LIPSCHITZ DOMAINS, DOMAINS WITH CORNERS, AND THE HODGE LAPLACIAN[1]

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Abstract. We define self-adjoint extensions of the Hodge Laplacian on Lipschitz domains in Riemannian manifolds, corresponding to either the absolute or the relative boundary condition, and examine regularity properties of these operators' domains and form domains. We obtain results valid for general Lipschitz domains, and stronger results for a special class of "almost convex" domains, which apply to domains with corners.

1. Introduction

Let Ω be an open Lipschitz domain in a smooth, compact Riemannian manifold M, equipped with a metric tensor g, which we will assume is of class C^2. As is customary, let d, δ stand, respectively, for the operator of exterior differentiation and its adjoint. We use the Friedrichs method to define a self-adjoint extension of the Hodge Laplacian $\Delta = -(d\delta + \delta d)$, with the absolute boundary condition (respectively, the relative boundary condition) on differential forms on Ω, which we denote $-H = -H_A$ or $-H_R$. We want to establish regularity properties of its domain $\mathcal{D}(H)$ and of its form domain (which coincides with $\mathcal{D}(H^{1/2})$). We obtain a circle of results valid for general Lipschitz domains, and then some stronger results valid for certain special classes of Lipschitz domains, including domains with corners. These results extend some of the work in [MMT] and [M2].

To set things up, we define

$$\begin{aligned} X_A(\Omega) &= \{u \in L^2(\Omega, \Lambda^*) : du, \delta u \in L^2(\Omega, \Lambda^*), \nu \vee u|_{\partial\Omega} = 0\}, \\ X_R(\Omega) &= \{u \in L^2(\Omega, \Lambda^*) : du, \delta u \in L^2(\Omega, \Lambda^*), \nu \wedge u|_{\partial\Omega} = 0\}. \end{aligned} \tag{1.1}$$

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Here $\Lambda^* = \oplus_k \Lambda^k$, with Λ^k denoting the k-th exterior power of the tangent bundle of M, and ν is the unit conormal to $\partial\Omega$, a Lipschitz section of $\Lambda^1 M|_{\partial\Omega}$; $\nu \vee u$ is an interior product and $\nu \wedge u$ an exterior product. An important ingredient in the proof that $X_A(\Omega)$ and $X_R(\Omega)$ are well defined is the following result:

$$\begin{aligned} &u, \delta u \in L^2(\Omega, \Lambda^*) \Longrightarrow \nu \vee u \in H^{-1/2}(\partial\Omega, \Lambda^*), \\ &u, du \in L^2(\Omega, \Lambda^*) \Longrightarrow \nu \wedge u \in H^{-1/2}(\partial\Omega, \Lambda^*), \end{aligned} \tag{1.2}$$

established in (11.9) of [MMT]. Hereafter, H^s will stand for the L^2-based Sobolev space of smoothness s, considered either on Ω or on $\partial\Omega$. More precisely, $H^s(\Omega)$ is defined for any $s \in \mathbb{R}$ as the restriction to Ω of elements from $H^s(M)$. Also, $H^s(\partial\Omega)$ can be defined for $-1 \leq s \leq 1$ by lifting the corresponding scale from $\mathbb{R}^{n-1}$ to the Lipschitz manifold $\partial\Omega$ using a smooth partition of unity and local charts. Finally, we set $H^s(\Omega, \Lambda^*) := H^s(\Omega) \otimes \Lambda^*$ if $s \in \mathbb{R}$ and $H^s(\partial\Omega, \Lambda^*) := H^s(\partial\Omega) \otimes \Lambda^*$ if $-1 \leq s \leq 1$, although in the sequel we shall occasionally drop the dependence of these, and other spaces, on the vector bundle.

If $(\cdot\,,\,\cdot)_{L^2(\Omega)}$ stands for the natural L^2-inner product of forms in Ω (again, the subscript may be dropped in subsequent occurrences) then $X_A(\Omega)$ and $X_R(\Omega)$ are Hilbert spaces, with the inner product

$$(du, dv)_{L^2} + (\delta u, \delta v)_{L^2} + (u, v)_{L^2} = Q(u, v) + (u, v)_{L^2}, \tag{1.3}$$

where the last equality defines the sesqui-linear form Q. The Friedrichs extension method then yields self-adjoint operators H_A and $H_R = -\Delta$ on $L^2(\Omega, \Lambda^*)$, with

$$\begin{gathered} \mathcal{D}(H_A) = \{u \in X_A(\Omega) : \; X_A(\Omega) \ni v \mapsto Q(u, v) \text{ is } L^2\text{-bounded}\}, \\ (H_A u, v)_{L^2} = Q(u, v), \end{gathered} \tag{1.4}$$

and $\mathcal{D}(H_R) \subset X_R(\Omega)$ similarly defined. As part of the standard theory, one has

$$\mathcal{D}(H_A^{1/2}) = X_A(\Omega), \quad \mathcal{D}(H_R^{1/2}) = X_R(\Omega). \tag{1.5}$$

In some cases, $X_A(\Omega)$ coincides with

$$H_A^1(\Omega, \Lambda^*) = \{u \in H^1(\Omega, \Lambda^*) : \nu \vee u|_{\partial\Omega} = 0\}, \tag{1.6}$$

with a similar result for $X_R(\Omega)$. This holds when $\partial\Omega$ is of class C^2, by a classical result of M. Gaffney [G] and K. Friedrichs [F]. Also, if we write

$$X_A(\Omega) = \bigoplus_k X_A^k(\Omega), \quad X_R(\Omega) = \bigoplus_k X_R^k(\Omega), \tag{1.7}$$

where $X_A^k(\Omega)$ (respectively, $X_R^k(\Omega)$) consists of k-forms in $X_A(\Omega)$ (respectively, in $X_R(\Omega)$), then standard regularity results for the Dirichlet and Neumann problems yield

$$X_A^k(\Omega) = H_A^1(\Omega, \Lambda^k), \quad X_R^k(\Omega) = H_R^1(\Omega, \Lambda^k), \tag{1.8}$$

for $k = 0$ and $k = n$, where $n = \dim \Omega$. However, for general Lipschitz Ω and $k \in [1, n-1]$, this identity fails.

Example. Take $k = 1,\ u = df$. Then $du = 0,\ \delta u = -\Delta f$, and $\nu \vee u = \partial_\nu f$, so

$$X_A^1(\Omega) \supset \{df :\ f \in H^1(\Omega),\ \Delta f \in L^2(\Omega), \partial_\nu f = 0\}.$$

Now the regularity result

$$f \in H^1(\Omega),\ \Delta f \in L^2(\Omega),\ \partial_\nu f = 0 \Longrightarrow f \in H^2(\Omega)$$

is true if Ω is convex, or more generally satisfies a strong exterior ball condition, but it fails for general Lipschitz Ω.

It was shown in [MMT] that, for general Lipschitz Ω,

$$X_R(\Omega),\ X_A(\Omega) \subset H^{1/2}(\Omega, \Lambda^*). \tag{1.9}$$

A closer study of the example above shows that the exponent $1/2$ cannot be improved in general. Furthermore, for a non-Lipschitz domain Ω, elements in $X_A(\Omega)$, $X_R(\Omega)$ may fail to exhibit this critical amount of regularity. An example of a domain between two cones with the same vertex and axis (thus not locally simply connected) is discussed in [CD2].

Various conditions ensuring the validity of (1.8) were given in [M2]. These include a "convexity" hypothesis on $\Omega \subset M$, and a strong exterior ball hypothesis, in case $\Omega \subset \mathbb{R}^n$. One of our main goals here is to extend that analysis, to include a broader class of Lipschitz domains for which (1.8) is valid. We define a class of "almost convex" Lipschitz domains Ω in a compact Riemannian manifold M. We show that this class contains the class of Lipschitz domains $\Omega \subset M$ satisfying a uniform exterior ball condition, which in turn contains the class of compact manifolds with corners. Furthermore we show that (1.8) holds for such almost convex domains. Part of the interest in obtaining such a result is the potential to extend the analysis of propagation of singularities in [Va] to the setting of the wave equation $(\partial_t^2 - \Delta)u = 0$ when Δ is the Hodge Laplacian and $u = u(t, x)$ a differential form, on a manifold with corners, satisfying the absolute or relative boundary condition. The regularity result (1.8) is also important in the variational treatment of the Maxwell system in the class of forms of finite L^2-energy. Cf., e.g., [CD2], [MM1], [MM2] for a discussion and references.

The rest of the paper is structured as follows. In §2 we present results on $\mathcal{D}(H^{1/2})$ and on $\mathcal{D}(H)$ valid for general Lipschitz domains. Some of these results are from [MMT], [MM1], and [MM2], and are collected here for convenience. Other results are new. In §3 we introduce the notion of almost convexity, and show that it holds whenever the uniform exterior ball condition holds, and in particular that domains with corners are almost convex. In §4 we show that (1.8) holds for almost convex domains, and establish further results on $\mathcal{D}(H)$ in this case.

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2. The Hodge Laplacian on Lipschitz domains

In this section we give further results on $X_A(\Omega)$ and on $\mathcal{D}(H_A)$ valid for general compact Lipschitz domains. Note that since the Hodge star operator is its own inverse, up to sign, satisfies $*\Delta = \Delta *$ and has the mapping properties

$$* : X_A(\Omega) \longrightarrow X_R(\Omega), \quad * : \mathcal{D}(H_A) \longrightarrow \mathcal{D}(H_R), \tag{2.1}$$

(plus a similar set with the roles of the subscripts A, R reversed), it would suffice to investigate the absolute boundary condition. We also consider some other spaces:

$$\begin{aligned} X^k(\Omega) &= \{u \in L^2(\Omega, \Lambda^k) : du, \delta u \in L^2(\Omega)\}, \\ X_b^k(\Omega) &= \{u \in X^k(\Omega) : \nu \vee u, \nu \wedge u \in L^2(\partial\Omega)\}. \end{aligned} \tag{2.2}$$

Recall that the result (1.2) makes $X_b^k(\Omega)$ well defined. We have the following trivial but occasionally useful observation:

Lemma 2.1. *The spaces $X_A^k(\Omega), X_R^k(\Omega), X^k(\Omega)$, and $X_b^k(\Omega)$ are all modules over $\mathrm{Lip}(\overline{\Omega})$.*

The following result was established in Theorem 11.2 of [MMT].

Proposition 2.2. *We have*

$$X_A^k(\Omega) \subset X_b^k(\Omega), \tag{2.3}$$

with an estimate

$$\|u\|^2_{L^2(\partial\Omega)} \leq C\big(\|du\|^2_{L^2(\Omega)} + \|\delta u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\big), \quad \forall\, u \in X^k_A(\Omega). \tag{2.4}$$

In fact, $X^k_A(\Omega)$ is a closed subspace of $X^k_b(\Omega)$.

This result leads to the inclusion (1.9), when coupled with the following, established in (11.20) of [MMT]:

Lemma 2.3. *Given $u \in X^k_b(\Omega)$, we have, on Ω,*

$$\begin{aligned} u = & -d\Pi_{k-1}(\delta u) - \delta\Pi_{k+1}(du) - \Pi_k(Vu) - \mathcal{Q}_{k-1}(\delta u) - \mathcal{R}_{k+1}(du) \\ & + \delta\mathcal{S}_{k+1}(\nu\wedge u) - d\mathcal{S}_{k-1}(\nu\vee u) + R_{k+1}(\nu\wedge u) - R_{k-1}(\nu\vee u). \end{aligned} \tag{2.5}$$

Precise definitions of these operators can be found in [MMT], particularly in (6.1)–(6.6) and (11.11), and here we shall largely restrict to reviewing their main properties and making some comments of general nature. First, $V \in L^\infty(M)$ is a suitable nonnegative function so that $\Delta - V : H^1(M,\Lambda^k) \to H^{-1}(M,\Lambda^k)$ is invertible for each $0 \leq k \leq n$ (the potential V is used to ensure a trivial kernel). Let $E_k(x,y)$ be the Schwartz kernel of the inverse of this operator and denote by Π_k the Newtonian-type potential operator associated with the integral kernel $E_k(x,y)$. Also, let $\mathcal{S}_k$ be the single layer-type integral operator $\mathcal{S}_k f(x) := \int_{\partial\Omega}\langle E_k(x,y), f(y)\rangle\, d\sigma_y$, mapping forms on $\partial\Omega$ into forms on Ω.

Then (2.5) can be though of as a higher-degree version of the classical Green's integral representation formula for scalar functions. The remaining operators, i.e. $\mathcal{Q}_k$, $\mathcal{R}_k$, mapping forms on Ω to forms on Ω, and R_k, mapping forms on $\partial\Omega$ to forms on Ω, are of residual nature and have as integral kernels commutators between $E_k(x,y)$, on the one hand, and d, δ on the other hand (in particular, they disappear altogether when $V \equiv 0$).

We will state some of the mapping properties of the above operators, for which we have further use below (complete proofs can be found in [MMT]). We have

$$\Pi_k : L^2(M) \longrightarrow H^2(M), \quad \mathcal{Q}_k, \mathcal{R}_k : L^2(M) \longrightarrow H^1(M), \tag{2.6}$$

and

$$\begin{gathered} \mathcal{S}_k : L^2(\partial\Omega) \longrightarrow H^{3/2}(\Omega), \quad R_k : L^2(\partial\Omega) \longrightarrow H^{1/2}(\Omega), \\ \mathcal{S}_k,\, R_k : L^2(\partial\Omega) \longrightarrow C^2_{\mathrm{loc}}(\Omega). \end{gathered} \tag{2.7}$$

Furthermore, for $p \in (1,\infty)$,

$$\varphi \in L^p(\partial\Omega,\Lambda^*) \Longrightarrow \mathcal{N}(d\mathcal{S}_k\varphi), \mathcal{N}(\delta\mathcal{S}_k\varphi), \mathcal{N}(R_k\varphi) \in L^p(\partial\Omega), \tag{2.8}$$

where $\mathcal{N}(\psi)$ denotes the nontangential maximal function associated to a function or form ψ on Ω. At every boundary point $x \in \partial\Omega$, the latter is defined by

$$\mathcal{N}(\psi)(x) = \sup\{|\psi(y)| : y \in \Omega, \operatorname{dist}(x, y) < \kappa \operatorname{dist}(y, \partial\Omega)\}$$

for some fixed, sufficiently large κ.

It should be mentioned that while (2.6) and the results on R_k are fairly straightforward, the results (2.7)–(2.8) on $\mathcal{S}_k$ require the fundamental results of [Ca] and [CMM], and their extension to the setting of potentials for variable coefficient operators, initiated in [MT] and carried out in the context needed here in Chapter 6 of [MMT].

It follows from (2.5)–(2.7) that

$$X_b^k(\Omega) \subset H^{1/2}(\Omega, \Lambda^k), \tag{2.9}$$

which together with (2.3) implies (1.9). Furthermore, we have the following:

Corollary 2.4. *Any $u \in X_b^k(\Omega)$ can be decomposed as $u = T_1 u + T_2 u$, where the (linear) operators T_1, T_2 enjoy the following regularity properties (up to the boundary):*

$$T_1 u \in H^1(\Omega, \Lambda^k), \qquad \mathcal{N}(T_2 u) \in L^2(\partial\Omega). \tag{2.10}$$

Also, in terms of interior regularity, $T_2 u \in C^2_{\mathrm{loc}}(\Omega, \Lambda^k)$.

Proof. Take $T_1 u$ to be the sum of the first five terms on the right side of (2.5), and take $T_2 u$ to be the sum of the last four terms. Then the desired properties are direct consequences of (2.6)-(2.8).

We will improve (2.4) and also (2.10) (for $u \in X_A(\Omega)$) later in this section, but for now we turn to other matters. The following denseness result generalizes work in [CD1], done there in the flat, three-dimensional Euclidean setting.

Proposition 2.5. *For each $k \in \{0, \dots, n\}$, the space $C^2(\overline{\Omega}, \Lambda^k)$ is dense in $X_b^k(\Omega)$.*

Proof. Via Lemma 2.1, we can assume Ω is a domain in $\mathbb{R}^n$, starlike about the origin (though with a variable coefficient, C^2 metric tensor). Then, given $u \in X_b^k(\Omega)$, write $u = T_1 u + T_2 u$ as in Corollary 2.4. Certainly $T_1 u$ is approximable in the H^1-norm, and a fortiori in the X_b^k-norm, by elements of $C^2(\overline{\Omega}, \Lambda^k)$. Furthermore, the dilates $w_r(x) = w(rx)$ for $r < 1$ of $w = T_2 u$ belong to $C^2(\overline{\Omega}, \Lambda^k)$ and we have $w_r \to w, dw_r \to dw$, and $\delta w_r \to \delta w$ in $L^2(\Omega)$ as $r \nearrow 1$, and also $w_r|_{\partial\Omega} \to w|_{\partial\Omega}$ in $L^2(\partial\Omega)$; hence $w_r \to w$ in $X_b^k(\Omega)$.

Proposition 2.5 is convenient for establishing some useful integration by parts formulas. Throughout the paper, we let dS denote the canonical surface measure on $\partial\Omega$. Also, $\langle \cdot\,, \cdot \rangle$ stands for the *pointwise* inner product of forms.

Proposition 2.6. *Given $v \in X^k(\Omega)$ with $\delta v \in X_b^{k-1}(\Omega)$, $dv \in X_b^{k+1}(\Omega)$, and $\varphi \in X_b^k(\Omega)$, we have*

$$(dv, d\varphi) + (\delta v, \delta\varphi) = -(\Delta v, \varphi) + \int_{\partial\Omega} \big[\langle \nu \vee dv, \varphi\rangle - \langle \delta v, \nu \vee \varphi\rangle\big]\, dS. \tag{2.11}$$

In particular, if $v \in H^2(\Omega, \Lambda^k)$, $\varphi \in X_A^k(\Omega)$,

$$(dv, d\varphi) + (\delta v, \delta\varphi) = -(\Delta v, \varphi) + \int_{\partial\Omega} \langle \nu \vee dv, \varphi\rangle\, dS. \tag{2.12}$$

Proof. The identity (2.11) follows by adding up

$$(dv, d\varphi) = (\delta dv, \varphi) + \int_{\partial\Omega} \langle \nu \vee dv, \varphi\rangle\, dS,$$
$$(\delta v, \delta\varphi) = (d\delta v, \varphi) - \int_{\partial\Omega} \langle \delta v, \nu \vee \varphi\rangle\, dS.$$

These identities, in turn, are easily justified by virtue of Proposition 2.5 and standard integration by parts formulas.

For applications below, it will be useful to complement Proposition 2.6 with the following result.

Lemma 2.7. *Given $\varphi \in X_b^k(\Omega)$ and $w \in C^2_{\mathrm{loc}}(\Omega, \Lambda^k)$, satisfying*

$$\Delta w = f \in L^2(\Omega, \Lambda^k), \quad \mathcal{N}(w),\ \mathcal{N}(dw),\ \mathcal{N}(\delta w) \in L^2(\partial\Omega), \tag{2.13}$$

with the boundary values taken in $L^2(\partial\Omega)$, we have

$$(dw, d\varphi) + (\delta w, \delta\varphi) = -(f, \varphi) - \int_{\partial\Omega} \langle \nu \vee dw, \varphi\rangle\, dS + \int_{\partial\Omega} \langle \delta w, \nu \vee \varphi\rangle\, dS. \tag{2.14}$$

Proof. Take a sequence $\Omega_\ell \subset\subset \Omega$ such that $\Omega_\ell \nearrow \Omega$ in a nice fashion and denote by ν_ℓ, dS, respectively, the unit conormal and surface measure on $\partial\Omega_\ell$. Then $w|_{\Omega_\ell} \in C^2(\overline{\Omega}, \Lambda^k)$ and $\varphi|_{\Omega_\ell} \in H^1(\Omega_\ell, \Lambda^k) \subset X_b^k(\Omega)$, so (2.12) applies, to give

$$\begin{aligned}(dw, d\varphi)_{L^2(\Omega_\ell)} + (\delta w, \delta\varphi)_{L^2(\Omega_\ell)} = -(f, \varphi)_{L^2(\Omega_\ell)} &- \int_{\partial\Omega_\ell} \langle \nu_\ell \vee dw, \varphi\rangle\, dS_\ell \\ &+ \int_{\partial\Omega_\ell} \langle \delta w, \nu_\ell \vee \varphi\rangle\, dS_\ell.\end{aligned} \tag{2.15}$$

It is elementary that the left side of (2.15) converges to the left side of (2.14) as $\ell \to \infty$. Now the boundary behaviors of φ and w given by hypothesis also yield convergence of the right side of (2.15) to the right side of (2.14) as $\ell \to \infty$, so we have the lemma.

It is of interest to look at the Dirac-type operator

$$D_A = d + \delta, \quad \mathcal{D}(D_A) = X_A(\Omega), \tag{2.16}$$

and its counterpart $D_R = d + \delta$, $\mathcal{D}(D_R) = X_R(\Omega)$. The Friedrichs construction of H_A and H_R entails

$$H_A = D_A^* D_A, \quad H_R = D_R^* D_R. \tag{2.17}$$

In light of this, it is valuable to have the following, which follows from Proposition 6.1 and Theorem 6.2 of [MM2]:

Proposition 2.8. *The operators D_A and D_R are self-adjoint.*

Hence we have

$$H_A = D_A^2, \quad H_R = D_R^2, \tag{2.18}$$

and consequently

$$\mathcal{D}(H_A) = \{u \in X_A(\Omega) : (d + \delta)u \in X_A(\Omega)\}. \tag{2.19}$$

Note that H_A takes k-forms to k-forms, and we can write

$$H_A = \bigoplus_k H_{A,k}, \quad \mathcal{D}(H_{A,k}) = \mathcal{D}(H_A) \cap L^2(\Omega, \Lambda^k). \tag{2.20}$$

We see that

$$\mathcal{D}(H_{A,k}) = \{u \in X_A^k(\Omega) :\, du \in X_A^{k+1}(\Omega),\, \delta u \in X_A^{k-1}(\Omega)\}. \tag{2.21}$$

In particular,

$$u \in \mathcal{D}(H_{A,k}) \Longrightarrow \nu \vee du = 0 \;\text{ on }\; \partial\Omega. \tag{2.22}$$

Membership of u to $\mathcal{D}(H_{A,k})$ also entails $\nu \vee \delta u = 0$ on $\partial\Omega$, though this is automatic from $\nu \vee u = 0$ and (6.15) of [MMT]. Another consequence of (2.21) is that

$$u \in \mathcal{D}(H_{A,k}) \Longrightarrow \delta du,\, d\delta u \in L^2(\Omega, \Lambda^k). \tag{2.23}$$

The following result gives important additional information on $\mathcal{D}(H_A)$. Related work, in the more general context of L^p with p close to 2, is given in §5 of [MM1]; cf. also [M1]. Let us also note here that, as far as the optimality of the range of possible p's is concerned, the case of three-dimensional manifolds is best understood at the moment. In this context, sharp estimates on Sobolev-Besov spaces for the Hodge Laplacian on Lipschitz domains have been recently proved in [M3].

Proposition 2.9. *There exists $p = p(\Omega) > 2$ with the following property. Let $u \in \mathcal{D}(H_{A,k})$. Assume $1 \le k \le n-1$ (since otherwise stronger results hold). Then $u = v - w$ with*

$$\begin{gathered} v \in H^2(\Omega), \quad w \in C^2_{\mathrm{loc}}(\Omega), \quad (\Delta - 1)w = 0, \\ \mathcal{N}(w),\ \mathcal{N}(dw),\ \mathcal{N}(\delta w) \in L^p(\partial\Omega). \end{gathered} \tag{2.24}$$

Also, the boundary values of w, dw and δw exist in $L^p(\partial\Omega)$.

Proof. Let $\mathcal{O}$ be an open neighborhood of $\overline{\Omega}$. Given $F \in L^2(\Omega, \Lambda^k)$, extend F to $\mathcal{O}$ and solve for v:

$$(\Delta - 1)v = F, \quad v \in H^2_{\mathrm{loc}}(\mathcal{O}). \tag{2.25}$$

Then

$$f = \nu \vee v,\ g = \nu \vee dv \in L^p(\partial\Omega) \tag{2.26}$$

for some $p > 2$ depending only on $n = \dim\Omega$. Furthermore, by (6.15) of [MMT], we have

$$f \in L^{p,\delta}_{\tan}(\partial\Omega, \Lambda^{k-1}TM), \tag{2.27}$$

a space defined by (5.2) of [MMT]. Hence, by (a simple variant of) Theorem 5.1 of [MMT], there exists $w \in C^2_{\mathrm{loc}}(\Omega, \Lambda^k)$ such that (possibly with smaller $p > 2$)

$$\mathcal{N}(w), \quad \mathcal{N}(dw), \quad \mathcal{N}(\delta w) \in L^p(\partial\Omega), \tag{2.28}$$

and

$$(\Delta - 1)w = 0, \quad \nu \vee w = f, \quad \nu \vee dw = g. \tag{2.29}$$

Set $u = v - w$. We see that

$$du = dv - dw \in L^2(\Omega), \quad \delta u = \delta v - \delta w \in L^2(\Omega), \tag{2.30}$$

and that

$$\nu \vee u = \nu \vee v - \nu \vee w = f - f = 0, \tag{2.31}$$

so

$$u \in X^k_A(\Omega). \tag{2.32}$$

Furthermore, given $\varphi \in X^k_A(\Omega)$, we have from (2.12) and (2.14) that

$$(dv, d\varphi) + (\delta v, \delta\varphi) = -(F + v, \varphi) - \int\limits_{\partial\Omega} \langle g, \varphi\rangle\, dS, \tag{2.33}$$

$$(dw, d\varphi) + (\delta w, \delta\varphi) = -(w, \varphi) - \int\limits_{\partial\Omega} \langle g, \varphi\rangle\, dS, \tag{2.34}$$

so that subtracting (2.34) from (2.33) yields

$$(du, d\varphi) + (\delta u, \delta\varphi) = -(F + u, \varphi), \quad \forall\, \varphi \in X_A^k(\Omega). \tag{2.35}$$

Thus $u \in \mathcal{D}(H_{A,k})$ and $H_{A,k}u + u = -F$.

Since this works for arbitrary $F \in L^2(\Omega, \Lambda^k)$ and since $1 + H_{A,k}$ has a bounded inverse on $L^2(\Omega, \Lambda^k)$, this proves the proposition.

We are now ready for the advertised improvements on (2.4) and (2.10). First, recall the operators T_1, T_2 introduced in Corollary 2.4.

Proposition 2.10. *There exists $p = p(\Omega) > 2$ such that*

$$u \in X_A(\Omega) \Longrightarrow u\big|_{\partial\Omega} \in L^p(\partial\Omega). \tag{2.36}$$

Furthermore, we have $u = T_1u + T_2u$ with $T_1u \in H^1(\Omega, \Lambda^*)$, $T_2u \in C^2_{\mathrm{loc}}(\Omega, \Lambda^*)$, and

$$\mathcal{N}(T_2u) \in L^p(\partial\Omega). \tag{2.37}$$

Proof. From Proposition 2.8 and (2.18) we see that any $u \in X_A(\Omega)$ can be written

$$u = (d + \delta)v + w, \quad v \in \mathcal{D}(H_A), \quad w \in \operatorname{Ker} H_A. \tag{2.38}$$

In fact, this is a manifestation of the fact that L^2-Hodge decompositions of forms are valid in any Lipschitz domain; cf. [MMT] and [MM1]. Thus (2.36) follows from (2.24). Then (2.37) follows from another application of (2.8), given the formula for T_2u.

Theorem 7.4 of [MT2] provides the following complement to (2.7):

$$\mathcal{S}_k : L^p(\partial\Omega) \longrightarrow B^{p,p^\#}_{1+1/p}(\Omega), \quad 1 < p < \infty, \tag{2.39}$$

where the target space is a Besov space and $p^\# := \max\{p, 2\}$. One has a corresponding result for R_k on $L^p(\partial\Omega)$. Hence (2.37) can be complemented by

$$T_2u \in B^{p,p}_{1/p}(\Omega), \tag{2.40}$$

for some $p > 2$. Consequently, we have the following.

Corollary 2.11. *There exists $p = p(\Omega) > 2$ such that*

$$X_A(\Omega),\, X_R(\Omega) \subset B^{p,p}_{1/p}(\Omega, \Lambda^*). \tag{2.41}$$

Let us remark that this regularity result is in the nature of best possible in the class of Lipschitz domains. Indeed, for $\omega \in (0, \pi)$ we let Ω_ω be the two-dimensional domain which coincide with the sector $\{z \in \mathbb{C} : |\arg z| < \omega\}$ near the origin, $w(z) = \mathrm{Re}\,(z^{\pi/2\omega})$, and finally set $u = dw$, suitably truncated near the origin. Then $u \in X_R(\Omega_\omega)$ and

$$u \in B^{p,p}_{1/p}(\Omega_\omega, \Lambda^1) \Longleftrightarrow p < \frac{2\omega}{2\omega - \pi}. \tag{2.42}$$

Note that $\omega \nearrow \pi$ forces $\frac{2\omega}{2\omega-\pi} \searrow 2$, justifying the claim about the sharpness of (2.41). This example can be modified to work in higher dimensions by adding extra dummy variables.

Closer inspection of the situation described above reveals that, nonetheless, $u \in H^1(\Omega_\omega, \Lambda^1)$ whenever $0 < \omega < \frac{\pi}{2}$, in which case Ω_ω is geometrically *convex*. This type of phenomenon is examined in the greater detail in Sections 3-4.

3. Almost convex domains and domains with corners

To set up our first definition, we assume we have a nested family of C^2 domains $\Omega_\ell \nearrow \Omega$. We take a neighborhood U of $\partial\Omega$ and assume $\partial\Omega_\ell \subset U$ for all ℓ. We assume each Ω_ℓ has a C^2 defining function ρ_ℓ, defined on U, strictly negative on $\Omega_\ell \cap U$ and vanishing on $\partial\Omega_\ell$, satisfying

$$C_1^{-1} \le \|d\rho_\ell(x)\| \le C_1, \quad \forall\, x \in \partial\Omega_\ell, \tag{3.1}$$

for some $C_1 \in (0, \infty)$. This ensures that each Ω_ℓ is Lipschitz, with Lipschitz constant independent of ℓ. The norm in (3.1) is defined by the metric tensor, but of course the condition (3.1) is independent of the choice of metric tensor. The *Hessian* of ρ_ℓ is defined by $\mathrm{Hess}(\rho_\ell) = \nabla d\rho_\ell$, where ∇ is the Levi-Civita connection of g. In local coordinates x this takes the form

$$\mathrm{Hess}(\rho_\ell) = \sum_{i,j} \left(\frac{\partial^2 \rho_\ell}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij} \frac{\partial \rho_\ell}{\partial x_k} \right) dx_i\, dx_j, \tag{3.2}$$

where Γ^k_{ij} are the Christoffel symbols of g. Note that by (3.1), the second term is uniformly bounded in ℓ over compact subsets of the coordinate patch, while the first term is the Hessian, $\mathrm{Hess}_x(\rho_\ell)$, of ρ_ℓ with respect to the Euclidean metric on the coordinate patch.

Our hypothesis of almost convexity is:

$$\mathrm{Hess}(\rho_\ell) \geq -C_2 g, \tag{3.3}$$

as quadratic forms on $T\partial\Omega_\ell$, for some $C_2 \in (0,\infty)$, independent of ℓ. In view of (3.2) and (3.1), an equivalent formulation is the following. Cover U by a finite number of coordinate systems (O^i, x^i), let $\mathcal{O}^i \subset O^i$ satisfy $\overline{\mathcal{O}^i} \subset O^i$, and $\cup_i \mathcal{O}^i \supset U$, and assume that in each of these local coordinates x, over $\mathcal{O}^i$,

$$\sum_{i,j} \frac{\partial^2 \rho_\ell}{\partial x_i \partial x_j} \xi_i \xi_j \geq -C_2 \sum_i \xi_i^2, \quad \text{whenever} \quad \rho_\ell = 0 \text{ and } \sum_i \frac{\partial \rho_\ell}{\partial x_i} \xi_i = 0, \tag{3.4}$$

for some (perhaps different) $C_2 \in (0,\infty)$, independent of ℓ.

This notion is independent of the choice of metric tensor, since for two Riemannian metrics g and g', by (3.2), $\nabla_g - \nabla_{g'}$ is a zeroth order differential operator from T^*M to $T^*M \otimes T^*M$, so $\mathrm{Hess}_g\, \rho_\ell - \mathrm{Hess}_{g'}\, \rho_\ell$ is uniformly bounded (from above as well as below) as a quadratic form by (3.1).

Alternatively, if we take the equivalent definition (3.4), which is clearly metric, but not coordinate, independent, then we can see directly that almost convexity is independent of the coordinate systems chosen. In fact, take two coordinate systems $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. By the chain rule, we have

$$\mathrm{Hess}_x(\rho_\ell) = \sum_{i,j,r,s,m,k} \frac{\partial y_r}{\partial x_i} \frac{\partial}{\partial y_r} \Big(\frac{\partial y_s}{\partial x_j} \frac{\partial \rho_\ell}{\partial y_s} \Big) \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_m}\, dy_k\, dy_m = I_\ell + II_\ell, \tag{3.5}$$

where the terms I_ℓ and II_ℓ arise when the derivative $\partial/\partial y_r$ falls on $\partial y_s/\partial x_j$ or $\partial \rho_\ell/\partial y_s$, respectively. By (3.1), the coefficient of each term $dy_k\, dy_m$ in I_ℓ is uniformly bounded in ℓ, so we have $-C_3 g \leq I_\ell \leq C_3 g$ for a suitable constant C_3. On the other hand,

$$II_\ell = \sum_{k,m} \frac{\partial^2 \rho_\ell}{\partial y_k \partial y_m}\, dy_k\, dy_m = \mathrm{Hess}_y(\rho_\ell), \tag{3.6}$$

so the coordinate independence of the definition of almost convexity is established.

We state another characterization of almost convexity. Let $l_\ell : T\partial\Omega_\ell \otimes T\partial\Omega_\ell \to \mathbb{R}$ denote the real-valued second fundamental form of $\partial\Omega_\ell$. Recall that l_ℓ depends on the choice of a unit normal vector field at $\partial\Omega_\ell$; we normalize it using the outward pointing normal vector n_ℓ to $\partial\Omega_\ell$. Then l_ℓ is given by $l_\ell(u,v) = g(\nabla_u n_\ell, v)$, $u, v \in T_p\partial\Omega_\ell$, where ∇ is the covariant derivative in M. This can be rephrased in terms of the outward pointing conormal, ν_ℓ. Namely, using $g(n_\ell, v) = 0 = \nu_\ell(v)$ for $v \in T\partial\Omega_\ell$, we deduce that $l_\ell(u,v) = (\nabla_u \nu_\ell)(v)$. Then $\nu_\ell = \frac{d\rho_\ell}{\|d\rho_\ell\|}$, and $l_\ell = \frac{\mathrm{Hess}\,\rho_\ell}{\|d\rho_\ell\|}$. Thus, almost convexity is equivalent to requiring that l_ℓ be bounded below, uniformly in ℓ.

We can compare the notion of almost convexity with that of k-convexity in the sense of [M2], which requires

$$\sum_{i,j} (\rho_\ell)_{ij} \langle w_i \wedge u, w_j \wedge u \rangle \geq 0 \tag{3.7}$$

on $\partial\Omega_\ell$, where the objects involved have the following significance. First, $1 \leq k \leq n$ and u is an k-form satisfying $\nu \wedge u = 0$ on $\partial\Omega_\ell$. Second, $\langle \cdot\,, \cdot \rangle$ is the (pointwise) inner product on $\Lambda^{k+1}T^*U$ induced by the metric tensor g. Third, $\{w_i : 1 \leq i \leq n\}$ is a local orthonormal frame field for T^*U, and, by definition, we take $\{\partial/\partial w^i\}_i$ to be the (local) orthonormal basis of TM which is dual to $\{w^i\}_i$. Finally, $(\rho_\ell)_{ij}$ stands for the expression $\frac{1}{2}\Big\{\frac{\partial^2 \rho_\ell}{\partial w^i \partial w^j} + \frac{\partial^2 \rho_\ell}{\partial w^j \partial w^i}\Big\}$. The proof of Proposition 3.2 of [M2] shows that if Ω is almost convex, then one has, in place of (3.7),

$$\sum_{i,j} (\rho_\ell)_{ij} \langle w_i \wedge u, w_j \wedge u \rangle \geq -C\langle u, u \rangle, \tag{3.8}$$

for some $C \in (0, \infty)$, for such u as above, and for all $k \leq n$.

We now discuss some important special classes of almost convex domains. We say that a Lipschitz domain $\Omega \subset M$ satisfies a local exterior ball condition, henceforth referred to as LEBC, if for every boundary point $x_0 \in \partial\Omega$ there exists a coordinate patch $\mathcal{O}$ which contains x_0 and which satisfies the following two conditions.

First, there exists a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\varphi(0) = 0$ and such that if D is the domain above the graph of φ then D satisfies the standard Euclidean uniform exterior ball condition (UEBC). Second, it is assumed that there exists a diffeomorphism Υ mapping $\mathcal{O}$ onto the unit ball $B_1(0)$ in $\mathbb{R}^n$ and such that $\Upsilon(x_0) = 0$, $\Upsilon(\mathcal{O} \cap \Omega) = B_1(0) \cap D$, $\Upsilon(\mathcal{O} \setminus \overline{\Omega}) = B_1(0) \setminus \overline{D}$.

Proposition 3.1. *If the Lipschitz domain $\Omega \subset M$ satisfies a LEBC then it is almost convex.*

Proof. We need to construct a nested sequence of approximating domains Ω_ℓ of bounded Lipschitz character, which have C^2 defining functions ρ_ℓ with the properties described above.

The construction is local in nature and, given that the original domain satisfies a LEBC, there is no loss of generality in assuming that a system of local coordinates $\{x_i\}_i$ has been selected for which $x_0 = 0$, g is the Euclidean metric in this system of coordinates, and Ω is the domain above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\varphi(0) = 0$ satisfying the following property. There exists $C_0 > 0$ such that for a.e. $a \in \mathbb{R}^{n-1}$ and $\forall\, v \in \mathbb{R}^{n-1}$, $|v| \leq C_0$, there holds

$$2\varphi(a) - \varphi(a+v) - \varphi(a-v) \leq C_0|v|^2. \tag{3.9}$$

That the latter condition can be assumed is a consequence of our definition of LEBC and Lemma 6.3 in [M2]. In this scenario, following the construction in §6 of [M2], we take, for each $\ell \geq 1$,

$$\rho_\ell(x) := \varphi_\ell(x') - x_n, \qquad x = (x', x_n) \in \mathbb{R}^n. \tag{3.10}$$

Above, for each $x' \in \mathbb{R}^{n-1}$,

$$\varphi_\ell(x') := \frac{C}{\ell} + \int\limits_{\mathbb{R}^{n-1}} \Phi_\ell(y')\varphi(x'-y')\, dy', \tag{3.11}$$

where $C > 0$ is a fixed constant (to be specified below), $\Phi \in C^\infty(\mathbb{R}^{n-1})$, $0 \leq \Phi \leq 1$, $\Phi \equiv 0$ for $|x'| > 1$, $\int_{\mathbb{R}^{n-1}} \Phi\, dx' = 1$ and, as is customary, $\Phi_\ell(x') := \ell^{n-1}\Phi(\ell x')$. If we now set

$$\Omega_\ell := \{x : \rho_\ell(x) < 0\} = \{(x', x_n) : \varphi_\ell(x') < x_n\}, \tag{3.12}$$

it follows that $\partial\Omega_\ell \in C^\infty$. Furthermore, if the constant C in (3.11) is sufficiently large, then the family (3.12) is nested, $\overline{\Omega}_\ell \subseteq \Omega$ and $\cup_{1<\ell<\infty}\Omega_\ell = \Omega$. More specifically, if $C > \|\nabla\varphi\|_{L^\infty}$ then

$$\frac{d}{d\mu}\Big[\varphi_{1/\mu}(x')\Big] = C - \int\limits_{\mathbb{R}^{n-1}} \Phi(z')\, z' \cdot (\nabla\varphi)(x' - \mu z')\, dz' > 0 \tag{3.13}$$

which ensures that the mapping $\ell \mapsto \varphi_\ell(x')$ is decreasing. Thus, $\varphi_\ell(x') \searrow \varphi(x')$ as $\ell \nearrow \infty$. In addition, by virtue of (3.9) – a feature also inherited by each φ_ℓ – the domain Ω_ℓ satisfies a UEBC with a constant independent of $\ell \in (1, \infty)$. Thanks to Lemma 6.3 in [M2], this last property further entails the existence of some $C > 0$ such that

$$\mathrm{Hess}(\varphi_\ell) \geq -C \tag{3.14}$$

uniformly in $\ell \in (1, \infty)$. Here $\mathrm{Hess}(\varphi_\ell)$ is the Hessian of φ_ℓ in the coordinates $\{x_i\}_{1\leq i\leq n-1}$, viewed as a symmetric $(n-1)\times(n-1)$ matrix.

It follows that for *all* vectors $t = (t', t_n) \in \mathbb{R}^n$,

$$\mathrm{Hess}_x(\rho_\ell)\, t \cdot t = \mathrm{Hess}(\varphi_\ell)\, t' \cdot t' \geq -C|t'|^2 \geq -C|t|^2, \tag{3.15}$$

uniformly in ℓ. This is one of the two conditions we set to check (recall that we have chosen g to be the Euclidean metric in the system of coordinates $\{x_i\}_i$) The remaining one, (3.1), is easily seen from (3.11). Indeed, $\|d\rho_\ell(x)\| \approx (1 + |\nabla\varphi_\ell(x)|^2)^{1/2}$ and $|\nabla\varphi_\ell(x)| \leq |\nabla\varphi(x)|$, uniformly in ℓ.

REMARK. What (3.9) says is that, in the approximation scheme $\Omega_\ell \nearrow \Omega$ we have constructed for a domain Ω satisfying a LEBC, the Hessians of the defining functions ρ_ℓ for Ω_ℓ are bounded from below on the *entire* tangent space to M, uniformly in ℓ, rather than just on $T\partial\Omega_\ell$ as required for almost convex domains. (In fact, if ρ_ℓ satisfy the almost convexity hypotheses (3.1) and (3.3), they can be replaced by some $\tilde\rho_\ell = F_\ell(\rho_\ell)$ so that for $\tilde\rho_\ell$, (3.1) still holds, and the lower bound (3.3) holds on the entire tangent space.) This gives a heuristic explanation as to why domains with LEBC happen to be almost convex.

A Lipschitz domain $\Omega \subset M$ is said to be a domain with corners provided that each $p \in \partial\Omega$ has a neighborhood $\mathcal{O}$ on which there are coordinates $x_1, \ldots, x_n$ such that $\overline{\Omega} \cap \mathcal{O}$ is defined by

$$x_j \geq 0 \quad \text{for} \quad 1 \leq j \leq m, \tag{3.16}$$

for some $m \in \{1, \ldots, n\}$. The following is apparent.

Proposition 3.2. *Every domain with corners satisfies a LEBC, and hence is almost convex.*

REMARK. It is an interesting exercise to prove *directly* that any domain with corners is almost convex. This can be shown locally, in particular in some local coordinate system $(x_1, ..., x_n)$ where it may be assumed that Ω is given by the system of inequalities (3.16) plus the requirement that all x_j's are bounded. Then one can define the family Ω_ℓ as $\{\rho_\ell < 0\}$ where $\rho_\ell = \tilde\rho_\ell/\|d\tilde\rho_\ell\|$ and $\tilde\rho_\ell = \ell^{-1} - x_1 \cdots x_m$ where $\ell > 0$ is a parameter which we will eventually allow to approach infinity. Thus, clearly, (3.1) holds. To verify (3.4) note that the condition $\sum_i(\partial\rho_\ell/\partial x_i)\xi_i = 0$ on $\partial\Omega_\ell$ entails $\sum_i \xi_i/x_i = 0$. On the other hand, the semi-boundedness condition on the Hessian amounts to checking that

$$\sum_{i\neq j} \frac{-1}{\ell}\frac{\xi_i\xi_j}{x_i x_j} \geq -C|\xi|^2\|d\tilde\rho_\ell\| \tag{3.16}$$

whenever the vector $\sum_i \xi_i(\partial/\partial x_i)$ is tangent to the zero set of ρ_ℓ. However, for such a vector,

$$0 = \Big(\sum_i \frac{\xi_i}{x_i}\Big)\Big(\sum_j \frac{\xi_j}{x_j}\Big) = \sum_{i,j} \frac{\xi_i \xi_j}{x_i x_j} = \sum_i \frac{\xi_i^2}{x_i^2} + \sum_{i \neq j} \frac{\xi_i \xi_j}{x_i x_j}. \tag{3.17}$$

As the first term in the rightmost expression is non-negative, the Hessian condition follows (with any $C \geq 0$).

4. The Hodge Laplacian on almost convex domains

We aim to prove the following.

Theorem 4.1. *If Ω is an almost convex domain, then*

$$\mathcal{D}(H_A^{1/2}) = H_A^1(\Omega, \Lambda^*) = \big\{u \in H^1(\Omega, \Lambda^*) : \nu \vee u\big|_{\partial\Omega} = 0\big\}, \tag{4.1}$$

and

$$\mathcal{D}(H_R^{1/2}) = H_R^1(\Omega, \Lambda^*) = \big\{u \in H^1(\Omega, \Lambda^*) : \nu \wedge u\big|_{\partial\Omega} = 0\big\}. \tag{4.2}$$

From this and Proposition 3.2 we may therefore readily conclude the following.

Corollary 4.2. *The identities (4.1)-(4.2) hold for any domain Ω with corners.*

A quick sketch of the proof of Theorem 4.1 is as follows. For a given form u, we follow the proof of Theorem 5.1 in [M2], noting that the only place where a modification is needed is equation (5.9). There, the boundary term

$$\sum_{i,j} \int_{\partial\Omega_\ell} (\rho_\ell)_{ij} \langle w_i \wedge v_\ell, w_j \wedge v_\ell \rangle \, dS_\ell$$

(in [M2], the index ℓ is actually denoted by μ) can be dropped from a subsequent estimate since it is non-negative by the convexity assumption (3.7). However, it suffices if one can estimate this term from below by $-C\|v_\ell\|_{H^1(\Omega_\ell)}\|v_\ell\|_{L^2(\Omega_\ell)}$ for some $C > 0$, for then the effect on (5.11) in [M2] is the same as the remainder term (5.10), which is already controlled. But this follows from (3.8) combined with

$$\|v_\ell\|^2_{L^2(\partial\Omega_\ell)} \leq C' \|v_\ell\|_{H^1(\Omega_\ell)} \|v_\ell\|_{L^2(\Omega_\ell)}, \tag{4.3}$$

with C' independent of ℓ (which holds as the domains Ω_ℓ are uniformly Lipschitz). This allows one to get (4.2) from the arguments used to prove Theorem 5.1 of [M2], and then (4.1) follows by applying the Hodge star operator.

For the convenience of the reader, we describe in more detail how the Hessian shows up in the proof of this theorem. The key point is to relate $\|dv\|^2_{L^2(\Omega_\ell)} + \|\delta v\|^2_{L^2(\Omega_\ell)}$ to the H^1-norm of v for v satisfying $\nu \wedge v = 0$, where $\nu = \nu_\ell$. By density –cf., e.g., Proposition 2.5– it suffices to do this analysis for $v \in C^2(\overline{\Omega}_\ell, \Lambda^*)$. We assume that this is the case and *henceforth drop the subscript* ℓ. More precisely, let $\widetilde{\nabla}$ be any first order differential operator on differential forms with the same principal symbol as the Levi-Civita connection ∇. (The only reason for not simply taking ∇ is to make the final estimate depend only on at most the first derivatives of g.) Then $d\delta + \delta d = -\Delta$ differs from $\widetilde{\nabla}^*\widetilde{\nabla}$ by a first order operator, so

$$(d\delta v, v)_{L^2} + (\delta dv, v)_{L^2} - (\widetilde{\nabla}^*\widetilde{\nabla} v, v)_{L^2} = R(v, v), \tag{4.4}$$

with $|R(v, v)| \leq C\|v\|_{L^2}\|v\|_{H^1}$. Now, for any first-order differential operator P on a smooth compact manifold with boundary X,

$$(Pu, v)_{L^2} = (u, P^*v)_{L^2} + \frac{1}{i}\int_{\partial X} \langle \sigma_P(x, \nu)u, v\rangle \, dS. \tag{4.5}$$

As $(1/i)\sigma_d(x, \xi)u = \xi \wedge u$, $(1/i)\sigma_\delta(x, \xi)u = -\xi \vee u$, and $(1/i)\sigma_\nabla(x, \xi)u = \xi \otimes u$, we deduce that

$$\begin{aligned}(d\delta v, v)_{L^2} &+ (\delta dv, v)_{L^2} - (\widetilde{\nabla}^*\widetilde{\nabla} v, v)_{L^2} \\ &= \|\delta v\|^2_{L^2} + \|dv\|^2_{L^2} - \|\widetilde{\nabla} v\|^2_{L^2} + \int_{\partial\Omega} \Big(\langle \nu \wedge \delta v, v\rangle - \langle \nu \vee dv, v\rangle + \langle \widetilde{\nabla}_\nu v, v\rangle \Big) \, dS.\end{aligned} \tag{4.6}$$

Note that $\langle \nu \vee dv, v\rangle = \langle dv, \nu \wedge v\rangle = 0$ since $\nu \wedge v = 0$, so the middle term in the boundary integral can be dropped.

To proceed further, extend ν to a 1-form on a neighborhood of $\partial\Omega$, so $\nu \wedge v$ is a form defined on a neighborhood of $\partial\Omega$ as well. Since $\nu \wedge v$ vanishes on $\partial\Omega$, $\nu \wedge v = \rho\tilde{v}$. Moreover,

$$\delta(f\tilde{v}) = f\delta\tilde{v} - df \vee \tilde{v}, \tag{4.7}$$

so on $\partial\Omega$ we have $\langle \delta(\rho\tilde{v}), v\rangle = -\langle \tilde{v}, d\rho \wedge v\rangle = 0$, since $d\rho$ is a multiple of ν and $\nu \wedge v = 0$ on $\partial\Omega$ by hypothesis. Thus we can add $\int_{\partial\Omega} \langle \delta(\nu \wedge v), v\rangle \, dS$ to the right side of (4.6), to obtain

$$\|\delta v\|^2_{L^2} + \|dv\|^2_{L^2} = \|\widetilde{\nabla} v\|^2_{L^2} - \int_{\partial\Omega} \langle \delta(\nu \wedge v) + \nu \wedge \delta v + \widetilde{\nabla}_\nu v, v\rangle \, dS + R(v, v), \tag{4.8}$$

with $R(v, v)$ as above.

To examine the integrand in (4.8), consider

$$P_\nu v = \delta(\nu \wedge v) + \nu \wedge \delta v + \widetilde{\nabla}_\nu v. \tag{4.9}$$

This is ostensibly a first-order differential operator, but its principal symbol satisfies

$$i\sigma_{P_\nu}(x,\xi)v = \xi \vee (\nu \wedge v) + \nu \wedge (\xi \vee v) - \langle \nu, \xi \rangle v = 0. \tag{4.10}$$

Hence P_ν is actually a zero-order operator. Moreover, the only term in P_ν that depends on derivatives of ν is the first one. Since (4.7) holds for functions f and forms $\tilde{v}$, if we write $\nu = \sum f_i\, dx_i$, then in fact

$$v \mapsto P_\nu v + \sum_i df_i \vee (dx_i \wedge v) \tag{4.11}$$

is not only zero order, but its norm is uniformly bounded as long as the functions f_i are uniformly bounded on $\partial\Omega$, i.e., as long as ν is uniformly bounded. Consequently, using

$$\sum_i \langle df_i \vee (dx_i \wedge v), v \rangle = \sum_i \langle dx_i \wedge v, df_i \wedge v \rangle = \sum_{i,j} \frac{\partial^2 \rho}{\partial x_i \partial x_j} \langle dx_i \wedge v, dx_j \wedge v \rangle, \tag{4.12}$$

we deduce that

$$\|\delta v\|_{L^2}^2 + \|dv\|_{L^2}^2 = \|\widetilde{\nabla} v\|_{L^2}^2 + \int\limits_{\partial\Omega} \sum_{i,j} \frac{\partial^2 \rho}{\partial x_i \partial x_j} \langle dx_i \wedge v, dx_j \wedge v \rangle \, dS + R'(v,v), \tag{4.13}$$

with $|R'(v,v)| \leq C\|v\|_{L^2}\|v\|_{H^1}$. For the estimates of the theorem, one wants the integral on the right to be positive, modulo terms that can be absorbed into R'. This is certainly satisfied for almost convex domains, and indeed this motivates our definition. If we add a large multiple of $\|v\|_{L^2}^2$ to both sides of (4.13), $R'(v,v)$ can be absorbed by reducing the constants in front of $\|\widetilde{\nabla} v\|_{L^2}^2$ and $\|v\|_{L^2}^2$, giving the desired uniform estimate for Ω (which, we recall, stands for Ω_ℓ here).

Parenthetically, we note that for $(n-1)$-forms v,

$$\langle dx_i \wedge v, dx_j \wedge v \rangle = (dx_i \vee *v)\overline{(dx_j \vee *v)}, \tag{4.14}$$

so the quadratic form on normal forms is equivalent to $\sum (\partial^2 \rho / \partial x_i \partial x_j)\, dx_i\, dx_j$ on the space of vectors tangent to $\partial\Omega$.

The final ingredient in the proof of Theorem 4.1, as described in the proof of Theorem 5.1 in [M2], is an approximation argument. Consider an approximating sequence $\Omega_\ell \nearrow \Omega$ as in the proof of Lemma 2.7 and, for $u \in \mathcal{D}(H_R^{1/2}) = X_R(\Omega)$, let u_ℓ be the solution of

$$(\Delta - 1)u_\ell = 0, \quad \delta u_\ell = 0, \quad \nu_\ell \wedge u_\ell = \nu_\ell \wedge u \ \text{ on } \ \partial\Omega_\ell, \tag{4.15}$$

satisfying

$$u_\ell,\ du_\ell \in L^2(\Omega_\ell, \Lambda^*). \tag{4.16}$$

Letting $v_\ell = u|_{\Omega_\ell} - u_\ell$, one shows, using (4.13), that v_ℓ converges to some $v \in H^1(\Omega, \Lambda^*)$ weakly in H^1. On the other hand, a direct argument using the solution of the auxiliary problem shows that $u_\ell \to 0$ weakly in L^2; this is quite natural since u_ℓ solves a homogeneous problem with boundary data going to 0 as $\ell \to \infty$. Combined, these two show that $u = v$, so $u \in H^1(\Omega, \Lambda^*)$. We refer to [M2] for more details on this sort of argument.

Remark. Known results on the Dirichlet and Neumann boundary problems imply that, when Ω satisfies the LEBC,

$$\mathcal{D}(H_{A,k}),\ \mathcal{D}(H_{R,k}) \subset H^2(\Omega, \Lambda^k), \quad \text{for } \ k = 0 \ \text{ or } \ n.$$

However, there is no $\sigma > 0$ for which one can say $\mathcal{D}(H_{A,1}) \subset H^{1+\sigma}(\Omega, \Lambda^1)$ for all such Ω. One can see this by considering the example introduced in §1. We have

$$\mathcal{D}(H_{A,1}) \supset \{df : f \in H^2(\Omega),\ \Delta f \in H^1(\Omega), \partial_\nu f = 0\}.$$

Simple counterexamples show that such f need not belong to $H^{2+\sigma}(\Omega)$ for any $\sigma > 0$.

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