Some recent advances in microlocal analysis

András Vasy

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In this talk I intend to explain how advances in microlocal analysis helped solve some problems. Some of these problems concern:

- The Laplacian on asymptotically hyperbolic (conformally compact) spaces, also related to the asymptotically de Sitter d'Alembertian. (V. '10)
- Wave propagation on Kerr-de Sitter space (black holes on a background with a cosmological constant). (V. '10, Nonlinear results joint with Hintz '13, '14)
- Meromorphic continuation of the Ruelle zeta function for Anosov flows. (Dyatlov and Zworski '13)

These are all achieved by placing these problems into a non-elliptic Fredholm framework using microlocal analysis.

Asymptotically hyperbolic spaces are *n*-dimensional compact manifolds with boundary X_0, with

- a preferred boundary defining function x,
- a complete Riemannian metric g_0 on the interior of X_0 such that $\hat{g}_0 = x^2 g_0$ is Riemannian on X_0 (i.e. up to the boundary)
- and $|dx|_{\hat{g}_0} = 1$ at ∂X_0.

Here $Y = \partial X_0$ is metric infinity, but it is useful to encode the structure near Y via this compactification.

For such metrics the Laplacian is essentially self-adjoint on $C_c^\infty(X_0^\circ)$, and is positive, and thus the modified resolvent

$$R(\sigma) = (\Delta_{g_0} - (n-1)^2/4 - \sigma^2)^{-1}$$

exists, as a bounded operator on $L^2(dg_0)$ for $\operatorname{Im}\sigma > 0$, $\sigma \notin \imath(0, (n-1)/2]$.

In a suitable product decomposition $[0,\epsilon)_x \times Y$ near ∂X_0 (perhaps new x), these metrics are of the form

$$g_0 = \frac{dx^2 + h}{x^2}$$

where h is a family of metrics on $Y = \partial X_0$. One calls the metric even if h depends on x in an even manner, i.e. all odd derivatives of h with respect to x vanish at Y.

Mazzeo and Melrose '87 have proved that, on functions, $R(\sigma)$ continues from $\operatorname{Im}\sigma > (n-1)/2$ to $\mathbb{C}$ with finite rank Laurent coefficients at the poles (called *resonances*), except possibly at certain potential essential singularities on the imaginary axis, and Guillarmou '03 has shown that the latter are not present if the metric is even.

Such a meromorphic continuation is of interest for a number of purposes, one of which is the decay of waves. For instance, one would like to expand solutions to the wave equation $(D_t^2 + (n-1)^2/4 - \Delta_g)u = 0$ as $t \to +\infty$ as

$$u(t,.) \sim \sum_{j:\operatorname{Im}\sigma_j \geq -C} e^{it\sigma_j} u_j + O(e^{-Ct})$$

in a suitable sense (there may also be some polynomial factors in t, and with σ_j the resonances); this can be done via a contour deformation using the meromorphic continuation of the resolvent of Δ_g as long as one also has high energy estimates for $R(\sigma)$ in strips $|\operatorname{Im}\sigma| < C'$.

Our method, which relies on a conformal, or more precisely projective, extension of the problem across $Y = \partial X_0$, yields new proof of the meromorphic continuation as well as high energy estimates. We also have a natural extension to differential forms.

Theorem (A.V. '10, '12)

Let (X_0, g_0) be an even *asymptotically hyperbolic space of dimension n.*

Then, on functions, $R(\sigma)$, continues meromorphically from $\operatorname{Im}\sigma > (n-1)/2$ to $\mathbb{C}$, and if the geodesic flow on (X_0, g_0) is non-trapping, i.e. all geodesics escape to infinity, then in strips $\operatorname{Im}\sigma > s$, $R(\sigma)$ satisfies non-trapping estimates $\|R(\sigma)\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \leq C|\sigma|^{-1}$, $\operatorname{Re}\sigma > C_1$, for suitable Hilbert spaces $\mathcal{X}, \mathcal{Y}$.

Analogous results hold on differential k-forms, with $(n-1)^2/4$ replaced by $(n-2k\pm 1)^2/4$, with the sign $+$ corresponding to closed, and $-$ corresponding to coclosed forms.

The basic idea is to connect a differential operator, P_σ, on a manifold without boundary extending X_0 to the spectral family of Δ_{g_0}.

Change the smooth structure on X_0 by declaring that only even functions of x are smooth, i.e. introducing $\mu = x^2$ as the boundary defining function. Then after a conjugation and division by a vanishing factor the resulting operator smoothly and non-degenerately continues across the boundary, i.e. continues to $X_{-\delta_0} = (-\delta_0, 0)_\mu \times Y \sqcup X_{0,\mathrm{even}}$, where $X_{0,\mathrm{even}}$ is the manifold X_0 with the new smooth structure:

$$P_\sigma = \mu^{-1/2}\mu^{\imath\sigma/2-(n+1)/4}\left(\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2\right)\mu^{-\imath\sigma/2+(n+1)/4}\mu^{-1/2}$$
$$= 4D_\mu \mu D_\mu - 4\sigma D_\mu + \Delta_h + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-1)/4).$$

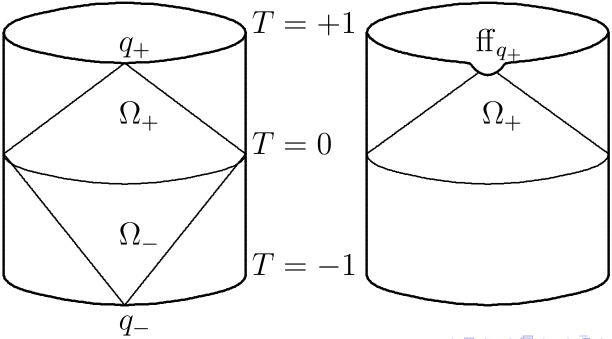
At the level of the principal symbol, i.e. the dual metric, the conjugation is irrelevant. Changing to (μ, y), $\mu = x^2$, as $x\partial_x = 2\mu\partial_\mu$,

$$G_0 = 4\mu^2\partial_\mu^2 + \mu H = \mu(4\mu\partial_\mu^2 + H),$$

so $\mu^{-1}G_0 = 4\mu\partial_\mu^2 + H$. This is a quadratic form that is positive definite for $\mu > 0$, is Lorentzian for $\mu < 0$, and has a transition at $\mu = 0$ that involves *radial points* we discuss later.

Similarly, in $\mu < 0$, this dual metric is obtained by analogous manipulations on an even *asymptotically de Sitter (Lorentzian) metric*, i.e. of the form $\tilde{x}^{-2}(d\tilde{x}^2 - h)$, with $\tilde{x}$ the boundary defining function, and h positive definite at $\tilde{x} = 0$. Then $\mu = -\tilde{x}^2$ gives this form of the metric. Here $-\tilde{x}^2$ and x^2 are formally the 'same', i.e. $\tilde{x}$ is formally like $\imath x$, which means that this extension across the boundary is a mathematically precise general realization of a 'Wick rotation'.

Left: the compactification of de Sitter space dS^n with the backward light cone from q_+ and forward light cone from q_-; Ω_+, resp. Ω_-, are the intersection of these light cones with $T > 0$, resp. $T < 0$. Right: the blow up of de Sitter space at q_+. This desingularizes the tip of the light cone, and the interior of the light cone inside the front face ff_{q_+} can be identified with a potential scattering problem on hyperbolic space $\mathbb{H}^{n-1}$. Domain of dependence properties: can solve the equation only in Ω_+.



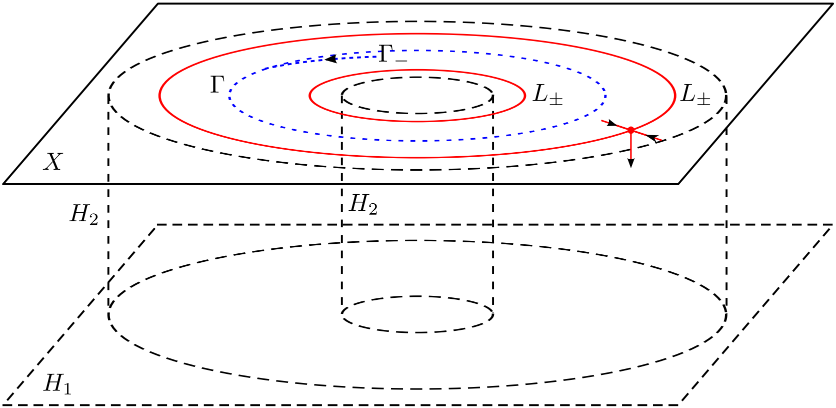
Kerr-de Sitter space (M°, g_0) is a Lorentzian space-time of $1+3$ dimension, which solves Einstein's equation with a cosmological constant. It models a rotating (with angular momentum a) black hole ('Kerr') of mass $M_\bullet$ in a space-time with cosmological constant Λ ('De Sitter').

In many ways it naturally generalizes the blown-up de Sitter picture just shown, with the backward light cone corresponding to the event horizons.

The second theorem concerns wave propagation on Kerr-de Sitter spaces. This is particularly interesting since the asymptotic behavior of waves involves resonances, which are poles of a family $\mathbb{C} \ni \sigma \mapsto P_\sigma^{-1}$, where P_σ is very similar to an operator in the asymptotically hyperbolic case; it is an operator on a manifold without boundary.

Concretely, Kerr-de Sitter space has a bordification, or partial compactification, $\overline{M}$, with a boundary defining function τ and P_σ is then an operator on ∂M.

The extra complication is that this operator is trapping, which here means that some null-geodesics do not escape to the event horizons, but the trapping is of a relatively weak type, called normally hyperbolic trapping, which has been analyzed by Wunsch and Zworski '10, Nonnenmacher and Zworski '13 and by Dyatlov '13 recently.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Γ
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Theorem (A.V. '10, cf. Melrose-Sá Barreto-V. '11 for $a = 0$.)

Let $(\overline{M}, g)$ be a Kerr-de Sitter type space with normally hyperbolic trapping. Then there is $\kappa > 0$ such that solutions of $(\Box_g - \lambda)u = 0$ have an asymptotic expansion $u \sim \sum_j \sum_{k \le k_j} \tau^{\imath\sigma_j} (\log|\tau|)^k a_{jk} + \tilde{u}$, where $\tilde{u} \in \tau^\kappa H^s_b(\overline{M})$; here σ_j are resonances *of the associated* normal operator. *For $\lambda = 0$ on Kerr-de Sitter space, the unique σ_j with $\operatorname{Im}\sigma_j \ge 0$ is 0, and the corresponding term is a constant, i.e. waves decay to constants.*

Further, this result is stable under b-perturbations of the metric, with the b-structure understood in the sense of Melrose.

In spatially compact parts of Kerr-de Sitter space, $\tau = e^{-t}$ for the usual time function t, i.e. this decay is exponential. For *Kerr* space-times *polynomial* decay has been shown by Dafermos, Rodnianski, Shlapentokh-Rothman ('05,..., '14) and Tataru and Tohaneanu ('08, '09,...).

In fact, in a slightly different way, the wave equation for Minkowski-type metrics, more specifically *Lorentzian scattering metrics*, can also be handled by similar techniques for both Cauchy problems (V. '10, work with Baskin, Wunsch '12, and with Hintz '13) and for the Feynman propagator. In fact, Klein-Gordon type equations, even in ultrahyperbolic settings, are also amenable to this type of analysis – in this case in Melrose's scattering framework.

There are extensions of this result both to semilinear and quasilinear PDE, in joint work with Hintz. For instance, one has global solvability, and a description of the asymptotic behavior, of certain quasilinear equations on M° of the form

$$\Box_{g(u,du)} = f + q(u, du),$$

where $g(0,0) = g_0$, for *small data* f.

The third theorem concerns the dynamical zeta function for Anosov flows: these are flows on compact manifolds with a *continuous* stable and unstable distribution transversal to the flow direction.

It was a conjecture of Smale's, proved by Giulietti, Liverani and Pollicott recently by dynamical systems techniques, but shortly afterwards Dyatlov and Zworski gave a new short proof.

This new proof uses microlocal analysis, inspired by ideas of Faure and Sjöstrand '10, which are analogous to the setup involved in proving the above theorems, as well as Guillemin's ('77) approach to trace formulae.

Theorem (Giulietti, Liverani and Pollicott '12, and Dyatlov and Zworski '13)

Let X be a compact manifold and $\phi_t : X \to X$ a C^∞ Anosov flow with orientable stable and unstable bundles. Let $\{\gamma^\sharp\}$ denote the set of primitive orbits of ϕ_t, and $T^\sharp_\gamma$ their periods. Then the Ruelle zeta function $\zeta_R(\lambda) = \prod_{\gamma^\sharp}(1 - e^{\imath\lambda T^\sharp_\gamma})$, which converges for $\operatorname{Im}\lambda \gg 0$, extends meromorphically to $\mathbb{C}$.

See also the work of Dyatlov, Faure and Guillarmou '14 on compact hyperbolic surfaces giving more precise results, Guillarmou '14 on invariant distributions and the X-ray transform, and Faure and Tsujii '13 on the distribution of resonances.

All of these problems reduce to Fredholm properties of partial differential operators P. Thus, we work with Hilbert spaces X, Y of distributions on a manifold M, with or without boundary or corners, such that

- $P : X \to Y$ continuously,
- $\operatorname{Ran} P$ closed
- $\operatorname{Ker} P$, $Y / \operatorname{Ran} P$ are finite dimensional.

The last two are guaranteed by the Fredholm estimates

$$\|u\|_X \leq C(\|Pu\|_Y + \|u\|_{Z_1})$$

and

$$\|v\|_{Y^*} \leq C(\|P^* v\|_{X^*} + \|u\|_{Z_2}),$$

where the inclusion maps $X \to Z_1$ and $Y^* \to Z_2$ are compact.

One often wants actual invertibility; another interesting question is the computation of the index of P.

- Typically this problem is considered in elliptic/Riemannian settings.
- However, one does solve e.g. wave equations as well – if the spaces are set up correctly, these should give a Fredholm statement.
- In this talk I discuss such non-elliptic problems on manifolds with or without boundary.
- In fact, due to the lack of time, the boundary setting (Melrose's b-analysis) will be ignored except for the already stated results. The main novelty is the appearance of resonances even in the statement of Fredholm properties: one needs to work on weighted Sobolev spaces in which the weight (decay order) is not the negative of the imaginary part of any resonance (see the lecture notes for more details).

Microlocal analysis is local in phase space, $T^*M \setminus o$, modulo dilations in the fibers, i.e. in $S^*M = (T^*M \setminus o)/\mathbb{R}^+$. This is done via *pseudodifferential operators*, $P \in \Psi^m(M)$; in local coordinates these are essentially *quantizations*,

$$(Pu)(x) = (2\pi)^{-n} \int e^{\imath(x-y)\cdot\xi} p(x,\xi)u(y)\,d\xi\,dy$$

of *symbols* p. Example: p which are asymptotically (as $|\xi| \to \infty$) homogeneous of degree m.

The basic examples of structures related to this are

- the *principal symbol*, $\sigma_m(P)$, which is a(n equivalence class of) function(s) on T^*M, capturing P modulo $\Psi^{m-1}(M)$, and
- the *wave front set*, $\mathrm{WF}'(P)$ which is a subset of S^*M describing where P is *not trivial*, i.e. in *not* $\Psi^{-\infty}(M)$.

For $P \in \mathrm{Diff}^m(M)$, $m \in \mathbb{N}$, $\sigma_m(P)$ captures the leading terms. If $P = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha$, $\sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$.

The simplest example is elliptic (pseudo)differential operators on compact manifolds without boundary M, acting between sections of vector bundles E, F, with basic geometric examples being the Laplacian on differential forms, and Dirac operators.

- $P \in \Psi^m(M)$ elliptic (at least principally classical), i.e. $\sigma_m(P) : T^*M \setminus o \to \mathrm{Hom}(E, F)$ invertible,
- $X = H^s = H^s(M; E)$, $Y = H^{s-m}(M; F)$, $s \in \mathbb{R}$,
- so $X^* = H^{-s}(M; E^*)$, $Y^* = H^{-s+m}(M; F^*)$,
- $Z_1 = H^{-N}(M; E)$, $Z_2 = H^{-N}(M; F^*)$, N large.

The Fredholm property follows from the elliptic estimate

$$\|\phi\|_{H^r} \leq C(\|L\phi\|_{H^{r-m}} + \|\phi\|_{H^{-N}}),$$

with $L = P$, $r = s$, resp. $L = P^*$, $r = -s + m$. Note that the choice of s is irrelevant here (elliptic regularity).

The non-elliptic problems we consider are problems in which the elliptic estimate is replaced by estimates of the form

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

i.e. with a loss of one derivative relative to the elliptic setting, and

$$\|v\|_{H^{s'}} \leq C(\|P^* v\|_{H^{s'-m+1}} + \|v\|_{H^{-N'}}),$$

with $s' = -s + m - 1$ being the case of interest.

Such estimates imply that $P : X \to Y$ is Fredholm if

$$X = \{u \in H^s : \; Pu \in H^{s-m+1}\}, \; Y = H^{s-m+1}.$$

Here X is a first order coisotropic space associated to the characteristic set of P.

A complication for non-elliptic Fredholm theory is that H^s is often a *variable order* Sobolev space (see e.g. Unterberger '71, Duistermaat '72), i.e. s is a real-valued function on S^*M. This space is defined by:

- Let $s_0 = \inf s$, and let $A \in \Psi^s_\delta(M) \subset \Psi^{\sup s}_\delta(M)$, $\delta \in (0, 1/2)$, be elliptic,
- $$H^s = \{u \in H^{s_0} :\ Au \in L^2\},$$
- for instance if g is a Riemannian metric, one can take the principal symbol of A to be $|\xi|^s_g$.

Then $P \in \Psi^m$ maps $P : H^s \to H^{s-m}$ continuously still.

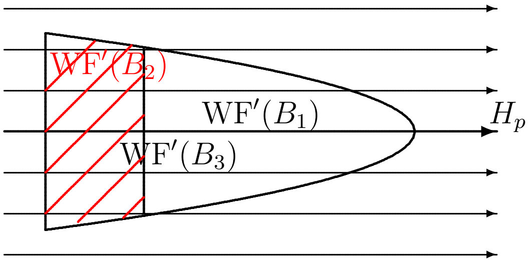
The most basic non-elliptic phenomenon, when $P \in \Psi^m$ has real scalar principal symbol p Id, is propagation of singularities along the bicharacteristics, i.e. integral curves of H_p (due to Hörmander '71):

- $\mathrm{WF}^s(u) \subset S^*M$ measures where u is microlocally *not* in H^s, i.e. a point $\alpha \in S^*M$ is *not* in $\mathrm{WF}^s(u)$ if there is $A \in \Psi^0(M)$ such that A is elliptic at α and $Au \in H^s$.
- Away from $\mathrm{Char}(P) = \{p = 0\}$, one has microlocal elliptic regularity, i.e. the Sobolev wave front set, $\mathrm{WF}^s(u)$, satisfies $\mathrm{WF}^s(u) \setminus \mathrm{Char}(P) \subset \mathrm{WF}^{s-m}(Pu)$.
- In $\mathrm{Char}(P) \setminus \mathrm{WF}^{s-m+1}(Pu)$, $\mathrm{WF}^s(u)$, is a union of maximally extended bicharacteristics.

This gives an estimate

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

$B_j \in \Psi^0$, provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$, and all bicharacteristics from points in $\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)$ reach the elliptic set $\mathrm{Ell}(B_2)$ of B_2 while remaining in $\mathrm{Ell}(B_3)$.

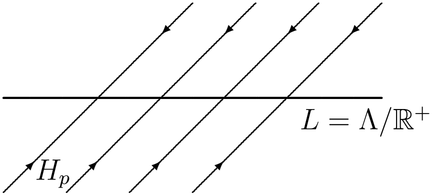


The same estimate is valid if s is variable, provided one either restricts to *forward* bicharacteristics and requires $H_p s \geq 0$, or to *backward* bicharacteristics and requires $H_p s \leq 0$.

The basic problem with this estimate is the term $\|B_2 u\|_{H^s}$ on the right hand side – how does one control this?

One option is *complex absorption*, this allows the use of constant order Sobolev spaces. The point is then that bicharacteristics reach the elliptic set of an operator Q with real principal symbol, and one works with $P - iQ$.

A more natural option is to have some structure of the bicharacteristic flow: we need that there are submanifolds L of S^*M which act as sources/sinks in the normal direction.



- The most natural place these arise is *radial sets*, i.e. points in T^*M where H_p is tangent to the dilation orbits. Note that Hörmander's theorem provides no information here.
- In non-degenerate settings, i.e. when H_p is non-zero, the biggest possible dimension of a radial set is that of M, in which case it is a conic Lagrangian submanifold of T^*M.
- In this case, they act as source or sink within $\mathrm{Char}(P)$; in the source case H_p flows to the zero section within Λ, in the sink case from the zero section.
- This also arises in scattering theory, where it was studied by Melrose '94.

Let $\tilde{p}$ be the principal symbol of $\frac{1}{2i}(P - P^*) \in \Psi^{m-1}$, and define $\tilde{\beta}$ by

$$\tilde{p}|_\Lambda = \tilde{\beta}\frac{H_p\rho}{\rho},$$

where ρ is an elliptic homogeneous degree 1 function, which is independent of choices (even that of the metric defining the adjoint!).

In this case there is an analogue of the propagation of singularity theorem, but there is a threshold, $(m-1)/2 - \tilde{\beta}$:

- if the Sobolev order is higher than this, one can propagate estimates from $L = \Lambda/\mathbb{R}^+$, without needing a priori control like B_2u,
- if the Sobolev order is below this, one can propagate estimates to L, needing control in a punctured neighborhood of L.

- If $s \geq s_0 > (m-1)/2 - \tilde{\beta}$, then

$$\|B_1 u\|_{H^s} \leq C(\|B_3 Pu\|_{H^{s-m+1}} + \|u\|_{H^{s_0}}),$$

$B_j \in \Psi^0$ elliptic on L, provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$, and all bicharacteristics from points in $\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)$ tend to L while remaining in $\mathrm{Ell}(B_3)$.

- If $s < (m-1)/2 - \tilde{\beta}$ then

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

$B_j \in \Psi^0$ elliptic on L, provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$, and all bicharacteristics from points in $(\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)) \setminus L$ reach the elliptic set $\mathrm{Ell}(B_2)$ of B_2 while remaining in $\mathrm{Ell}(B_3)$.

Replacing P by P^* changes the sign of $\tilde{\beta}$, and it naturally leads to estimates on the required dual spaces.

As a consequence, if there are radial sets L_1, L_2 such that all bicharacteristics in $\mathrm{Char}(P) \setminus (L_1 \cup L_2)$ escape to L_1 in one of the directions along the bicharacteristics and to L_2 in the other, one has the required Fredholm estimate provided one can arrange the Sobolev spaces so that

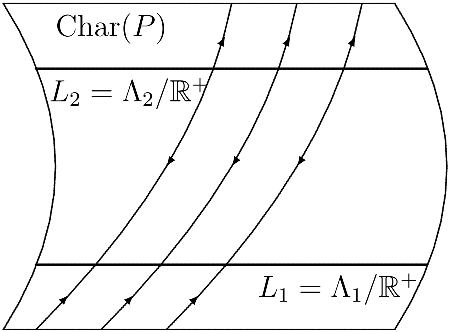
- at L_1 the Sobolev order is above the threshold for P,
- at L_2 the Sobolev order is above the threshold for P^*,
- the Sobolev order is monotone decreasing from L_1 to L_2.

Namely,

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

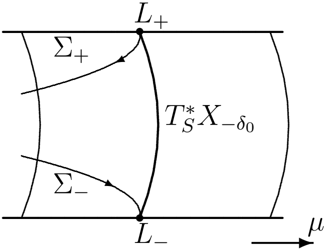
$$\|v\|_{H^{s'}} \leq C(\|P^* v\|_{H^{s'-m+1}} + \|v\|_{H^{-N'}}),$$

with $s' = -s + m - 1$.


Char(P)
$L_2 = \Lambda_2/\mathbb{R}^+$
$L_1 = \Lambda_1/\mathbb{R}^+$

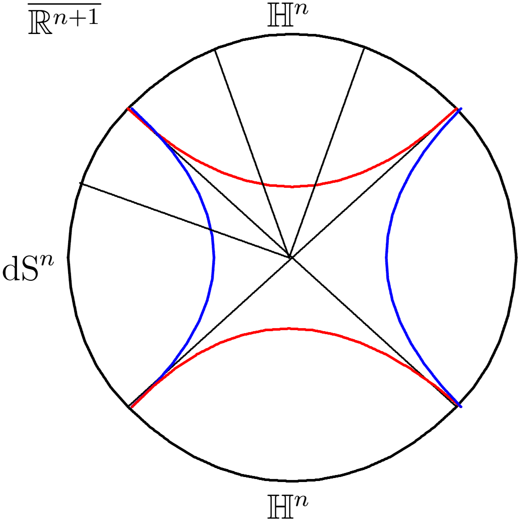
As an example, in the asymptotically hyperbolic setting, the cosphere bundle, i.e. the quotient of the cotangent bundle by dilations, $S^*X_{-\delta_0}$ of $X_{-\delta_0}$ near $S = \{\mu = 0\}$ is shown.

$S^*X_{-\delta_0}$ is the surface of the cylinder shown. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. They lie in $\mu \leq 0$, only meeting $S^*_S X_{-\delta_0}$ at $L_\pm$. Here we need to add complex absorption (or a boundary) near $\mu = -\delta_0$.



A different way to look at this, which also explains the projective relation, and has no complex absorption, is as follows:

- Let $\tilde{M} = \mathbb{R}^{n+1}$ with the Minkowski metric and $\Box$ be the wave operator.
- Let ρ be a homogeneous degree 1 positive function, e.g. a Euclidean distance from the origin.
- The conjugate of $\rho^2\Box$ by the Mellin transform along the dilation orbits gives a family of operators P_σ, σ the Mellin dual parameter, on $\mathbb{S}^n$ (smooth transversal to the dilation orbits).
- P_σ is elliptic inside the light cone, but are Lorentzian outside the light cone.
- The conormal bundle of the light cone consists of radial points.
- The characteristic set has two components, and there are four components of the radial set: a future and a past component within each component of the characteristic set.


$\overline{\mathbb{R}^{n+1}}$
$\mathbb{H}^n$
dS^n
$\mathbb{H}^n$

In one component Σ_+ of the characteristic set, the bicharacteristics go from the past component of the radial set L_{+-} to the future one L_{++}; in the other component Σ_- they go from the future component of the radial set L_{-+} to the past one L_{--}.

Reasonable choices of Fredholm problems:

- Make the Sobolev spaces high regularity at the past radial sets and low at the future radial sets: this is the *forward propagator*.
- Make the Sobolev spaces low regularity at the past radial sets and high at the future radial sets: this is the *backward propagator*.
- Make the Sobolev spaces high regularity at the sources L_{+-} and L_{-+} and low regularity at the sinks, or vice versa. These are the Feynman propagators, and they propagate estimates for P_σ in the direction of the Hamilton flow in the first case, and against the Hamilton flow in the second.
- Note that none of these problems are self-adjoint: the adjoint always propagates estimates in the *opposite* direction as the operator itself.

In this case the interior of the light cone is naturally identified with hyperbolic space, while the exterior with de Sitter space.

It is easy to see that the backward propagator gives rise to the resolvent of the hyperbolic Laplacian in the future copy of the hyperbolic space, while the forward propagator does so in the past copy of hyperbolic space.

This gives rise to the new, Fredholm, construction of the analytic continuation of the resolvent on hyperbolic space.

There is a completely analogous construction for general asymptotically hyperbolic spaces.