamma, \quad \xi \rightsquigarrow \xi.$$

These changes yield the symbol

$$\xi^2-4(1-v)\xi\gamma-4v(1-v)\gamma^2+O(\rho).$$

Thus we find that in the notation of (3.5), for the Kerr metric, we may read off the coefficients of the dual metric in normal form as:

$$\omega|_{\rho=v=0}=1,\ \alpha|_{\rho=v=0}=2,\ \beta|_{\rho=v=0}=4. \tag{A.6}$$

Appendix B. Explicit log terms

In this section we describe how to explicitly compute the leading order log singularity in the expansion at the radiation field face, and verify that its coefficient is nonzero for the Kerr metric, whenever the radiation field does not vanish identically.

Recall that we know a priori that if $\Box u = f$ with $f \in \dot{\mathcal{C}}^\infty$, then

$$w \equiv \rho^{-(n-2)/2} u,$$

which solves

$$\rho^{-(n-2)/2-2} \Box \rho^{(n-2)/2} w = \rho^{-(n-2)/2-2} f \in \dot{\mathcal{C}}^\infty$$

has an expansion at the radiation field front face, i.e., locally in the variables $s = \mathrm{v}/\varrho, \varrho, \mathrm{y}$ beginning

$$w \sim w_0(s,\mathrm{y}) + w_1^0(s,\mathrm{y})\varrho + w_1^1(s,\mathrm{y})\varrho\log\varrho + w_1^2(s,\mathrm{y})\varrho\log^2\varrho. \tag{B.1}$$

To explicitly find these terms (at least in principle) we recall that we may write

$$L = \rho^{-(n-2)/2-2} \Box \rho^{(n-2)/2} = L_0 + \mathcal{N}^2,$$

with

$$L_0 = 4\partial_\mathrm{v}\left(\varrho\partial_\varrho + \mathrm{v}\partial_\mathrm{v}\right).$$

We will need to analyze the module term more closely to obtain the explicit singularity.

To begin, we return to our original coordinate system ρ, v, y and note that if we look at the module vector fields $\rho\partial_v, v\partial_v, \rho\partial_\rho$, when we change to logified coordinates these become respectively

$$\varrho\partial_\varrho + \chi m\varrho(1+\log\varrho)\partial_\mathrm{v},\ (\mathrm{v} - \chi m\varrho\log\varrho)(1+\chi' m\varrho\log\varrho)\partial_\mathrm{v},\ \varrho(1+\chi' m\varrho\log\varrho)\partial_\mathrm{v}.$$

We then perform the radiation field blowup $s = \mathrm{v}/\varrho$ and note that the terms we get in this manner are spanned by

$$\partial_s, \log\varrho\partial_s, \varrho\partial_\varrho, \varrho\log\varrho\partial_\varrho.$$

Of these terms, the important one for our purposes is $\log\varrho\partial_s$, as it is the only one that can produce a $\log\varrho$ term when applied to a series of the form (B.1). We now note the crucial fact that in changing to log coordinates followed by lifting vector fields from $\mathcal{M}$, we have

$$\begin{aligned} \rho\partial_\rho &\rightsquigarrow \varrho\partial_\varrho + (\chi m - s)\partial_s + \chi m\log\varrho\partial_s, \\ v\partial_v &\rightsquigarrow (s - \chi m\log\varrho)(1 + \chi' m\varrho\log\varrho)\partial_s, \\ \rho\partial_v &\rightsquigarrow (1 + \chi' m\varrho\log\varrho)\partial_s, \end{aligned}$$

hence isolating the crucial term, we simply remark that

$$\begin{aligned} \rho\partial_\rho &\rightsquigarrow \chi m\log\varrho\partial_s + \dots, \\ v\partial_v &\rightsquigarrow -\chi' m\log\varrho\partial_s + \dots, \\ \rho\partial_v &\rightsquigarrow \chi' m\varrho\log\varrho\partial_s + \dots. \end{aligned} \tag{B.2}$$

(In dealing with $\mathcal{C}^\infty_{\log}$ coefficients of such terms, meanwhile, we note that since every factor $\log\varrho$ also comes with a factor of ϱ, in analyzing the coefficient of $\log\varrho\partial_s$ in the lift, it suffices to freeze these coefficients at $\rho = 0$.) We also recall that the operator

$$4\partial_v(\rho\partial_\rho + v\partial_v) - 4m\rho\partial_v^2$$

lifts under this transformation to precisely

$$L_0 = 4\partial_s\partial_\varrho.$$

Now we return to the form of a general long-range scattering metric. Following the proof of Lemma 5.2, we can more precisely write, using the notation of (3.5) for the dual metric components,

$$L = 4\partial_v(\rho\partial_\rho + v\partial_v) - 4m\rho\partial_v^2 + \omega(\rho\partial_\rho)^2 + 2\alpha\rho\partial_\rho v\partial_v + \beta(v\partial_v)^2 + E$$

where E consists of first order terms in the module, second order terms vanishing to higher order at $\rho = 0$, and terms involving ∂_y. Now lifting this expression to the logified, blown-up space, using (B.2), it becomes

$$L = 4\partial_s\partial_\varrho + m^2(\overline{\omega} - 2\overline{\alpha} + \overline{\beta})\log^2\varrho\partial_s^2 + E'$$

where E' consists of terms up to second order in $\varrho\partial_\varrho$, $\varrho\log\varrho\partial_\varrho$, $\partial_s, \log\varrho\partial_s$ with log-smooth coefficients, *but containing at most one factor of this last vector field*, and $\overline{\omega}, \overline{\alpha}, \overline{\beta}$ denote the respective restrictions of these functions to $\rho = v = 0$. Because the derivative of χ is supported away from the radial set, E' also includes the error terms from dropping the factors of χ and χ' in the above expression.

Now we apply this expression for L to the series Ansatz (B.1). Matching the resulting coefficients of $\log^2\varrho$ yields

$$4\partial_s w_1^2 + m^2(\overline{\omega} - 2\overline{\alpha} + \overline{\beta})\partial_s^2 w_0 = 0,$$

hence, since all the coefficients vanish for $s \to -\infty$, we may integrate to find

$$w_1^2 = -\frac{m^2}{4}(\overline{\omega} - 2\overline{\alpha} + \overline{\beta})\partial_s w_0.$$

For the particular case of the Kerr metric, (A.6) now gives

$$w_1^2 = -\frac{m^2}{4}\partial_s w_0.$$

The function w_0 cannot be constant unless it is zero (again since it vanishes for $s \ll 0$), so in general, we find that $w_1^2 \neq 0$. (Note that $\partial_s w_0$ is in fact exactly the Friedlander radiation field in this context.)

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