Quantum fields from global propagators on asymptotically Minkowski and extended de Sitter spacetimes

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Abstract. We consider the wave equation on asymptotically Minkowski spacetimes and the Klein-Gordon equation on even asymptotically de Sitter spaces. In both cases we show that the extreme difference of propagators (i.e. retarded propagator minus advanced, or Feynman minus anti-Feynman), defined as Fredholm inverses, induces a symplectic form on the space of solutions with wave front set confined to the radial sets. Furthermore, we construct isomorphisms between the solution spaces and symplectic spaces of asymptotic data. As an application of this result we obtain distinguished Hadamard two-point functions from asymptotic data. Ultimately, we prove that the corresponding Quantum Field Theory on asymptotically de Sitter spacetimes induces canonically a QFT beyond the future and past conformal horizon, i.e. on two even asymptotically hyperbolic spaces. Specifically, we show this to be true both at the level of symplectic spaces of solutions and at the level of Hadamard two-point functions.

1. Introduction and summary of results

1.1. **Introduction.** As understood nowadays, the rigorous construction of a non-interacting Quantum Field Theory associated to a hyperbolic differential operator P on a given spacetime (M°, g) is crucially based on two ingredients. The first one is the existence of advanced and retarded (also called backward and forward) propagators $P_\pm^{-1}$, i.e. inverses of P that solve the inhomogeneous problem $Pu = f$ for f vanishing at respectively future or past infinity[1]. The relevant properties of the propagators that one seeks to prove crucially rely on decay estimates (or support properties) of $P_\pm^{-1} f$ given decay (or compact support) of f. Specifically, one needs for instance to show that the formal adjoint of P_+^{-1} is P_-^{-1}, so that $P_+^{-1} - P_-^{-1}$ is anti-hermitian (or, by abuse of terminology, symplectic) when identified with a sesquilinear form using the volume density. Then by acting with $P_+^{-1} - P_-^{-1}$ on say, test functions, one gets a space of solutions equipped with the induced symplectic form. One obtains this way a *symplectic space*[2] *of solutions* of P that physically represents the classical field theory.

The second ingredient one needs is a way to specify a quantum state. Without going into details (cf. Appendix A), this can be conveniently reformulated as the problem of constructing *two-point functions* (here more specifically *bosonic* ones), which in the

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[1] The convention for the signs in $P_\pm^{-1}$ is taken to be different from the one used typically in the QFT literature, for the sake of consistency with e.g. [52].

[2] By abuse of terminology we consider symplectic forms to be sesquilinear.

present setup will be pairs of operators $\Lambda^{\pm}$ acting, say, on test functions, such that

$$P\Lambda^{\pm} = \Lambda^{\pm}P = 0, \quad \Lambda^{+} - \Lambda^{-} = \mathrm{i}(P_{+}^{-1} - P_{-}^{-1}), \quad \Lambda^{\pm} \geq 0, \tag{1.1}$$

where positivity refers to the canonical sesquilinear pairing obtained from the volume form. In the case of globally hyperbolic spacetimes (cf. recent reviews [31, 39]), the present consensus is that physically reasonable two-point functions should in addition satisfy the *Hadamard condition*

$$\mathrm{WF}'(\Lambda^{\pm}) = \textstyle\bigcup_{t\in\mathbb{R}} \Phi_t(\mathrm{diag}_{T^*M^\circ}) \cap \pi^{-1}\Sigma^{\pm}, \tag{1.2}$$

where $\bigcup_{t\in\mathbb{R}} \Phi_t(\mathrm{diag}_{T^*M^\circ})$ is the flowout of the diagonal in $(T^*M^\circ \times T^*M^\circ) \setminus o$ by the bicharacteristic flow of the wave operator $\Box_g$ (Φ_t acts on the left component), $\Sigma^{\pm}$ are the two connected components of its characteristic set and π projects to the left component. Such operators do exist indeed in the case of the Klein-Gordon and wave equation [20, 22] and are unique modulo smooth terms (i.e. modulo operators with smooth kernel) [46]. This key result is fundamentally based on Duistermaat and Hörmander's real principal type propagation of singularities theorem [16]. Since one is however interested in setting up QFTs on more general manifolds [37, 58], potentially with boundary [38, 59], one is naturally led to revisit propagation of singularities theorems and their connections to inverses of P.

Incidentally, all these ingredients are reassembled in a recent approach to propagation estimates that uses microlocal analysis in a *global* setup [53, 30, 29, 25]. The main technical feature are propagation of singularities theorems that (in contrast to Hörmander's work) are also valid near *radial sets*, where the bicharacteristic flow degenerates. These are expressed as estimates microlocalized along the bicharacteristic flow, which then can be combined to yield a global estimate at least if one can get around potential issues induced by trapping. Ultimately, if this is the case, the estimate in question translates to the Fredholm property of P acting between several choices of Hilbert spaces $\mathcal{X}_I$, $\mathcal{Y}_I$, whose precise definition depends on the details of the setup and refers in particular to the bicharacteristic flow. One obtains this way generalized inverses P_I^{-1}, whose wave front set can be deduced from their mapping properties.

Before discussing this in a more specific setup, let us point out the main difficulty in adapting this strategy to the construction of two-point functions. Although one could fairly easily define a pair of operators $\Lambda^{\pm}$ satisfying the Hadamard condition (1.2) by taking the difference of two adequately chosen inverses of P, one would not expect the positivity condition $\Lambda^{\pm} \geq 0$ to hold apart from exceptional cases (even though under quite general assumptions it is actually possible to get this way $\Lambda^{+} + \Lambda^{-} \geq 0$, cf. [52]). One possible alternative is to define $\Lambda^{\pm}$ by specifying its asymptotic data, in terms of which positivity can be hoped to be realized explicitly. In fact, this strategy has already been successfully applied indeed in the case of the conformal wave equation on a class of asymptotically flat spacetimes [44, 45, 21] (see also [10, 13, 14] for other classes of spacetimes), where one can consider as data at future null infinity the characteristic Cauchy data for a conformally rescaled metric. Recent advances show also that one can define Hadamard states for some asymptotically flat spacetimes using tools from scattering theory [24]. An additional motivation for this point of view is that in QFT one is interested in constructing two-point functions with specific global or asymptotic properties (including symmetries): this has been a very active field of study recently [10, 12, 13, 44, 48] and is still the subject of many conjectures.

In the present paper we consider the (rescaled, see below) wave operator P on asymptotically Minkowski spacetimes and the Klein-Gordon operator $\hat{P}_X$ on a class of asymptotically de Sitter spacetimes. Asymptotic data of solutions will be realized by regarding solutions as conormal distributions of a certain type, and then global inverses of P and $\hat{P}_X$ (also called propagators) will serve us to construct the associated *Poisson operators*, i.e. the maps that assign to given asymptotic data the corresponding solution.

QFT on asymptotically Minkowski spacetimes. As an illustration of our setup, we start with the special case of the radial compactification of Minkowski space.

Namely, if $M^\circ = \mathbb{R}^{1+d}$ is Minkowski space with its metric $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$, we replace it by a compact manifold with boundary M by making the change of coordinates $z_i = \rho\vartheta_i$ (with ϑ_i coordinates on the sphere $\mathbb{S}^d$) away from the origin, and then gluing a sphere at infinity, i.e. the boundary of M is $\partial M = \{\rho = 0\}$ with $\rho = (z_0^2 + z_1^2 + \cdots + z_d^2)^{-1/2}$. In the setup of Melrose's b-analysis [41], which lies at the heart of our approach, regularity and decay are measured relatively to weighted b-Sobolev spaces $H_{\rm b}^{m,l}(M) = \rho^l H_{\rm b}^m(M)$, where (away from the origin, and in a particular spherical coordinate chart U_i, say ϑ_j, $j = 0, \dots, n$, $j \neq i$) the b-Sobolev space $H_{\rm b}^m(M)$ is essentially the Sobolev space $H^m(\mathbb{R}^{1+d})$ in coordinates $(-\log\rho, \{\vartheta_j : \ j \neq i\}) \in \mathbb{R} \times U \subset \mathbb{R} \times \mathbb{R}^d$. The space of smooth functions vanishing to arbitrary order at the boundary can be conveniently characterized as $\dot{\mathcal{C}}^\infty(M) = \bigcap_{m,l\in\mathbb{R}} H_{\rm b}^{m,l}(M)$ and its dual provides a useful space of distributions denoted $\mathcal{C}^{-\infty}(M)$.

The definition of $H_{\rm b}^{m,l}(M)$ can be modified to allow for orders m that vary on M and in the dual variables [55]. Specifically, we will need here m to be monotone along the (suitably reinterpreted, cf. Subsect. 2.3) bicharacteristic flow and for each of the two connected components $\Sigma^\pm$, m needs to be larger than the *threshold value* $\frac{1}{2} - l$ near one of the ends and smaller than $\frac{1}{2} - l$ near the other. This gives in total four distinct choices that we label by a subset $I \subset \{+, -\}$ that indicates the components of $\Sigma^+ \cup \Sigma^-$ along which m is taken to be increasing. For any such (m, l), the choice of m is actually immaterial in terms of the Fredholm/invertibility properties discussed below, as long as the properties described above, including the ends at which the particular inequalities hold, are kept unchanged. The main outcome of the recent work of Gell-Redman, Haber and Vasy [25] that we use here is that the rescaled wave operator

$$P := \rho^{-(d-1)/2}\rho^{-2}\Box_g\rho^{(d-1)/2} : \mathcal{X}_I \to \mathcal{Y}_I$$

is Fredholm as an operator acting on the Hilbert spaces

$$\mathcal{X}_I := \left\{ u \in H_{\rm b}^{m,l}(M) : \ Pu \in H_{\rm b}^{m-1,l}(M) \right\}, \quad \mathcal{Y}^{m,l} := H_{\rm b}^{m,l}(M),$$

for *any* m, l consistent with the choice of $I \subset \{+, -\}$, apart from a discrete set of values of l; P is actually invertible for $|l|$ small; and the same holds true if M is a small perturbation of (radially compactified) Minkowski spacetime. With the conventions used in the present paper, the operators $P_{\{\pm\}}^{-1}$, denoted also $P_\pm^{-1}$, are precisely the retarded/advanced propagators. On the other hand, the remaining two, $P_\emptyset^{-1}$ and

$P^{-1}_{\{+,-\}}$ are named Feynman and anti-Feynman propagator [25] and we show that they have indeed the same wave front set as the Feynman/anti-Feynman parametrices of Duistermaat and Hörmander [16].

Our first result directly relevant for QFT on perturbations of Minkowski space is that, for l not in the discrete set above, the extreme propagator difference defines a bijection

$$P_I^{-1} - P_{I^{\rm c}}^{-1} : \frac{H_{\rm b}^{\infty,l}(M)}{PH_{\rm b}^{\infty,l}(M)} \longrightarrow {\rm Sol}(P), \tag{1.3}$$

where $H_{\rm b}^{\infty,l}(M) = \bigcap_{m\in\mathbb{R}} H_{\rm b}^{m,l}(M)$ and ${\rm Sol}(P)$ consists of solutions of P that are smooth in the interior M° of M (more precisely, with b-wave front set only at the radials sets). Furthermore, $P_I^{-1} - P_{I^{\rm c}}^{-1}$ is formally anti self-adjoint [52], therefore by (1.3), for $l = 0$ this induces a symplectic form on ${\rm Sol}(P)$. In the advanced/retarded case $I = \{\pm\}$ the resulting symplectic space of solutions represents the classical (bosonic) field theory (in fact, in our setup it plays the same role as the space of smooth space-compact solutions in standard formulations, cf. [3]). On the other hand, the validity of (1.3) in the Feynman/anti-Feynman case ($I = \emptyset/\{+,-\}$) is far more puzzling as it seems to have no direct analogue in well-known QFT constructions. Let us point out, however, that by a recent result [52], ${\rm i}^{-1}(P_I^{-1} - P_{I^{\rm c}}^{-1})$ is positive for $I = \emptyset$ (when appropriately identified with a sesquilinear form, cf. Subsect. 4.1), therefore in that case the spaces (1.3) meet all the formal requirements for a *fermionic* classical field theory (indeed from the mathematical point of view one needs a pre-Hilbert space, as opposed to bosonic field theory where a symplectic space suffices, cf. for instance [15]). While we restrain ourselves from interpreting this observation too literally, the use of fermionic terminology will turn out to be helpful in our discussion of two-point functions constructed from asymptotic data.

Before discussing the latter in more detail, let us point out that after suitable modifications our result (1.3) also applies to the class of asymptotically Minkowski spacetimes considered by Baskin, Vasy and Wunsch [6] and Hintz and Vasy [29], which includes (globally hyperbolic) small perturbations of Minkowski space, but is also believed to include some non globally hyperbolic examples[3], cf. Section 2 for the precise assumptions. In this greater generality, the work of Gell-Redman, Haber and Vasy gives the Fredholm property of $P_I := P : \mathcal{X}_I \to \mathcal{Y}_I$ rather than its invertibility (unless for instance $I = \{\pm\}$ and M° is globally hyperbolic) for all $l \in \mathbb{R}$ except for a discrete subset corresponding to *resonances*. Consequently P_I^{-1} makes sense merely as a generalized inverse, mapping from the range of P_I to a predefined complement of the kernel of P_I. Nevertheless, the spaces in (1.3) can be modified by removing some finite dimensional subspaces in such way that one still gets an isomorphism of symplectic spaces and thus a reasonable field theory. Furthermore, we show that the generalized inverses P_I^{-1} are distinguished parametrices in the sense of Duistermaat and Hörmander [16] (i.e. have the correct wave front set) provided one has a smooth kernel, specifically

$$\operatorname{Ker} P_I \subset H_{\rm b}^{\infty,l}(M), \tag{1.4}$$

[3] The problem of constructing examples of such non globally hyperbolic spacetimes is the subject of ongoing work.

where strictly speaking $\operatorname{Ker} P_I$ is the intersection of the kernel of P_I over all choices of the orders m compatible with I. Although this assumption still needs to be better understood in the advanced/retarded case (unless M° is globally hyperbolic, in which case (1.4) is trivial), we prove that (1.4) is actually automatically satisfied in the Feynman/anti-Feynman case at least for $l = 0$.

Four types of asymptotic data. Our construction of distinguished Hadamard two-point functions (as well as the proof of (1.4) in the (anti-)Feynman case) is based on making explicit an isomorphism between the space of solutions $\mathrm{Sol}(P)$ and the symplectic space of their asymptotic data, to a large extent basing on the work of Baskin, Vasy and Wunsch on asymptotics of the radiation field [6]. If M° is actual Minkowski space, we thus introduce the coordinate $v = \rho^2(z_0^2 - (z_1^2 + \cdots + z_d^2))$ and then the submanifold $\{\rho = 0, v = 0\}$ is the union of two connected components denoted $S_\pm$ and representing the *lightcone at future/past null infinity*. More generally, on asymptotically Minkowski spacetimes there is a coordinate v with similar features, with two components of $\{\rho = 0, v = 0\}$ also denoted $S_\pm$.

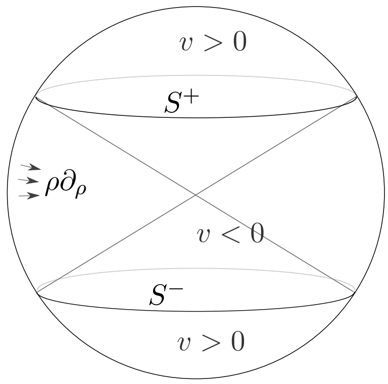

FIGURE 1. Radially compactified Minkowski space M.

Completing the coordinates ρ,v with some y and denoting γ the dual variable of v, one has as a direct consequence of [6] that near S_+ (and similarly near S_-), any solution $u \in \mathrm{Sol}(P)$ can be written as the sum of two integrals of the form

$$\int \rho^{\mathrm{i}\sigma} \mathrm{e}^{\mathrm{i}v\gamma} |\gamma|^{\mathrm{i}\sigma-1} a_+^\pm(\sigma, y)\chi^\pm(\gamma) d\gamma d\sigma$$

modulo terms with above-threshold regularity (i.e. in $H^{m,l}(M)$ for some $m > \frac{1}{2} - l$), with $\chi^\pm$ smooth and supported in $\pm[0,\infty)$. Here $a_+^\pm(\sigma, y)$ are holomorphic functions of σ in a half plane with values in $\mathcal{C}^\infty(S_+)$, rapidly decaying in $\operatorname{Re}\sigma$, and they define a pair of asymptotic data of u that we denote $\varrho_+ u$. Similarly one can define data at past null infinity $\varrho_- u = (a_-^+, a_-^-)$, or consider one piece of data at future infinity and the other at past infinity: we call this *Feynman* $\varrho_\emptyset u := (a_+^+, a_-^+)$ and *anti-Feynman data* $\varrho_{\{+,-\}} u := (a_+^-, a_-^-)$. Note that in all cases $\gamma > 0$ corresponds to sinks, $\gamma < 0$ to sources, of the bicharacteristic flow, so in the Feynman case the data are at the sinks, while in the anti-Feynman case at the sources. The corresponding propagators P_I^{-1} are then used to construct Poisson operators $\mathcal{P}_I$, i.e. inverses of ϱ_I. Most importantly, for any choice of I, if any of the two pieces of ϱ_I-data of a solution $u \in \mathrm{Sol}(P)$ vanishes then u has wave front set only in one of the two connected components $\Sigma^{\pm}$ of the characteristic set of P (in the sense of the usual wave front set in the interior M°). (This is related to (a_+^+, a_-^-) not being appropriate data: they are at the sink and source in the *same* component of Σ.) As a consequence, denoting $\pi^{\pm}$ the projections to the respective piece of data, by letting

$$\Lambda_I^{\pm} := (P_I^{-1} - P_{I^c}^{-1})^* \varrho_I^* \pi^{\pm} \varrho_I (P_I^{-1} - P_{I^c}^{-1}) \tag{1.5}$$

we eventually obtain pairs of operators that satisfy $\Lambda_I^{\pm} \geq 0$, $P\Lambda_I^{\pm} = \Lambda_I^{\pm} P = 0$ and the Hadamard condition (1.2). Moreover, by means of a pairing formula we show that they satisfy the relation

$$\Lambda_I^{+} - \Lambda_I^{-} = \mathrm{i}(P_+^{-1} - P_-^{-1}) \tag{1.6}$$

exactly if $I = \{\pm\}$, and modulo possible terms smooth in M° if $I = \emptyset$ or $I = \{+,-\}$, and thus we conclude:

Theorem 1.1. *The operators $\Lambda_I^{\pm}$ with $I = \{+\}$ and $I = \{-\}$ are Hadamard two-point functions, i.e. they satisfy* (1.1) *and* (1.2).

These should be interpreted as the analogues of two-point functions constructed in [44, 45, 21] from data at future or past infinity in the case of the conformal wave equation, and in [24] from scattering data in the case of the massive Klein-Gordon equation, even though the methods are very different. On the other hand, the interpretation of $\Lambda_I^{\pm}$ in the Feynman/anti-Feynman case is less obvious. One possibility is to view $\Lambda_I^{\pm}$ as fermionic two-point functions: we show indeed that $\Lambda_I^{+} + \Lambda_I^{-} = \mathrm{i}^{-1}(P_I^{-1} - P_{I^c}^{-1})$, which is, as already discussed, positive. A perhaps more conventional alternative is to regard $\Lambda_I^{\pm}$ as (bosonic) two-point functions for a non-standard field theory, where $P_+^{-1} - P_-^{-1}$ is modified by a smooth term. On exact Minkowski space this smooth term actually vanishes, so although in general the non-standard theory would be 'non-causal' (meaning that the integral kernel of the operator that replaces $P_+^{-1} - P_-^{-1}$ could fail to vanish at causally separated points), the departure would be presumably small (and at very low frequencies), and it would be thus interesting to study these non-local effects on a separate basis.

QFT on extended asymptotically de Sitter spacetimes. Our results for asymptotically de Sitter spacetimes are to some extent analogous to the case of asymptotically Minkowski ones, thanks to the duality between the Klein-Gordon equation on the former and the wave equation on the latter, made explicit in [56] by means of a Mellin transform in ρ. Considering for simplicity the case of exact (radially compactified) Minkowski space M, recall that de Sitter spacetime (X_0, g_{X_0}) is by definition the hyperboloid $z_0^2 - (z_1^2 + \cdots + z_d^2) = -1$ in M equipped with the induced metric. In the compactified picture it can be conveniently identified with the subregion $\{\rho = 0, v < 0\}$ of the sphere at infinity (i.e. of the boundary $\partial M = \{\rho = 0\} = \mathbb{S}^{d-1}$). In a similar vein, the hyperboloids $z_0^2 - (z_1^2 + \cdots + z_d^2) = 1$ with either $z_0 > 0$ or $z_0 < 0$ are two copies of hyperbolic space $(X_{\pm}, g_{X_{\pm}})$ (also called 'Euclidean AdS' in the physics literature) and are identified with the two connected components of the region $\{\rho = 0, v > 0\}$. Here we consider (X_0, g_{X_0}), resp. $(X_{\pm}, g_{X_{\pm}})$ as compact manifolds with boundary, i.e.

$\partial X_0 = S_+ \cup S_-$ and $\partial X_\pm = S_\pm$. The boundary of de Sitter, ∂X_0, is called traditionally the *conformal horizon*, thus the whole boundary of M,

$$\partial M = X_+ \cup X_0 \cup X_-, \tag{1.7}$$

represents de Sitter spacetime extended across the conformal horizon (which we simply call extended de Sitter spacetime).

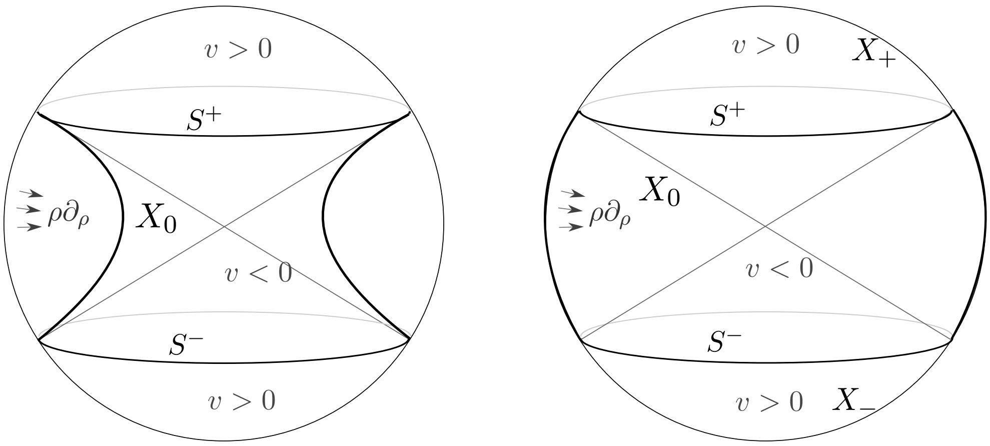

FIGURE 2. The de Sitter hyperboloid X_0 before and after identification with the 'equatorial belt' region of the boundary $\{\rho = 0\}$ of radially compactified Minkowski space. The two other regions are two copies $X_\pm$ of hyperbolic space.

Following [56], we consider the differential operator on X

$$\hat{P}_X(\sigma) := \mathcal{M}_\rho P \mathcal{M}_\rho^{-1} = \mathcal{M}_\rho \rho^{-(d-1)/2} \rho^{-2} \Box_g \rho^{(d-1)/2} \mathcal{M}_\rho^{-1},$$

obtained from P by conjugating it with the Mellin transform[4] $\mathcal{M}_\rho$ in ρ and thus depending on a complex variable σ. The crucial ingredient in our analysis are the two identities

$$\begin{aligned} \hat{P}_X\restriction_{X_0} &= x_{X_0}^{-\mathrm{i}\sigma-(d-1)/2-2}(\Box_{X_0} - \sigma^2 - (d-1)^2/4)x_{X_0}^{\mathrm{i}\sigma+(d-1)/2}, \\ \hat{P}_X\restriction_{X_\pm} &= x_{X_\pm}^{-\mathrm{i}\sigma-(d-1)/2-2}(\Delta_{X_\pm} - \sigma^2 - (d-1)^2/4)x_{X_\pm}^{\mathrm{i}\sigma+(d-1)/2}, \end{aligned} \tag{1.8}$$

to the very best of our knowledge made explicit the first time in [56], where

$$x_{X_0} = \left(\frac{z_1^2 + \cdots + z_d^2 - z_0^2}{z_1^2 + \cdots + z_d^2 + z_0^2}\right)^{\frac{1}{2}}, \quad x_{X_\pm} = \left(\frac{z_0^2 - (z_1^2 + \cdots + z_d^2)}{z_1^2 + \cdots + z_d^2 + z_0^2}\right)^{\frac{1}{2}}.$$

As the first identity in (1.8) connects P with the Klein-Gordon operator on X_0, this suggests a sort of duality between QFT on M and QFT on de Sitter space X_0 and one can wonder if that would mean that there is also a duality between QFT on M and a hypothetical QFT on hyperbolic space X_+ (or X_-). In the present paper we provide evidence for this claim by setting up a QFT on $X_\pm$ indeed (in fact, on the whole extended de Sitter space X) and by relating it to QFT on de Sitter. Beside

[4]Recall that the Mellin transform of $u \in \mathcal{C}_c^\infty((0,\infty))$ is defined by $(\mathcal{M}_\rho u)(\sigma) := \int_0^\infty \rho^{-\mathrm{i}\sigma-1} u(\rho) d\rho$.

the case of exact de Sitter space, our results do also apply to even asymptotically de Sitter spacetimes (Definition 6.1), introduced in [56] (as well as even asymptotically hyperbolic space, cf. the work of Guillarmou [27]), where a direct analogue of (1.7) and (1.8) is available in terms of some asymptotically Minkowski spacetime M.

The relevant feature of the operator $\hat{P}_X$ on extended asymptotically de Sitter spacetimes is that it fits into the framework of [53, 30] and thus possesses various inverses in a similar way as P does (here as meromorphic functions of σ), the main difference being that one only needs to consider regularity in the sense of Sobolev spaces $H^s(X)$ (note that X is a closed manifold). This allows to obtain in a very analogous way an isomorphism

$$\hat{P}_{X,I}^{-1} - \hat{P}_{X,I^c}^{-1} : \frac{\mathcal{C}^\infty(X)}{\hat{P}_X \mathcal{C}^\infty(X)} \longrightarrow \mathrm{Sol}(\hat{P}_X) \tag{1.9}$$

with $\mathrm{Sol}(\hat{P}_X)$ the space of solutions of $\hat{P}_X u = 0$ such that $\mathrm{WF}(u) \subset N^*(S_+ \cup S_-)$. Moreover, the definition of Hadamard two-point functions transports directly to this case, thus once their existence is proved one gets a perfectly reasonable QFT on X (at least if $\sigma \in \mathbb{R}$ so that $\hat{P}_{X,I^c}^{-1}$ is the formal adjoint of $\hat{P}_{X,I}^{-1}$), despite it being governed by a differential operator $\hat{P}_X$ that is hyperbolic only in the asymptotically de Sitter region $\{v < 0\}$. In order to understand the relation of this new QFT with the well-established theory on X_0, let us recall that the latter is based on the isomorphism

$$\hat{P}_{X_0,+}^{-1} - \hat{P}_{X_0,-}^{-1} : \frac{\mathcal{C}_\mathrm{c}^\infty(X_0^\circ)}{\hat{P}_{X_0} \mathcal{C}_\mathrm{c}^\infty(X_0^\circ)} \longrightarrow \mathrm{Sol}(\hat{P}_{X_0})$$

where $\mathrm{Sol}(\hat{P}_{X_0})$ is the space of solutions of $\hat{P}_{X_0}$ that are smooth in the interior X_0°. On the other hand, we prove that the map

$$\restriction_{X_0} \circ\, x_{X_0}^{\mathrm{i}\sigma + (d-1)/2} : \mathrm{Sol}(\hat{P}_X) \to \mathrm{Sol}(\hat{P}_{X_0}) \tag{1.10}$$

is an isomorphism (i.e. symplectomorphism), which allows to conclude that QFT on X_0 *extends across the boundary.* Even more specifically, we show:

Theorem 1.2. *Any pair of Hadamard two-point functions $\Lambda_{X_0}^{\pm}$ on an even asymptotically de Sitter spacetime (X_0, g_{X_0}) extends canonically to Hadamard two-point functions $\Lambda_X^{\pm}$ on X via the isomorphism* (1.10).

Furthermore, we construct Hadamard two-point functions $\Lambda_{X_0,I}^{\pm}$ on X_0 from asymptotic data in a similar fashion as in the Minkowski case: these then extend to Hadamard two-point functions on X and we give a direct formula for the latter in terms of the X_0 asymptotic data.

QFT on asymptotically hyperbolic space. Intriguingly, since the two-point functions on asymptotically de Sitter space X_0 give rise to two-point functions on the extended space X, in particular one gets 'two-point functions' $\Lambda_{X_\pm}^{\pm}$ on the two copies $X_\pm$ of asymptotically hyperbolic space (by restricting to $X_\pm$ and conjugating with $x_{X_\pm}^{\mathrm{i}\sigma+(d-1)/2}$). We show that in fact there is an underlying QFT on X_+ and on X_-, given by the isomorphism

$$\hat{P}^{-1}_{X_\pm,+} - \hat{P}^{-1}_{X_\pm,-} : \frac{\dot{\mathcal{C}}^\infty(X_\pm)}{\hat{P}_{X_\pm}\dot{\mathcal{C}}^\infty(X_\pm)} \longrightarrow \mathrm{Sol}(\hat{P}_{X_\pm}) \tag{1.11}$$

where $\hat{P}^{-1}_{X_\pm,+}$, $\hat{P}^{-1}_{X_\pm,-}$ are defined by analytic continuation of the resolvent of Δ_{X_0} starting from positive, resp. negative large values of the imaginary part of complex parameter σ, and $\mathrm{Sol}(\hat{P}_{X_\pm})$ is a space of solutions (defined more precisely in Subsect. 5.2) of $\hat{P}_{X_\pm}$ that are smooth in the interior $X^\circ_\pm$.

Let us stress that the QFT obtained this way, although of course defined with fundamentally Euclidean objects, is crucially different from Euclidean QFTs often considered in the physics literature and obtained by a Wick rotation (i.e. complex scaling) of the time variable in a relativistic QFT, cf. [33, 34, 35] for the case of curved spacetimes and other recent developments. For instance, our two-point functions on $X_\pm$ are subject to a positivity condition reminiscent of relativistic QFT, as opposed to the reflection positivity in Euclidean QFT.

Outlook. Since the two-point functions $\Lambda^\pm_{X_+}$ that we consider on asymptotically hyperbolic spaces are smooth, we expect that this could serve as a basis to construct a very regular interacting (i.e. non-linear) QFT. We plan to follow on this idea in a future work.

One can also wonder whether the strategy adopted in the present paper extends to other classes of spacetimes, possibly with trapping; it is plausible that this question could be addressed using the recent advances in [30, 53, 7, 17].

A further aspect to look into is the relation of the Feynman and anti-Feynman asymptotic data that we consider with the Atiyah-Patodi-Singer and anti-Atiyah-Patodi-Singer boundary data adapted recently to the Lorentzian case by Bär and Strohmaier [4, 5] in the context of the Dirac equation on a manifold which is the product of a finite interval with a compact Riemannian manifold. Although the setup is clearly different, there are many striking analogies to be explored [24], in particular it would be thus beneficial to have a Dirac version of our results. (Cf. the differential forms setup of [51].)

1.2. Summary of results.

Our main results can be summarized as follows.

In the case of the wave equation on an asymptotically Minkowski spacetime M, we assume that $l = 0$ is not a resonance (i.e., of the Mellin transformed normal operator family of the relevant function space setup corresponding to I, $I^{\rm c}$, see Subsect. 3.1), and we assume 'smoothness of kernels' (1.4).

1) In Proposition 4.2 we prove that the propagator difference $P_I^{-1} - P_{I^{\rm c}}^{-1}$ induces an isomorphism that generalizes (1.3).

2) In Proposition 5.4 we show bijectivity of the maps ϱ_I that assign to a solution its asymptotic data (strictly speaking, in order to have a bijection we consider a space of solutions $\mathrm{Sol}_I(P)$ with elements of $\operatorname{Ker} P_I$, $\operatorname{Ker} P_{I^{\rm c}}$ removed) and then Theorem 5.5 provides an explicit formula for the induced symplectic form on asymptotic data.

3) In Theorem 5.7 we prove the Hadamard property (1.2) of the operators $\Lambda_I^{\pm}$ constructed from asymptotic data (1.5) and in particular we get two pairs of Hadamard two-point functions $\Lambda_-^{\pm}$, $\Lambda_+^{\pm}$ from data at past and future null infinity.

Then, for any even asymptotically de Sitter spacetime X_0, we consider the Klein-Gordon operator $\hat{P}_{X_0} = \Box_{X_0} - \sigma^2 - (d-1)^2/4$ and the associated operators on the extended space X and on the asymptotically hyperbolic spaces $X_{\pm}$. We assume that $\sigma \in \mathbb{R} \setminus \{0\}$ is not a pole of $\hat{P}_{X,I}^{-1}(\sigma)$.

4) In Propositions 6.2 and 6.6 we prove isomorphisms (1.9), (1.11) induced by respective propagator differences, and the isomorphism (1.10) between solution spaces on X and on X_0.

5) In Theorem 6.7 we give a formula in terms of asymptotic data for Hadamard two-point functions in the asymptotically de Sitter region X_0 and for the induced Hadamard two-point functions on $X_{\pm}$.

In particular, the latter two results mean that non-interacting scalar fields on even asymptotically de Sitter spacetime canonically extend across the conformal horizon.

2. Asymptotically Minkowski spacetimes and propagation of singularities

2.1. **Notation.** If M is a smooth manifold with boundary ∂M, we denote M° its interior. We denote $\mathcal{C}^{\infty}(M)$ the space of smooth functions on M (in the sense of extendability across the boundary). The space of smooth functions vanishing with all derivatives at the boundary ∂M are denoted $\dot{\mathcal{C}}^{\infty}(M)$ and their dual $\mathcal{C}^{-\infty}(M)$. The signature of Lorentzian metrics is taken to be $(+,-,\dots,-)$. We adopt the convention that sesquilinear forms $\langle\cdot,\cdot\rangle$ are linear in the second argument.

2.2. **Geometrical setup.** The spacetime of interest is modelled by an n-dimensional smooth manifold M with boundary ∂M ($n \geq 2$), equipped with a *Lorentzian scattering metric* g.

To define this class of metrics, let ρ be a boundary-defining function of ∂M, meaning that $\partial M = \rho = 0$ and $d\rho \neq 0$ on ∂M, and let $w = (w_1, \dots, w_{n-1})$ be coordinates on ∂M. Then ${}^{\rm sc}T^*M$ is the bundle whose sections are locally given by the $\mathcal{C}^{\infty}(M)$-span of the differential forms $\rho^{-2}d\rho, \rho^{-1}dw = (\rho^{-1}dw_1, \dots, \rho^{-1}dw_{n-1})$. Lorentzian scattering metrics are by definition non-degenerate sections of ${\rm Sym}^{2{\rm sc}}T^*M$ of Lorentzian signature [42], and they define an open subset of $\mathcal{C}^{\infty}(M; {\rm Sym}^2\,{}^{\rm sc}T^*M)$ (equipped with the $\mathcal{C}^{\infty}$ topology).

A more refined structure near the boundary ∂M can be specified as follows [6, 29, 25].

Definition 2.1. *One says that* (M,g) *is a* Lorentzian scattering space *if there exists* $v \in \mathcal{C}^{\infty}(M)$ *s.t.* $v\restriction_{\partial M}$ *has non-degenerate differential at* $S := \{\rho = 0, v = 0\}$ *and moreover:*

- *on* ∂M, $g(\rho^2\partial_\rho, \rho^2\partial_\rho)$ *has the same sign as* v*;*
- g *has the form*

$$g = v\frac{d\rho^2}{\rho^4} - \left(\frac{d\rho}{\rho^2} \otimes \frac{\alpha}{\rho} + \frac{\alpha}{\rho} \otimes \frac{d\rho}{\rho^2}\right) - \frac{\tilde{g}}{\rho^2}, \tag{2.1}$$

where $\tilde{g} \in \mathcal{C}^\infty(M; \mathrm{Sym}^2 T^*M)$ *with* $\tilde{g}\restriction_{(d\rho, dv)^{\mathrm{ann}}}$ *positive definite*[5] *at* S*, and* α *is a one-form on* M *of the form* $\alpha = dv/2 + O(v) + O(\rho)$ *near* S*.*

The zero-set $S = \{v = 0, \rho = 0\}$ is called the *light-cone at infinity* and is in fact a submanifold of M.

The example of primary importance of a Lorentzian scattering space is the radial compactification of $n = 1 + d$-dimensional Minkowski space $\mathbb{R}^{1,d}$ outlined in the introduction. Namely, writing the Minkowski metric as $dz_0^2 - (dz_1^2 + \dots + dz_d^2)$, a manifold M with boundary $\partial M = \{\rho = 0\}$ is obtained by making the change of coordinates $z_0 = \rho^{-1}\cos\theta$, $z_i = \rho^{-1}\omega_i \sin\theta$, (valid near $\rho = 0$), where $\rho = (z_0^2 + z_1^2 + \dots + z_d^2)^{-1/2}$ and ω_i are coordinates on the sphere $\mathbb{S}^{d-2}$. Then a further change of coordinates

$$v = \cos 2\theta = \rho^2(z_0^2 - (z_1^2 + \dots + z_d^2))$$

brings the metric into the form

$$g = v\frac{d\rho^2}{\rho^4} - \frac{v}{4(1-v^2)}\frac{dv^2}{\rho^2} - \frac{1}{2}\left(\frac{d\rho}{\rho^2} \otimes \frac{dv}{\rho} + \frac{dv}{\rho} \otimes \frac{d\rho}{\rho^2}\right) + \frac{1-v}{2}\frac{d\omega^2}{\rho^2},$$

which is a special case of (2.1) with $\alpha = dv/2$.

2.3. **Wave operator and** b**-geometry.** The main object of interest is the wave operator $\Box_g \in \mathrm{Diff}^2(M)$. It is convenient to introduce at once the conformally related operator

$$P := \rho^{-(n-2)/2}\rho^{-2}\Box_g \rho^{(n-2)/2}. \tag{2.2}$$

With this definition, P is a b-*differential operator*, that is $P \in \mathrm{Diff}^2_{\mathrm{b}}(M)$ where $\mathrm{Diff}^k_{\mathrm{b}}(M)$ consists of differential operators of order k which are in the algebra $\mathcal{C}^\infty(M)$-generated by $\rho\partial_\rho, \partial_w$, using as before coordinates (ρ, w) near ∂M. The operator P is formally self-adjoint with respect to the b-*density* (i.e., smooth section of the density bundle of ${}^{\mathrm{b}}TM$) $\rho^n |dg|$. We denote $\langle \cdot, \cdot \rangle_{\mathrm{b}}$ the corresponding pairing and $L^2_{\mathrm{b}}(M)$ the Hilbert space it defines.

Let us now introduce the notions relevant for the description of the bicharacteristic flow in the b-setting. To start with, the $\mathcal{C}^\infty(M)$-module generated by the vector fields $\rho\partial_\rho$, ∂_w can be viewed as the space of smooth sections of a bundle ${}^{\mathrm{b}}TM$, called the b-tangent bundle. The dual bundle ${}^{\mathrm{b}}T^*M$ is called the b-conormal bundle and locally near ∂M its sections are the $\mathcal{C}^\infty(M)$-span of $\rho^{-1}d\rho$, dw. Since vector fields (i.e., sections of TM) can also be considered as sections of ${}^{\mathrm{b}}TM$, there is a canonical embedding $\mathcal{C}^\infty({}^{\mathrm{b}}TM) \to \mathcal{C}^\infty(TM)$ and a corresponding dual map on covectors. Now for a submanifold $S \subset M$, the b-*conormal bundle* ${}^{\mathrm{b}}N^*S$ is defined as the image in ${}^{\mathrm{b}}T^*M$ of covectors in T^*M that annihilate the image of TS in TM.

Specifically, in the setting of Lorentzian scattering spaces, the b-conormal bundle of $S = \{\rho = 0, v = 0\}$ is easily seen to be generated by dv. Indeed, the vectors in TS are annihilated by dv and $d\rho$, and their image in ${}^{\mathrm{b}}T^*M$ is respectively dv, $\rho(\rho^{-1}d\rho)$ with the latter vanishing above $\{\rho = 0\}$.

[5] Here $\tilde{g}\restriction_{(d\rho, dv)^{\mathrm{ann}}}$ denotes the restriction of $\tilde{g}$ to the annihilator of the span of $d\rho, dv$.

The bundles ${}^{\mathrm{b}}T^*M \setminus o$, ${}^{\mathrm{b}}N^*S \setminus o$ have their 'spherical' versions ${}^{\mathrm{b}}S^*M$ and ${}^{\mathrm{b}}SN^*S$, defined as the quotients

$$ {}^{\mathrm{b}}S^*M := ({}^{\mathrm{b}}T^*M \setminus o)/\mathbb{R}_+, \quad {}^{\mathrm{b}}SN^*S := ({}^{\mathrm{b}}N^*S \setminus o)/\mathbb{R}_+. $$

by the fiberwise $\mathbb{R}_+$-action of dilations, where o is the zero section.

Let now $p \in \mathcal{C}^\infty(T^*M \setminus o)$ be the principal symbol of P (in this paragraph the specific form of P is irrelevant, only the fact that it belongs to $\mathrm{Diff}^m_{\mathrm{b}}(M)$ and that p is real). By homogeneity, the Hamiltonian vector field of p on $T^*M \setminus o$ extends to a vector field on ${}^{\mathrm{b}}T^*M \setminus o$, which is tangent to the boundary. Specifically, it is given by (and could be defined by) the local expression

$$ H_p = (\partial_\varsigma p)(\rho\partial_\rho) - (\rho\partial_\rho p)\partial_\varsigma + \textstyle\sum_i \left((\partial_{\zeta_i} p)\partial_{w_i} - (\partial_{w_i} p)\partial_{\zeta_i}\right), $$

in b-covariables (ς, ζ) in which sections of ${}^{\mathrm{b}}T^*M$ read $\varsigma(\rho^{-1}d\rho) + \sum_i \zeta_i dw_i$.

In order to keep track of the behavior of H_p along the orbits of the $\mathbb{R}_+$ action it is actually convenient to view ${}^{\mathrm{b}}S^*M$ as the boundary of the so-called radial compactification ${}^{\mathrm{b}}\overline{T}^*M$ of ${}^{\mathrm{b}}T^*M$. Without giving the details of the construction (cf. [43, Ch. 1.8]), the relevant feature here is that it comes with a function $\tilde\rho \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*M \setminus o)$, homogeneous of degree -1, that serves as a boundary defining function. Since p is homogeneous of degree m, $\tilde\rho^m p$ can be restricted to fiber infinity and thus identified with a smooth function on ${}^{\mathrm{b}}S^*M$. Now, the *characteristic set* Σ (of P) is the zero-set of the rescaled principal symbol $\tilde\rho^m p \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$. The *bicharacteristic flow* of P is defined in the present setup as the flow Φ_t of the rescaled Hamilton vector field $\mathrm{H}_p := \tilde\rho^{m-1} H_p$ in Σ. Accordingly, the (reparametrized) projections of the integral curves of H_p by the quotient map in ${}^{\mathrm{b}}T^*M \setminus o \to {}^{\mathrm{b}}S^*M$ are called bicharacteristics[6].

2.4. **Non-trapping assumptions.** In contrast to standard real principal type estimates that are entirely local and are therefore not invalidated by the presence of trapping, the estimates that we use here to obtain the Fredholm property of P on appropriate function spaces are *global*, i.e. depend on what happens at infinite times, therefore issues related with trapping are very likely to produce difficulties. To eliminate these we make use of the non-trapping geometrical setup considered in [6, 25, 53] (of which radially compactified Minkowski space is an example again):

Hypothesis 2.1. *We assume that g is* non-trapping *in the following sense.*

(1) $S = \{v = 0, \rho = 0\}$ *is the disjoint sum of two components* $S = S_+ \cup S_-$ *and moreover:*
(2) $\{v > 0\} \subset \partial M$ *splits into disjoint components* $X_\pm$ *with* $S_\pm = \partial X_\pm$
(3) *all maximally extended bicharacteristics flow from* ${}^{\mathrm{b}}SN^*S_+$ *to* ${}^{\mathrm{b}}SN^*S_-$ *or vice-versa.*

The Lorentzian scattering space (M, g) is then called an asymptotically Minkowski spacetime and the submanifold S_+ is the *lightcone at future null infinity* and S_- the *lightcone at past null infinity*.

The characteristic set $\Sigma \subset {}^{\mathrm{b}}S^*M$ of P splits into two connected components $\Sigma^\pm$. Accordingly, each of the radial sets ${}^{\mathrm{b}}SN^*S_\pm$ splits further into two components ${}^{\mathrm{b}}SN^{*\pm}S_\pm$

[6]These are often called *null* bicharacteristics in the literature.

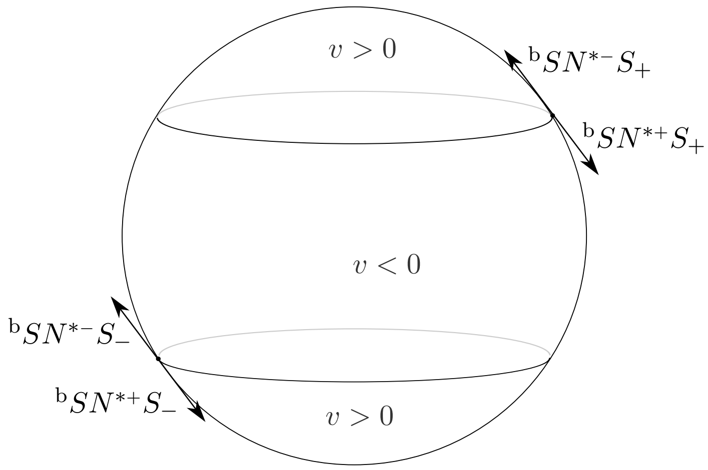

FIGURE 3. An asymptotically Minkowski spacetime M. The radial sets are located above $S = S_+ \cup S_-$ and split into sources and sinks $^{\mathrm{b}}SN^{*\pm}S_\pm$.

which act as sources (-) or sinks (+) for the bicharacteristic flow, meaning specifically that

$$\mathrm{H}_p \tilde{\rho} = \tilde{\rho}\beta_0,$$

where $\pm\beta_0 > 0$ for sources, resp. sinks [6].

We introduce the short-hand notation $\mathcal{R} := \bigcup_\pm {}^{\mathrm{b}}SN^{*\pm}S$ for the whole radial set.

Let us remark that in this setup, a time orientation of (M, g) can be fixed as follows: one specifies the future lightcone to be the one from which forward bicharacteristics (in the sense of the H_p-flow) tend to S_+. Moreover, it was shown in [30] that if ρ can be chosen in such way that $\rho^{-1}d\rho$ is timelike near $X_+ \cup X_-$ (with respect to $\rho^2 g$) then the interior of M, M°, is globally hyperbolic, we will however not use this assumption unless specified otherwise.

2.5. b**-regularity and propagation of singularities.** Recall that the algebra of b-differential operators $\mathrm{Diff}_{\mathrm{b}}(M)$ is generated by vector fields tangent to the boundary (and the identity), thus setting

$$H_{\mathrm{b}}^{k,0}(M) = \{u \in \mathcal{C}^{-\infty}(M) : \ Au \in L^2_{\mathrm{b}}(M) \ \forall A \in \mathrm{Diff}^k_{\mathrm{b}}(M)\},$$

for $k \in \mathbb{N}$ gives a space of distributions (the b-*Sobolev space of order* k) that have usual Sobolev regularity of order k in M°, the interior of M, and are moreover regular of order k at the boundary in the sense of conormality. In the above expression $\mathrm{Diff}^k_{\mathrm{b}}(M)$ can be replaced by b-pseudodifferential operators of order k, $\Psi^k_{\mathrm{b}}(M)$ — here we will not give the precise definition (instead we refer the reader to [41, 54, 55]), though formally one can simply think of those as operators of the form $A = a(\rho, w; \rho\partial_\rho, \partial_w)$, with a a symbol in the usual sense. By analogy this allows to define b-Sobolev spaces of arbitrary order $m \in \mathbb{R}$, and at the same time we introduce weighted ones:

$$H_{\rm b}^{m,0}(M) = \{u \in \mathcal{C}^{-\infty}(M) : \ Au \in L^2_{\rm b}(M) \ \forall A \in \Psi_{\rm b}^m(M)\},$$
$$H_{\rm b}^{m,l}(M) = \rho^l H_{\rm b}^{m,0}(M),$$

so that m corresponds to usual Sobolev regularity in M° and conormal regularity at the boundary, whereas l corresponds to decay near the boundary (and this agrees with the definition sketched in the introduction). The dual of $H_{\rm b}^{m,l}(M)$ can be identified with $H_{\rm b}^{-m,-l}(M)$ using the $L^2_{\rm b}(M)$ pairing $\langle \cdot, \cdot \rangle_{\rm b}$. We have correspondingly spaces of distributions of arbitrarily low and arbitrarily high b-Sobolev regularity

$$H_{\rm b}^{-\infty,l}(M) := \textstyle\bigcup_{m\in\mathbb{R}} H_{\rm b}^{m,l}(M), \quad H_{\rm b}^{\infty,l}(M) := \textstyle\bigcap_{m\in\mathbb{R}} H_{\rm b}^{m,l}(M),$$

endowed with their canonical Fréchet topologies, one of which is the dual of the other if l is replaced by $-l$.

There is a notion of b-Sobolev wave front set of a distribution $u \in H_{\rm b}^{-\infty,l}(M)$, denoted $\mathrm{WF}_{\rm b}^{m,l}(u) \subset {}^{\rm b}S^*M$, which consists of the points in phase space in which u is not in $H_{\rm b}^{m,l}(M)$. Concretely, the definition says that for $\alpha \in {}^{\rm b}S^*M$, $\alpha \notin \mathrm{WF}_{\rm b}^{m,l}(u)$ if there exists $A \in \Psi_{\rm b}^{0,0}(M)$ elliptic at α and such that $Au \in H_{\rm b}^{m,l}(M)$, where ellipticity refers to invertibility of the principal symbol, cf. [41, 55, 54]. Note that locally in the interior of M, b-Sobolev regularity and standard Sobolev regularity are just the same, so the b-Sobolev wave front set coincides with the standard wave front set there. We refer to [54, Sec. 2 & 3] for a more detailed discussion.

The definitions of $\Psi_{\rm b}^{m,l}(M)$, $H_{\rm b}^{m,l}(M)$ and $\mathrm{WF}_{\rm b}^{m,l}(u)$ can be extended to allow for *varying* Sobolev orders $m \in \mathcal{C}^\infty({}^{\rm b}S^*M)$, cf. for instance [6, App. A]. This is particularly convenient for the formulation of propagation of singularities theorems near radial sets. We will use in particular the following result from [53], cf. also the discussion in [25].

Theorem 2.2. *Let (M, g) be a Lorentzian scattering space. Let P be the rescaled wave operator (2.2), let us denote by $\mathcal{R}_i$ any of the components of the radial sets, and let $u \in H_{\rm b}^{-\infty,l}(M)$.*

(1) *If $m < \frac{1}{2} - l$ and m is nonincreasing along the bicharacteristic flow in the direction approaching $\mathcal{R}_i$, then*

$$\mathrm{WF}_{\rm b}^{m,l}(u) \cap \mathcal{R}_i = \emptyset \quad \textit{if} \quad \mathrm{WF}_{\rm b}^{m-1,l}(Pu) \cap \mathcal{R}_i = \emptyset$$

and provided that $(U \setminus \mathcal{R}_i) \cap \mathrm{WF}_{\rm b}^{m,l}(u) = \emptyset$ for some neighborhood $U \subset \Sigma \cap {}^{\rm b}S^*M$ of $\mathcal{R}_i$.

(2) *If $m_0 > \frac{1}{2} - l$, $m \geq m_0$ and m is nonincreasing along the bicharacteristic flow in the direction going out from $\mathcal{R}_i$ then*

$$\mathrm{WF}_{\rm b}^{m,l}(u) \cap \mathcal{R}_i = \emptyset \quad \textit{if} \quad \big(\mathrm{WF}_{\rm b}^{m_0,l}(u) \cup \mathrm{WF}_{\rm b}^{m-1,l}(Pu)\big) \cap \mathcal{R}_i = \emptyset.$$

Thus, there is a threshold value $m = \frac{1}{2} - l$, and in the 'below-threshold' case $m < \frac{1}{2} - l$ one has a propagation of singularities statement similar to real principal type estimates, while in the 'above-threshold' case one gets arbitrarily high regularity at the radial set provided Pu is regular enough.

3. Propagators

3.1. **Inverses of the wave operator.** Theorem 2.2 is deduced from (and is in fact equivalent to) a priori estimates involving $H_{\rm b}^{m,l}$ norms of u and $H_{\rm b}^{m-1,l}$ norms of Pu (plus a weaker norm of u in $H_{\rm b}^{m',l}$, $m'<m$), microlocalized using b-pseudodifferential operators accordingly with the stated direction of propagation. These estimates give a global statement if for each component Σ^j of the characteristic set ($j\in\{+,-\}$) one takes m to be above-threshold at one radial set within Σ^j and below-threshold at the other [29, 25], one gets namely

$$\|u\|_{H_{\rm b}^{m,l}(M)} \leq C(\|Pu\|_{H_{\rm b}^{m-1,l}(M)} + \|u\|_{H_{\rm b}^{m',l}(M)}). \tag{3.1}$$

Thus, in other words, (3.1) is obtained by 'propagating estimates from one radial set to another'. Defining then

$$\mathcal{Y}^{m,l} := H_{\rm b}^{m,l}(M), \quad \mathcal{X}^{m,l} := \left\{ u\in H_{\rm b}^{m,l}(M) :\ Pu \in H_{\rm b}^{m-1,l}(M)\right\}, \tag{3.2}$$

by analogy to some elliptic problems [55] one would like to conclude a statement about P being Fredholm as a map $\mathcal{X}^{m,l}\to\mathcal{Y}^{m-1,l}$ (using a standard argument from functional analysis, see [32, Proof of Thm. 26.1.7]). The problematic point (as explained in more detail in [25]) is however that $H_{\rm b}^{m,l}$ is not compactly included in $H_{\rm b}^{m',l}$ (as opposed for instance to $H_{\rm b}^{m,l}\hookrightarrow H_{\rm b}^{m',l'}$ for $m'<m$, $l'<l$) and therefore the corresponding remainder term is not negligible. Improved estimates (with better control on the decay of remainder terms) can be however derived by a careful analysis of the *Mellin transformed normal operator* of P, defined as follows.

Recall that any $P\in {\rm Diff}_{\rm b}^k(M)$ is locally given by

$$P = \sum_{i+|\alpha|\leq k} a_{i,\alpha}(\rho,w)(\rho\partial_\rho)^i\partial_w^\alpha.$$

Its Mellin transformed normal operator family is then

$$\widehat{N}(P)(\sigma) := \sum_{i+|\alpha|\leq k} a_{i,\alpha}(0,x)\sigma^i\partial_x^\alpha.$$

A direct computation shows that in our specific case of interest, $\widehat{N}(P)(\sigma)\in{\rm Diff}^2(\partial M)$ takes the form

$$\widehat{N}(P)(\sigma) = 4\left((v+O(v^2))\partial_v^2 + ({\rm i}\sigma+1+O(v))\partial_v\right) + O(1)\partial_y^2 + O(1)\partial_y + O(v)\partial_v\partial_y \tag{3.3}$$

near $\{v=0\}$ modulo terms $O(\sigma^2)$, cf. [6] for more explicit expressions. The crucial property is that $\widehat{N}(P)(\sigma)$ is hyperbolic on $\{v<0\}$ (and elliptic elsewhere, which is the easiest part) and its characteristic set splits into two connected components $\hat\Sigma^\pm$ with bicharacteristics starting and ending at radial sets. Fredholm estimates combined with a semiclassical analysis with small parameter $|\sigma|^{-1}$ are then used in [25] to prove that $\widehat{N}(P)(\sigma)^{-1}$ exists as a meromorphic family and the structure of its poles determines the Fredholm (or invertibility) property of $P:\mathcal{X}^{m,l}\to\mathcal{Y}^{m-1,l}$. In particular the following assumption is made use of.

Hypothesis 3.1. *The weight l is assumed to satisfy $l \neq -\operatorname{Im}\sigma_i$ for any resonance*[7] *$\sigma_i \in \mathbb{C}$ of the Mellin transformed normal operator family $\widehat{N}(P)(\sigma)$ of P.*

Concerning the possible choices of the order defining function m, different choices of directions along which m is increasing give different (generalized, see below) inverses of P. Specifically, for each of the two sinks ${}^{\mathrm{b}}SN^{*+}S_{\pm}$, we can choose whether estimates are propagated *from* it or *to* it. Following the convention in [52], let us label this choice by a set of indices $I \subset \{+,-\}$ indicating the sinks *from* which we propagate, i.e. where high regularity is imposed (and thus also the components of the characteristic set $\Sigma = \Sigma^+ \cup \Sigma^-$ along which m is increasing). Then the complement I^{c} indicates the sinks *to* which we propagate. We denote correspondingly $\mathcal{R}_I^-$ the components of the radial set from which the estimates are propagated, and $\mathcal{R}_I^+$ the remaining others. Note that by definition $\mathcal{R}_{I^{\mathrm{c}}}^{\mp} = \mathcal{R}_I^{\pm}$.

With these definitions at hand, the main result of [25] states that $P : \mathcal{X}^{m,l} \to \mathcal{Y}^{m-1,l}$ is Fredholm for any m such that

$$\pm m > 1/2 - l \quad \text{near} \quad \mathcal{R}_I^{\mp}, \tag{3.4}$$

with m monotone along the bicharacteristic flow as long as l satisfies Hypothesis 3.1. Moreover, it is shown that $P : \mathcal{X}^{m,l} \to \mathcal{Y}^{m-1,l}$ is invertible if $|l|$ is small and (M,g) is a perturbation of the radial compactification of Minkowski space in the sense of Lorentzian scattering metrics $\mathcal{C}^\infty(M; \operatorname{Sym}^2 {}^{\mathrm{sc}}T^*M)$, within the closed subset of Lorentzian scattering spaces (cf. Definition 2.1).

We will use the shorthand notation $\mathcal{X}_I$, $\mathcal{Y}_I$ for the spaces $\mathcal{X}^{m,l}, \mathcal{Y}^{m-1,l}$ with any choice of orders and weights m,l as in (3.4). We will also write occasionally P_I for P understood as an operator $\mathcal{X}_I \to \mathcal{Y}_I$.

A consequence of the Fredholm property is that one can define a *generalized inverse* of $P_I : \mathcal{X}_I \to \mathcal{Y}_I$ as follows. First, one makes a choice of complementary spaces $\mathcal{W}_I$, $\mathcal{Z}_I$, to respectively $\operatorname{Ker} P_I$, $\operatorname{Ran} P_I$ in $\mathcal{X}_I$, $\mathcal{Y}_I$, with $\mathcal{W}_I$ of finite codimension and $\mathcal{Z}_I$ of finite dimension. We define P_I^{-1} to be the unique extension of the inverse of $P : \mathcal{W}_I \to \operatorname{Ran} P_I$ to $\mathcal{Y}_I \to \mathcal{X}_I$ such that

$$\operatorname{Ker} P_I^{-1} = \mathcal{Z}_I, \quad \operatorname{Ran} P_I^{-1} = \mathcal{W}_I.$$

In what follows we will choose a complementary space $\mathcal{Z}_I$ consisting of $\dot{\mathcal{C}}^\infty(M)$ functions, which is always possible since $\operatorname{Ran} P_I$ is of finite codimension and $\dot{\mathcal{C}}^\infty(M)$ is dense in $\mathcal{Y}_I$. The property $\mathcal{Z}_I \subset \dot{\mathcal{C}}^\infty(M)$ then ensures that PP_I^{-1} equals $\mathbf{1}$ on $\mathcal{Y}_I$ modulo smoothing terms. To make sure that P_I^{-1} is also a left parametrix[8], one needs the following additional property.

Hypothesis 3.2. *Assume that* $\operatorname{Ker} P_I \subset H_{\mathrm{b}}^{\infty,l}(M)$.

We will refer to Hypothesis 3.2 simply as *smoothness of the kernel*, we will actually see in Proposition 5.6 that it is in fact automatically satisfied in the Feynman and anti-Feynman case (the argument we use therein does however not apply to the retarded and advanced case).

The (generalized) inverses P_I^{-1} corresponding to the four possible choices of I are named as follows:

[7] This is synonym for σ_i being a pole of the meromorphic family $\widehat{N}(P)(\sigma)^{-1}$.

[8] By say, left parametrix, we mean that $P_I^{-1}P$ equals $\mathbf{1}$ modulo terms that have smooth kernel in M°.

(1) $I=\emptyset$ (i.e., $\mathcal{R}_I^- = {}^{\mathrm{b}}SN^{*-}S$) — Feynman propagator,
(2) $I=\{+,-\}$ (i.e., $\mathcal{R}_I^- = {}^{\mathrm{b}}SN^{*+}S$) — anti-Feynman propagator,
(3) $I=\{-\}$ (i.e., $\mathcal{R}_I^- = {}^{\mathrm{b}}SN^{*}S_-$) — retarded (or forward) propagator,
(4) $I=\{+\}$ (i.e., $\mathcal{R}_I^- = {}^{\mathrm{b}}SN^{*}S_+$) — advanced (or backward) propagator.

The terminology for $I=\{-\}$, resp. $I=\{+\}$ is motivated by the fact that due to its mapping properties, the corresponding inverse P_I^{-1} solves the forward, resp. backward problem in the interior M° of M, and thus equals the advanced, resp. retarded propagator defined in the usual way as in the introduction (modulo smoothing terms if P_I^{-1} is just a parametrix). The name *Feynman propagator* for $P_\emptyset^{-1}$ can be justified by relating it to a Feynman parametrix in the sense of Duistermaat and Hörmander [16], as pointed out in [25, 52] (and analogously for the anti-Feynman one). Here we make this precise by proving that the Schwartz kernel of $P_\emptyset^{-1}$ (considered as a distribution on $M^\circ\times M^\circ$) has wave front set of precisely the same form as the Feynman parametrix' of Duistermaat and Hörmander, and therefore the two operators coincide modulo smoothing terms (at least provided (M°,g) is globally hyperbolic so that the assumptions in [16] are satisfied).

Such statement is closely related to the propagation of singularities along the bicharacteristic flow Φ_t. In the present setting it can be formulated as follows. If $I\subset\{+,-\}$, m,l are chosen consistently with I, $m_0>\frac{1}{2}-l$ is a fixed constant and $u\in\mathcal{X}^{m,l}$ then (3.5)

$$\begin{aligned}\big(\mathrm{WF}_{\mathrm{b}}^{m_0,l}(u)\cap\Sigma\big)\setminus\mathcal{R}_I^+ \subset \mathrm{WF}_{\mathrm{b}}^{m_0-1,l}(Pu)&\cup{\textstyle\bigcup_{j\in I}}\big(\cup_{t\geq 0}\Phi_t(\mathrm{WF}_{\mathrm{b}}^{m_0-1,l}(Pu)\cap\Sigma^j)\big)\\ &\cup{\textstyle\bigcup_{j\in I^{\mathrm{c}}}}\big(\cup_{t\leq 0}\Phi_t(\mathrm{WF}_{\mathrm{b}}^{m_0-1,l}(Pu)\cap\Sigma^j)\big)\end{aligned}$$

provided that $\mathrm{WF}_{\mathrm{b}}^{m_0-1,l}(Pu)\cap\mathcal{R}_I^-=\emptyset$. The latter condition is trivially satisfied if for instance $\operatorname{supp}Pu\subset\subset M^\circ$, then in the interior of M (3.5) reduces to

$$\begin{aligned}\mathrm{WF}^{m_0}(u)\cap\Sigma\subset\mathrm{WF}^{m_0-1}(Pu)&\cup{\textstyle\bigcup_{j\in I}}\big(\cup_{t\geq 0}\Phi_t(\mathrm{WF}^{m_0-1}(Pu)\cap\Sigma^j)\big)\\ &\cup{\textstyle\bigcup_{j\in I^{\mathrm{c}}}}\big(\cup_{t\leq 0}\Phi_t(\mathrm{WF}^{m_0-1}(Pu)\cap\Sigma^j)\big),\end{aligned}\tag{3.6}$$

in terms of the standard Sobolev wave front set $\mathrm{WF}^{m_0}(u)\subset S^*M^\circ$ (since the restriction of $\mathrm{WF}_{\mathrm{b}}^{m_0-1,l}$ to M° is precisely WF^{m_0}). Therefore, disregarding singularities lying on $\mathrm{diag}_{T^*M^\circ}$ (the diagonal in $T^*M^\circ\times T^*M^\circ$), one expects that the primed wave front set of the Schwartz kernel of P_I^{-1}, denoted $\mathrm{WF}'(P_I^{-1})$, is contained in

$$\mathscr{C}_I:={\textstyle\bigcup_{j\in I}}\big(\cup_{t\geq 0}\Phi_t(\mathrm{diag}_{T^*M^\circ})\cap\pi^{-1}\Sigma^j\big)\cup{\textstyle\bigcup_{j\in I^{\mathrm{c}}}}\big(\cup_{t\leq 0}\Phi_t(\mathrm{diag}_{T^*M^\circ})\cap\pi^{-1}\Sigma^j\big),$$

where Φ_t operates on the left factor and $\pi:\Sigma\times\Sigma\to\Sigma$ is the projection to the left factor[9].

In other words $\mathscr{C}_I$ consists of pairs of points $((y,\eta),(x,\xi))$ such that (y,η), $(x,\xi)\in\Sigma$ are connected by a bicharacteristic and such that on the component Σ^j, (y,η) comes after (x,ξ) respective to the Hamilton flow if $j\in I$ and (x,ξ) comes after (y,η) otherwise.

Proposition 3.1. *Assume smoothness of the kernel (Hypothesis 3.2) and global hyperbolicity of (M°,g). Then:*

[9] Here one can equivalently take the projection to the right factor.

(1) $\mathrm{WF}'(P_I^{-1}) = (\mathrm{diag}_{T^*M^\circ}) \cup \mathscr{C}_I$ *for* $I \subset \{+,-\}$;
(2) $\mathrm{WF}'(P_\emptyset^{-1} - P_\pm^{-1}) = \cup_{t\in\mathbb{R}} \Phi_t(\mathrm{diag}_{T^*M^\circ}) \cap \pi^{-1}\Sigma^\pm$.

Proof. In the case of retarded/advanced propagators, statement (1) follows from [16], so we only have to show (1) in the (anti-)Feynman case. We start by proving (2).

Let δ_x be the Dirac delta distribution supported at some point $x \in M^\circ$. For any I we can choose the order defining function m in $\mathcal{X}_I = \mathcal{X}^{m,l}$ in such way that $\delta_x \in \mathcal{Y}_I$. Even more, we can arrange that δ_x is at the same time in $\mathcal{Y}_\emptyset$ and in $\mathcal{Y}_+$. Then $P_I^{-1}\delta_x \in \mathcal{X}_I$ for $I = \emptyset$ and $I = \{+\}$. Consequently, the distribution $(P_\emptyset^{-1} - P_+^{-1})\delta_x$ has above-threshold regularity microlocally in Σ^- near S_+. Since it also solves the wave equation (modulo smooth terms), this implies by propagation of singularities

$$\mathrm{WF}((P_\emptyset^{-1} - P_+^{-1})\delta_x) \subset \Sigma^+. \tag{3.7}$$

In fact, this holds in the sense of the uniform wave front set for the family

$$\{(P_\emptyset^{-1} - P_+^{-1})\delta_x :\ x \in K\}, \tag{3.8}$$

K compact in M°, by propagation of singularities estimates (which are uniform estimates), i.e. that for $A \in \Psi^0(M)$ of compactly supported Schwartz kernel and with $\mathrm{WF}'(A) \cap \Sigma_+ = \emptyset$,

$$\{A(P_\emptyset^{-1} - P_+^{-1})\delta_x :\ x \in K\} \text{ is bounded in } \mathcal{C}^\infty. \tag{3.9}$$

On the level of the Schwartz kernel $(P_\emptyset^{-1} - P_+^{-1})(y,x) = ((P_\emptyset^{-1} - P_+^{-1})\delta_x)(y)$, which holds in a distributional sense, (3.9) yields

$$\mathrm{WF}'(P_\emptyset^{-1} - P_+^{-1}) \subset (\Sigma^+ \cup o) \times T^*M^\circ, \tag{3.10}$$

as can be seen e.g. by using the explicit Fourier transform characterization of the wave front set, using appropriate pseudodifferential operators in (3.9). We now use [52, Thm. X], which states (for parametrices, which our inverses are) that $\mathrm{i}^{-1}(P_\emptyset^{-1} - P_\pm^{-1})$ differs from a positive operator by a smooth term. Disregarding this smooth error, one can write a Cauchy-Schwarz inequality for $|\langle f, (P_\emptyset^{-1} - P_+^{-1})g\rangle_{\mathrm{b}}|$ in terms of $|\langle f, (P_\emptyset^{-1} - P_+^{-1})f\rangle_{\mathrm{b}}|$, $|\langle g, (P_\emptyset^{-1} - P_+^{-1})g\rangle_{\mathrm{b}}|$ for any test functions f, g. This allows to get estimates for the wave front set in $o \times (T^*M^\circ \setminus o)$ from estimates in $(T^*M^\circ \setminus o) \times (T^*M^\circ \setminus o)$, and also to get a symmetrized form of the wave front set[10], in particular (3.10) gives

$$\mathrm{WF}'(P_\emptyset^{-1} - P_+^{-1}) \subset \Sigma^+ \times \Sigma^+. \tag{3.11}$$

The analogous argument gives correspondingly

$$\mathrm{WF}'(P_\emptyset^{-1} - P_-^{-1}) \subset \Sigma^- \times \Sigma^-. \tag{3.12}$$

Observe that the two wave front sets (3.11), (3.12) are disjoint. In view of the identity

$$(P_\emptyset^{-1} - P_+^{-1}) - (P_\emptyset^{-1} - P_-^{-1}) = P_-^{-1} - P_+^{-1}$$

this implies that $\mathrm{WF}'(P_\emptyset^{-1} - P_\pm^{-1})$ equals $(\Sigma^\pm \times \Sigma^\pm) \cap \mathrm{WF}'(P_-^{-1} - P_+^{-1})$. On the other hand, using the exact form of $\mathrm{WF}'(P_\pm^{-1}) \setminus \mathrm{diag}_{T^*M^\circ} = \mathscr{C}_\pm$ one obtains $\mathrm{WF}'(P_-^{-1} -$

[10] It is worth mentioning that this sort of argument was already used for instance in [18, 49, 47].

$P_+^{-1}) = \mathscr{C}_+ \cup \mathscr{C}_-$, thus

$$\begin{aligned} \mathrm{WF}'(P_\emptyset^{-1} - P_\pm^{-1}) &= (\Sigma^\pm \times \Sigma^\pm) \cap (\mathscr{C}_+ \cup \mathscr{C}_-) \\ &= \cup_{t\in\mathbb{R}} \Phi_t(\mathrm{diag}_{T^*M^\circ}) \cap \pi^{-1}\Sigma^\pm. \end{aligned} \tag{3.13}$$

The exact form of $\mathrm{WF}'(P_\emptyset^{-1})$ is concluded from (3.13) and $\mathrm{WF}'(P_\pm) = \mathrm{diag}_{T^*M^\circ} \cup \mathscr{C}_\pm$ by means of the two identities $P_\emptyset^{-1} = (P_\emptyset^{-1} - P_\pm^{-1}) + P_\pm^{-1}$. □

Concerning the b-wave front set, it would require more work to make precise statements about the Schwartz kernel of P_I^{-1} (in the sense of manifolds with boundaries), we still have however at our disposal information on $\mathrm{WF}_\mathrm{b}^{m,l}(P_I^{-1}f)$ given the b-wave front set of f. For our purposes it is sufficient to observe that P_I^{-1} adds singularities only at the radial set, specifically by propagation of singularities (3.5)

$$\mathrm{WF}_\mathrm{b}^{m_0,l}(P_I^{-1}f) \subset \mathcal{R}_I^+, \quad f \in \mathrm{Ran} P_I \tag{3.14}$$

for $m_0 < m$, where m, l are the orders corresponding to I, so in particular if $f \in H_\mathrm{b}^{\infty,l}(M)$ then $\mathrm{WF}_\mathrm{b}^{\infty,l}(P_I^{-1}f) \subset \mathcal{R}_I^+$.

4. Symplectic spaces of smooth solutions

4.1. Solutions smooth away from $\mathcal{R}$.

A particularly useful way to construct solutions of $Pu = 0$ is to take $u = (P_I^{-1} - P_{I^\mathrm{c}}^{-1})f$ for $f \in \mathrm{Ran}P_I \cap \mathrm{Ran}P_{I^\mathrm{c}}$. For such solutions, by (3.14) and Hörmander's propagation of singularities we have $\mathrm{WF}_\mathrm{b}^{m_0,l}(u) \subset \mathcal{R}$ for $m_0 \leq \max(m, m^\mathrm{c})$, where m, l, resp. m^c, l are the orders corresponding to I, resp. I^c.

We will see that the so-obtained space of solutions can be equivalently defined as

$$\mathrm{Sol}_I(P) := \big\{ u \in \mathcal{W}_I + \mathcal{W}_{I^\mathrm{c}} : \ Pu = 0, \ \mathrm{WF}_\mathrm{b}^{m_0,l}(u) \subset \mathcal{R}, \ m_0 = \max\{m, m^\mathrm{c}\} \big\}. \tag{4.1}$$

Note that by definition $\mathrm{Sol}_I(P) = \mathrm{Sol}_{I^\mathrm{c}}(P)$. If P_I is invertible then the condition $u \in \mathcal{W}_I + \mathcal{W}_{I^\mathrm{c}}$ in (4.1) reduces to $u \in \mathcal{X}_I + \mathcal{X}_{I^\mathrm{c}}$ (recall that $\mathcal{W}_I$ is a complement of $\mathrm{Ker}\, P_I$). In the case when P_I is merely a Fredholm operator, the main reason to use $\mathcal{W}_I$ in the definition is the validity of the following lemma.

Lemma 4.1. *Assume Hypothesis 3.2. If $u \in \mathrm{Sol}_I(P)$ is microlocally in $H_\mathrm{b}^{m,l}(M)$ near $\mathcal{R}_I^-$ for $m > \frac{1}{2} - l$ then $u = 0$.*

Proof. By assumption $u \in \mathcal{X}_I$ and $Pu = 0$, hence $u \in \mathrm{Ker}\, P_I$ by definition of $P_I : \mathcal{X}_I \to \mathcal{Y}_I$. Using Hypothesis 3.2 this implies $u \in \mathcal{X}_{I^\mathrm{c}}$, and repeating the previous argument one gets $u \in \mathrm{Ker}\, P_{I^\mathrm{c}}$. This contradicts that $u \in \mathcal{W}_I + \mathcal{W}_{I^\mathrm{c}}$ unless $u = 0$. □

We will use Lemma 4.1 repeatedly. For instance, let $Q_I \in \Psi_\mathrm{b}^{0,0}$ be microlocally the identity near $\mathcal{R}_I^-$ and microlocally vanishing near the remaining components $\mathcal{R}_I^+$ of the radial set. For any $u \in \mathrm{Sol}_I(P)$,

$$u = Q_I u + (\mathbf{1} - Q_I)u = Q_I u + P_I^{-1}P(\mathbf{1} - Q_I)u + (\mathbf{1} - P_I^{-1}P)(\mathbf{1} - Q_I)u.$$

Since $(\mathbf{1} - Q_I)u$ belongs to $\mathcal{X}_I$, the term $(\mathbf{1} - P_I^{-1}P)(\mathbf{1} - Q_I)u$ is in the null space of P, so in fact we have

$$u = Q_I u - P_I^{-1}PQ_I u \tag{4.2}$$

modulo a term in $\operatorname{Ker} P_I$, and hence in $\mathcal{X}_I \cap \mathcal{X}_{I^c}$ by Hypothesis 3.2. Rewriting now (4.2) in the form $u = Q_I u - P_I^{-1}[P, Q_I]u$ (modulo irrelevant terms) we conclude that $-P_I^{-1}[P, Q_I]u$ agrees with u microlocally at $\mathcal{R}_I^+$, and so does $P_{I^c}^{-1}[P, Q_I]u - P_I^{-1}[P, Q_I]u$. The latter is in $\mathrm{Sol}_I(P)$ (because $[P, Q_I]u = PQ_I u = -P(\mathbf{1} - Q_I)u \in \mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}$), therefore by Lemma 4.1 (using $\mathcal{R}_I^+ = \mathcal{R}_{I^c}^-$) we obtain

$$(P_{I^c}^{-1} - P_I^{-1})[P, Q_I] = \mathbf{1} \quad \text{on} \quad \mathrm{Sol}_I(P). \tag{4.3}$$

For the sake of compactness of notation we define $G_I := P_I^{-1} - P_{I^c}^{-1}$, in terms of which the above identity reads

$$-G_I[P, Q_I] = \mathbf{1} \quad \text{on} \quad \mathrm{Sol}_I(P). \tag{4.4}$$

Proposition 4.2. *Assume Hypothesis 3.2. Then the map G_I induces a bijection*

$$\frac{\mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}}{P(\mathcal{X}_I \cap \mathcal{X}_{I^c})} \xrightarrow{[G_I]} \mathrm{Sol}_I(P) \tag{4.5}$$

Proof. We first need to check that G_I induces a well-defined map on the quotient, i.e. $G_I(\mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}) \subset \mathrm{Sol}_I(P)$ (which we already know) and $G_I P(\mathcal{X}_I \cap \mathcal{X}_{I^c}) = 0$. The latter follows from the identity

$$P(\mathcal{W}_I \cap \mathcal{W}_{I^c}) = P(\mathcal{X}_I \cap \mathcal{X}_{I^c}), \tag{4.6}$$

(this is true because the spaces $\mathcal{W}_I \cap \mathcal{W}_{I^c}$ and $\mathcal{X}_I \cap \mathcal{X}_{I^c}$ differ only by elements of $\operatorname{Ker} P_I$ and $\operatorname{Ker} P_{I^c}$) and the fact that $P_I^{-1} P = \mathbf{1}$ on $\mathcal{W}_I$.

Surjectivity of $[G_I]$ means

$$G_I(\mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}) \supset \mathrm{Sol}_I(P).$$

but this follows readily from (4.4), taking into account that $[P, Q_I]$ is smoothing near the radial set. Injectivity of $[G_I]$ means that the kernel of G_I acting on $\mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}$ equals $P(\mathcal{W}_I \cap \mathcal{W}_{I^c})$. Indeed if $u \in \mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}$ and $G_I u = 0$ then setting $w = P_I^{-1} u$ we have $u = Pw$, with $w \in \mathcal{W}_I$. On the other hand $w = P_{I^c}^{-1} u$ hence it is also in $\mathcal{W}_{I^c}$. □

To simplify the discussion further it is convenient to eliminate the dependence of the spaces $\mathcal{X}_I$, $\mathrm{Sol}_I(P)$ and $\mathrm{Ran} P_I$ on the specific choice of Sobolev orders m, m^c by taking the intersection over all possible orders. With this redefinition, $\mathrm{WF}_{\mathrm{b}}^{\infty,l}(u) \subset \mathcal{R}$ for all $u \in \mathrm{Sol}_I(P)$. Furthermore, Proposition 4.2 remains valid and in the special case when P_I and P_{I^c} are invertible (this is true for instance when (M°, g) is globally hyperbolic) one gets instead of (4.5) the more handy statement that there is a bijection

$$\frac{H_{\mathrm{b}}^{\infty,l}(M)}{P H_{\mathrm{b}}^{\infty,l}(M)} \xrightarrow{[G_I]} \mathrm{Sol}_I(P). \tag{4.7}$$

The case $I = \{-\}$ in (4.7) is the analogue of the well-known characterization of smooth space-compact[11] solutions of the wave equation on globally hyperbolic spacetimes as the range of the difference of the advanced and retarded propagator acting on test functions, cf. [3, Thm. 3.4.7].

[11] By space-compactness one means that the restriction to a Cauchy surface has compact support.

In what follows we will consider pairings between elements of spaces such as $\mathcal{X}_I$, $\mathcal{X}_{I^c}$ and for that purpose we fix $l = 0$ for the weight respective to decay. As shown in [52], the formal adjoint of P_I^{-1} is $P_{I^c}^{-1}$, possibly up to some obstructions caused by the lack of invertibility of $P : \mathcal{X}_I \to \mathcal{Y}_I$ in the case when it is merely Fredholm. In addition to that, there is a positivity statement in the Feynman case, more precisely:

Theorem 4.3 ([52])**.** *As a sesquilinear form on* $\mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}$*,* $G_I = P_I^{-1} - P_{I^c}^{-1}$ *is formally skew-adjoint. Moreover if* $I = \emptyset$ *then* $\mathrm{i}^{-1}\langle\cdot, G_I \cdot\rangle_{\mathrm{b}}$ *is positive on* $\mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}$*.*

The relevance of Proposition 4.2 and Theorem 4.3 in QFT stems from the conclusion that $\langle\cdot, G_I \cdot\rangle_{\mathrm{b}}$ induces a well-defined symplectic form (in particular non-degenerate, thanks to the injectivity statement of Proposition 4.2) on the quotient space

$$\mathscr{V}_I := \mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c} / P(\mathcal{W}_I \cap \mathcal{W}_{I^c}),$$

which can be then transported to $\mathrm{Sol}_I(P)$ using the isomorphism in (4.5). In the case $I = \{-\}$ the resulting structure is interpreted as the canonical symplectic space of the classical field theory and is the first ingredient in the construction of non-interacting quantum fields. The next step is to specify a pair of *two-point functions* on $\mathscr{V}_I$, defined in the very broad context below.

Definition 4.4. *Let* $\mathscr{V}$ *be a complex vector space equipped with a (complex) symplectic form* G*. One calls a pair of sesquilinear forms* $\Lambda^{\pm}$ *on* $\mathscr{V}$ bosonic (resp. fermionic) two-point functions *if* $\Lambda^+ - \Lambda^- = \mathrm{i}^{-1} G$ *(resp.* $\Lambda^+ + \Lambda^- = \mathrm{i}^{-1} G$*) and* $\Lambda^{\pm} \geq 0$ *on* $\mathscr{V}$*.*

Note that in the fermionic case one needs to have necessarily $\mathrm{i}^{-1} G \geq 0$. Once $\Lambda^{\pm}$ are given, the standard apparatus of quasi-free states and algebraic QFT can be used to construct quantum fields, see Appendix A or [15, 28, 39], here we will rather focus on the two-point functions themselves.

In the literature on QFT on globally hyperbolic spacetimes one considers usually the symplectic space $\mathscr{V}_I$ with $I = \{-\}$ or equivalently $I = \{+\}$ (or strictly speaking an analogous quotient space defined in terms of test functions) and bosonic two-point functions $\Lambda_I^{\pm}$ on it. The physical reason is that for $I = \{\pm\}$ the Schwartz kernel $G_I(x, y) = \pm(P_-^{-1}(x, y) - P_+^{-1}(x, y))$ vanishes for space-like related $x, y \in M^\circ$ and in consequence the relation $\Lambda_I^+ - \Lambda_I^- = \mathrm{i}^{-1} G_I$ translates to the property that quantum fields commute in causally disjoint regions.

In contrast, two-point functions on $\mathscr{V}_I$ in the cases $I = \emptyset$, $I = \{+, -\}$ have not been considered before to the best of our knowledge. We argue that since $\mathrm{i}^{-1} G_I$ is positive in the Feynman case, it is natural to consider then *fermionic* two-point functions $\Lambda_I^{\pm}$. In later sections we will indeed construct fermionic two-point functions (in particular satisfying $\Lambda_I^+ + \Lambda_I^- = \mathrm{i}^{-1} G_I$ for $I = \emptyset$) for which the quantity $\Lambda_I^+ - \Lambda_I^-$ equals $\mathrm{i}(P_+^{-1}(x, y) - P_-^{-1}(x, y))$ modulo terms smooth in M°. In the special case of Minkowski space one finds $\mathrm{i}(P_+^{-1}(x, y) - P_-^{-1}(x, y))$ exactly, i.e. the smooth remainders are absent.

In our setup, rather than with abstract sesquilinear forms on $\mathscr{V}_I$ it is much more convenient to work with operators $\Lambda_I^{\pm}$ that map continuously, say, $H_{\mathrm{b}}^{m',0} \to H_{\mathrm{b}}^{-m',0}$ for large m', these then define a pair of (hermitian) sesquilinear forms $\langle\cdot, \Lambda_I^{\pm} \cdot\rangle_{\mathrm{b}}$ on $\mathscr{V}_I$ if $\Lambda_I^{\pm}$ is formally self-adjoint on $\mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}$ with respect to $\langle\cdot, \cdot\rangle_{\mathrm{b}}$ and

$$\langle \phi, \Lambda_I^{\pm} P \psi \rangle_{\mathrm{b}} = 0 \quad \forall\, \phi \in \mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c},\ \psi \in \mathcal{W}_I \cap \mathcal{W}_{I^c}. \tag{4.8}$$

The sesquilinear forms $\langle\cdot,\Lambda_I^{\pm}\cdot\rangle_{\rm b}$ are two-point functions on $\mathscr{V}_I$ if they satisfy

$$(-1)^{I(+)}\Lambda_I^{+}+(-1)^{I(-)}\Lambda_I^{-}={\rm i}G_I,\quad \langle\cdot,\Lambda_I^{\pm}\cdot\rangle_{\rm b}\geq 0\quad\text{on}\quad {\rm Ran}P_I\cap{\rm Ran}P_{I^{\rm c}} \tag{4.9}$$

where we employed the notation

$$(-1)^{I(\pm)}:=\begin{cases}1 \text{ if } \ \pm\in I,\\ -1 \text{ otherwise,}\end{cases}$$

so that one gets bosonic two-point functions in the retarded/advanced case and fermionic ones in the Feynman/anti-Feynman case.

In QFT on curved spacetime one is primarily concerned about the subclass of Hadamard two-point functions, which in the present setup can be defined as follows (conforming to the discussion above, two-point functions will be considered to be operators instead of sesquilinear forms).

Definition 4.5. *We say that* $\Lambda_I^{\pm}:H_{\rm b}^{m',0}(M)\to H_{\rm b}^{-m',0}(M)$ *are* Hadamard two-point functions *for* P *if they satisfy (4.8), (4.9) and if moreover*

$$\mathrm{WF}'(\Lambda_I^{\pm})=\textstyle\bigcup_{t\in\mathbb{R}}\Phi_t(\mathrm{diag}_{T^*M^\circ})\cap\pi^{-1}\Sigma^{\pm} \tag{4.10}$$

over $M^\circ\times M^\circ$.

Remark 4.6. *In practice if we are given a pair of operators* $\Lambda_I^{\pm}$ *satisfying (4.8), and* $\Lambda_I^{+}-\Lambda_I^{-}={\rm i}G_I$, $\Lambda_I^{\pm}\geq 0$ *w.r.t.* $\langle\cdot,\cdot\rangle_{\rm b}$, *then to ensure the Hadamard condition (4.10) it is sufficient to have* $\mathrm{WF}'(\Lambda_I^{\pm})\subset(\Sigma^{\pm}\cup o)\times T^*M^\circ$, *as can be shown by the same arguments as in the proof of Proposition 3.1.*

The wave front set condition (4.10) will be called the *Hadamard condition*, in agreement with the terminology used on globally hyperbolic spacetimes, cf. [46, 47, 49] for the various equivalent formulations. From the point of view of applications in QFT (renormalization in particular, see [31, 39, 11] and references therein), one of the key properties of Hadamard two-point functions is that any two differ by an operator whose kernel is smooth in $M^\circ\times M^\circ$. This statement (known on globally hyperbolic spacetimes as Radzikowski's theorem [46]) is easily shown using the identity

$$(-1)^{I(+)}(\Lambda_I^{+}-\tilde\Lambda_I^{+})+(-1)^{I(-)}(\Lambda_I^{-}-\tilde\Lambda_I^{-})={\rm i}G_I-{\rm i}G_I=0$$

for any two pairs of Hadamard two-point functions $\Lambda_I^{\pm}$, $\tilde\Lambda_I^{\pm}$. Indeed, the terms in parentheses have disjoint primed wave front sets in the interior of M, so in fact $\Lambda_I^{+}-\tilde\Lambda_I^{+}$ and $\Lambda_I^{-}-\tilde\Lambda_I^{-}$ have smooth kernel in M°.

4.2. **Time-slice property.** Let us consider again the identity

$$G_I[P,Q_I]=\mathbf{1}\quad\text{on}\quad \mathrm{Sol}_I(P), \tag{4.11}$$

which we proved to be true for any pseudo-differential operator $Q_I\in\Psi_{\rm b}^{0,0}(M)$ that is microlocally the identity near $\mathcal{R}_I^{-}$ and microlocally vanishes near $\mathcal{R}_I^{+}$. In the cases $I=\{+\}$, $I=\{-\}$, Q_I can actually be chosen to be a multiplication operator and one can ensure that $[P,Q_I]$ vanishes in a neighborhood of $S=S_+\cup S_-$, so this way one can characterize $\mathrm{Sol}_I(P)$ as the range of G_I acting on functions supported away from S.

Proposition 4.7. *Suppose* $I = \{+\}$ *or* $I = \{-\}$ *and let* $Q_I \in \mathcal{C}^\infty(M)$ *be equal* 0 *near* S_- *and* 1 *near* S_+. *Then for any* $u \in \mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}$ *there exists* $\tilde{u} \in \mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c}$ *s.t.* $[u] = [\tilde{u}]$ *in* $\mathrm{Ran} P_I \cap \mathrm{Ran} P_{I^c} / P(\mathcal{X}_I \cap \mathcal{X}_{I^c})$ *and*

$$\mathrm{supp}(\tilde{u}) \subset \mathrm{supp}(Q_I) \cap \mathrm{supp}(\mathbf{1} - Q_I). \tag{4.12}$$

Proof. It suffices to set $\tilde{u} = [P, Q_I] G_I u$, then it is clear that this has the requested support properties. Furthermore $G_I(\tilde{u} - u) = 0$ by (4.11), thus $\tilde{u} - u \in P(\mathcal{X}_I \cap \mathcal{X}_{I^c})$ by the injectivity statement of Proposition 4.2. □

In the case when M° is globally hyperbolic this statement implies that for any $[u] \in H_{\mathrm{b}}^{\infty,0}(M)/P H_{\mathrm{b}}^{\infty,0}(M)$ one can find a representative $\tilde{u}$ supported in an arbitrary neighborhood of a Cauchy surface. This fact (with $\mathcal{C}_{\mathrm{c}}^\infty(M^\circ)$ in place of $H_{\mathrm{b}}^{\infty,0}(M)$) is known as the *time-slice property*, a particularly useful consequence is that this allows to construct two-point functions by specifying their restriction to a small neighborhood of a Cauchy surface.

5. Parametrization of solutions on the lightcone at infinity

5.1. Mellin transform.

In what follows we collect some elementary facts on the Mellin transform that will be needed later on.

Recall that for $u \in \mathcal{C}_{\mathrm{c}}^\infty((0,\infty))$ the Mellin transform is defined by the integral

$$(\mathcal{M}_\rho u)(\sigma) := \int_0^\infty \rho^{-\mathrm{i}\sigma - 1} u(\rho) d\rho.$$

It extends to a unitary operator $\rho^l L_{\mathrm{b}}^2(\mathbb{R}_+) \to L^2(\{\mathrm{Im}\, \sigma = -l\})$ whose inverse can be expressed using the integral formula

$$u(\rho) = (2\pi)^{-1} \int_{\{\mathrm{Im}\, \sigma = -l\}} \rho^{\mathrm{i}\sigma} (\mathcal{M}_\rho u)(\sigma) d\sigma, \tag{5.1}$$

and it intertwines the generator of dilations ρD_ρ with multiplication by σ, i.e. $\rho D_\rho = \mathcal{M}_\rho^{-1} \sigma \mathcal{M}_\rho$.

Let us denote $\mathscr{S}(\{\mathrm{Im}\ \sigma = -l\})$ the space of complex functions with boundary value $\mathrm{Im}\ \sigma = -l$ rapidly decreasing as $\sigma \to \infty$. If the Mellin transform of u is in that space then by (5.1) $\rho^{-l} u$ is bounded near $\rho = 0$, and by a simple reduction to this case we get the following estimate.

Lemma 5.1. *If* $\mathcal{M} u \in \mathscr{S}(\{\mathrm{Im}\ \sigma = -l\})$ *then* $\rho^{-l} (\log \rho)^k (\rho \partial_\rho)^j u(\rho)$ *is bounded near* $\rho = 0$ *for any* $j, k \in \mathbb{N}$.

5.2. Asymptotic data of solutions.

Let now $l \geq 0$ be any order satisfying Hypothesis 3.1. For a brief moment let us consider the space of all solutions with wave front set only in the radial set, i.e.

$$\mathrm{Sol}(P) := \{u \in H_{\mathrm{b}}^{-\infty,l}(M) : \ \mathrm{WF}_{\mathrm{b}}^{\infty,l}(u) \subset \mathcal{R}\}. \tag{5.2}$$

This is simply the space $\mathrm{Sol}_I(P)$ considered in Subsect. 4.1 plus possible elements of $\mathrm{Ker}\, P_I$ and $\mathrm{Ker}\, P_{I^c}$. These solutions enjoy the following properties:

(1) by below-threshold propagation of singularities they belong to $H_{\mathrm{b}}^{m,l}(M)$ for all $m < \frac{1}{2} - l$;

(2) as proved in [6] they are 'b-Lagrangian' distributions[12] associated to $\mathcal{R}$ in the sense that

$$A_1 A_2 \dots A_k \mathrm{Sol}(P) \subset H_{\mathrm{b}}^{m,l}(M), \quad \forall k \in \mathbb{N}, A_j \in \mathfrak{M}(M),$$

where $\mathfrak{M}(M) \subset \Psi_{\mathrm{b}}^1(M)$ is the space of b-pseudodifferential operators whose principal symbols vanish on the radial set $\mathcal{R}$. More explicitly, $\mathfrak{M}(M)$ can be characterized as the $\Psi_{\mathrm{b}}^0(M)$-module generated by $\rho\partial_\rho$, $\rho\partial_v$, $v\partial_y$, ∂_y and $\mathbf{1}$.

Let $\eta_\pm \in \mathcal{C}^\infty(M)$ be smooth cutoff functions of a neighborhood of $S_\pm$ in M. For the moment we restrict our attention to S_+, keeping in mind that the discussion for S_- is analogous.

For a solution $u \in \mathrm{Sol}(P)$, cutting it off with η_+ and taking the Mellin transform[13] in ρ one obtains a family of functions $\mathcal{M}(\eta_+ u)(\sigma)$ that is holomorphic in $\mathrm{Im}\ \sigma > -l$ with boundary value at $\mathrm{Im}\ \sigma = -l$ lying in the H^m-based Lagrangian space

$$\{f \in H^m(\partial M) : \ A_1 A_2 \dots A_k f \in H^m(\partial M), \ A_j \in \mathfrak{M}(\partial M)\},$$

and such that $\mathcal{M}(\eta_+ u)(\sigma)$ rapidly decreases as $\sigma \to \infty$ (where $\mathfrak{M}(\partial M)$ is generated by $v\partial_y$, ∂_y). Furthermore, as shown in [6], $\mathcal{M}(\eta_+ u)(\sigma)$ is necessarily a classical conormal distribution in the sense that it is given by the sum of two oscillatory integrals of the form

$$\int e^{iv\gamma} |\gamma|^{i\sigma - 1} \tilde{a}^\pm(\sigma, v, y, \gamma) d\gamma$$

modulo Schwartz functions of σ, on $\mathrm{Im}\ \sigma = -l$, holomorphic in the upper half plane, with values in $\mathcal{C}^\infty(\partial M)$, with $\tilde{a}^\pm$ (Schwartz function of σ with values in) classical symbols[14] of order 0 in γ. Here $\tilde{a}^\pm$ are supported in $\pm\gamma > 0$, corresponding to the half of ${}^{\mathrm{b}}SN^*S_+$ considered (${}^{\mathrm{b}}SN^{*+}S_+$ versus ${}^{\mathrm{b}}SN^{*-}S_+$). Thus, inverting the Mellin transform, and absorbing a factor of 2π into a newly defined $\tilde{a}^\pm$, $\eta_+ u$ itself is of the form

$$J(\tilde{a}^\pm) = \int_{\mathrm{Im}\,\sigma = -l} \int \rho^{i\sigma} e^{iv\gamma} |\gamma|^{i\sigma - 1} \tilde{a}^\pm(\sigma, v, y, \gamma)\, d\gamma\, d\sigma,$$

modulo elements of $H_{\mathrm{b}}^{\infty,l}$. We call such distributions weight l *b-conormal distributions* of symbolic order 0 associated to the half of ${}^{\mathrm{b}}SN^*S_+$ considered (${}^{\mathrm{b}}SN^{*+}S_+$ versus ${}^{\mathrm{b}}SN^{*-}S_+$). Note that if $\tilde{a}^\pm$ vanishes to order k at $v = 0$ then integration by parts in γ allows one to conclude that $J(\tilde{a}^\pm) = J(\tilde{b}^\pm)$ where $\tilde{b}^\pm$ now take values of classical conormal symbols of order $-k$. Then, by an asymptotic summation argument (which is just the σ-dependent version of the standard argument for conormal distributions, conormal to $v = 0$) one sees that the v dependence of $\tilde{a}^\pm$ can be essentially completely eliminated in that one can write the integrand as $\chi_0(v)$ times a v independent symbol, with $\chi_0 \equiv 1$ near 0 and of compact support, again modulo Schwartz functions of σ, on $\mathrm{Im}\ \sigma = -l$, holomorphic in the upper half plane, with values in $\mathcal{C}^\infty(\partial M)$. In particular,

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[12]Note that components of ${}^{\mathrm{b}}SN^*S$ are not even Legendre in ${}^{\mathrm{b}}S^*M$ since the symplectic structure degenerates at ∂M in the b-normal directions, so ${}^{\mathrm{b}}SN^*S$ has dimension $n-2$ if n is the dimension of M: both the boundary defining function ρ and its b-dual variable σ vanish on ${}^{\mathrm{b}}SN^*S$.

[13]Near the boundary M admits a product decomposition of the form $[0, \epsilon)_\rho \times \partial M$, we can then take η_+ supported in, say, $\rho < \epsilon/2$, which makes the Mellin transform of $\eta_+ u$ well defined.

[14]Here we use L^∞-based symbols, so a symbol a of order 0 satisfies $|D_y^\alpha D_v^k D_\gamma^N a| \le C_{\alpha k N} \langle\gamma\rangle^{-N}$ for all α, k, N.

the leading term of the asymptotic expansion of $\tilde{a}^{\pm}$ as $\gamma \to \pm\infty$ is recovered by simply taking the Fourier transform of the Mellin transform of $\eta_+ u$ and letting $\gamma \to \pm\infty$. Furthermore, analogous statements apply if $\tilde{a}^{\pm}$ is a classical symbol of order s. In particular, the isomorphism properties of the Fourier and Mellin transforms show that when $\tilde{a}^{\pm}$ is a classical symbol of order s, $J(\tilde{a}_{\pm})$ is in $H_{\rm b}^{m,l}(M)$ if $m < \frac{1}{2} - l - s = -\frac{1}{2} - (l + s - 1)$, with $l + s - 1$ being the symbolic order of the symbol $|\gamma|^{{\rm i}\sigma-1}\tilde{a}_{\pm}$.

In terms of $u \in \mathrm{Sol}(P)$, this means that for v and ρ near 0, $\eta_+ u$ is the sum of two integrals of the form

$$\int \rho^{{\rm i}\sigma} {\rm e}^{{\rm i}v\gamma} |\gamma|^{{\rm i}\sigma-1} a^{\pm}(\sigma, y)\chi^{\pm}(\gamma) d\gamma d\sigma \tag{5.3}$$

with $a^{\pm}$ (Schwartz function of σ with values in) smooth functions, modulo terms that belong to $H_{\rm b}^{m',l}(M)$ for some $m' > \frac{1}{2} - l$ (indeed, any $m' < \frac{3}{2} - l$) and for this reason will turn out to be irrelevant for the analysis that follows. Above, $\chi^{\pm}$ are smooth functions with support in $\pm[0, \infty)_{\gamma}$.

In the reverse direction, taking the inverse Mellin and Fourier transform yields two maps

$$\mathrm{Sol}(P) \ni u \mapsto a^{+}(\sigma, y) \in \tilde{\mathcal{I}}^l_{+}, \quad \mathrm{Sol}(P) \ni u \mapsto a^{-}(\sigma, y) \in \tilde{\mathcal{I}}^l_{+}, \tag{5.4}$$

where we have introduced the notation

$$\begin{aligned} \tilde{\mathcal{I}}^l_{\pm} := \big\{ a \in & \,\mathcal{C}^{\infty}(\overline{\mathbb{C}_{-l}} \times S_{\pm}) : \ \overline{\partial} a = 0, \\ & \forall M, N, k \in \mathbb{N}, \ B \in \mathrm{Diff}(S_{\pm}), \ \langle\sigma\rangle^N \partial_{\sigma}^k B a \restriction_{\{\sigma:\ \mathrm{Im}\ \sigma \in (-l, M)\}} \in L^{\infty} \big\} \end{aligned}$$

for the principal symbols of conormal distributions considered here. Above, $\mathbb{C}_{-l} = \{\sigma \in \mathbb{C} : \mathrm{Im}\ \sigma > -l\}$ and the Cauchy–Riemann operator $\overline{\partial}$ acts in the first variable (i.e., σ) in the domain where l is such that no resonances of the Mellin transformed inverse of P have imaginary part in $[-l, l]$.

Now, we make a choice of components $\mathcal{R}_I^-$ in the radial set from which the estimates are propagated from, labelled as usual by $I \subset \{+, -\}$ and set

$$\tilde{\mathcal{I}}_I := \tilde{\mathcal{I}}^l_{\pm} \oplus \tilde{\mathcal{I}}^l_{\pm},$$

where the signs are chosen in such way that the number of pluses (resp. minuses) reflects the number of components of $\mathcal{R}_I^+$ in S_+ (resp. S_-). Accordingly, we have a map (denoted ϱ_I) that assigns to a solution its pair of data on $\mathcal{R}_I^+$

$$\mathrm{Sol}(P) \ni u \mapsto \varrho_I u = (a, a') \in \tilde{\mathcal{I}}_I. \tag{5.5}$$

We will show that the map $\varrho_I : \mathrm{Sol}_I(P) \to \tilde{\mathcal{I}}_I$ is in fact bijective, possibly after removing a finite-dimensional subspace from $\tilde{\mathcal{I}}_I$.

Injectivity is a consequence of Lemma 4.1 (note that the hypotheses of this lemma are the reason why we consider here the restricted solution space $\mathrm{Sol}_I(P)$ instead of $\mathrm{Sol}(P)$), so we focus on surjectivity. Let $\tilde{\mathcal{P}}_I^0$ be the map defined for $(a, a') \in \tilde{\mathcal{I}}_I$, by applying formula (5.3) to a and a' (with the signs chosen consistently with I), multiplying the resulting distributions by η_+ or η_- (consistently with I), and then adding them up. Then $w = \tilde{\mathcal{P}}_I^0(a, a')$ belongs to $H_{\rm b}^{m,l}(M)$ for $m < \frac{1}{2} - l$ and its wave front set is in $\mathcal{R}$. Moreover, w is regular under $\mathfrak{M}$. The especially non-obvious part of this statement is regularity with respect to ρD_v, which uses the holomorphicity: ρD_v applied to (5.3) yields indeed

$$\begin{aligned}&\int_{\mathrm{Im}\ \sigma=-l} \rho^{\mathrm{i}(\sigma-\mathrm{i})} \mathrm{e}^{\mathrm{i}v\gamma} |\gamma|^{\mathrm{i}(\sigma-\mathrm{i})-1} a^{\pm}(\sigma, y) \chi^{\pm}(\gamma) d\gamma d\sigma \\ &= \int_{\mathrm{Im}\ \sigma=-l+1} \rho^{\mathrm{i}(\sigma-\mathrm{i})} \mathrm{e}^{\mathrm{i}v\gamma} |\gamma|^{\mathrm{i}(\sigma-\mathrm{i})-1} a^{\pm}(\sigma, y) \chi^{\pm}(\gamma) d\gamma d\sigma \\ &= \int_{\mathrm{Im}\ \sigma=-l} \rho^{\mathrm{i}\sigma} \mathrm{e}^{\mathrm{i}v\gamma} |\gamma|^{\mathrm{i}\sigma-1} a^{\pm}(\sigma+\mathrm{i}, y) \chi^{\pm}(\gamma) d\gamma d\sigma.\end{aligned} \tag{5.6}$$

One also gets that $Pw \in H_{\mathrm{b}}^{m,l}$ (two orders better than a priori expected, this follows from P being equal to $-4D_v(vD_v + \rho D_\rho)$ modulo $\mathfrak{M}^2$). We can improve this further:

Lemma 5.2. *Suppose $l \in \mathbb{R}$. There is a continuous linear map $\tilde{\mathcal{P}}_I : \tilde{\mathcal{I}}_I \to H_{\mathrm{b}}^{m,l}$, for all $m < \frac{1}{2} - l$, such that $P \circ \tilde{\mathcal{P}}_I : \tilde{\mathcal{I}}_I \to H_{\mathrm{b}}^{\infty,l}$ and $\tilde{\mathcal{P}}_I - \tilde{\mathcal{P}}_I^0 : \tilde{\mathcal{I}}_I \to H_{\mathrm{b}}^{m+1,l}$ for all $m < \frac{1}{2} - l$.*

Proof. This is a standard construction in microlocal analysis; see the proof of [6, Lemma 6.4] for a similar argument, but phrased without the explicit use of oscillatory integrals.

For $\tilde{a}$ holomorphic in $\mathrm{Im}\, \sigma > -l$, with boundary value Schwartz taking values in classical symbols of order s, consider a distribution w that modulo $H_{\mathrm{b}}^{\infty,l}$ is given an oscillatory integral of the form

$$J(\tilde{a}) = \int \rho^{\mathrm{i}\sigma} \mathrm{e}^{\mathrm{i}v\gamma} |\gamma|^{\mathrm{i}\sigma-1} \tilde{a}(\sigma, v, y, \gamma) d\gamma d\sigma.$$

Then for any $Q \in \mathrm{Diff}_{\mathrm{b}}^j(M)$, Qw is a distribution of the same form (modulo $H_{\mathrm{b}}^{\infty,l}$), but with s replaced by $s + j$, namely:

$$Qw = \int \rho^{\mathrm{i}\sigma} \mathrm{e}^{\mathrm{i}v\gamma} |\gamma|^{\mathrm{i}\sigma-1} \tilde{b}(\sigma, v, y, \gamma) d\gamma d\sigma$$

modulo $H_{\mathrm{b}}^{\infty,l}$, where $\tilde{b}$ is of order $s+j$, and $\tilde{b}$ differs from $\sigma_{\mathrm{b},j}(Q)(0, 0, y, \sigma, \gamma, 0)\tilde{a}(\sigma, v, y, \gamma)$ by a classical symbol of order $s + j - 1$ (where the variables are the local coordinates $(\rho, v, y, \sigma, \gamma, \eta)$ on the b-cotangent bundle). Indeed, this is straightforward to see for multiplication operators by $\mathcal{C}^\infty$ functions on ∂M, as well as for the vector fields $\rho D_\rho, D_v, D_{y_j}$: indeed, due to the Mellin transform this amounts to a σ-dependent version of the standard regularity statement for conormal distributions, conormal to $v = 0$. In addition, the statement holds for multiplication by powers ρ^k of ρ which in fact increase the domain of holomorphy, and indeed on $\mathrm{Im}\, \sigma = -l$ (and in the corresponding upper half plane) yields a similar term but with $\tilde{b}$ now of order $s - k$ by a contour shift argument similar to (5.6). Thus, for finite Taylor expansions of arbitrary $\mathcal{C}^\infty$ functions on ∂M one has the same multiplication property, with the symbolic order improving as one increases the power of ρ, so in fact the symbols arising from the full formal Taylor series can be asymptotically summed. One also sees by rewriting multiplication by ρ^k times an element ϕ of $\mathcal{C}^\infty(M)$ of support in $\rho < \epsilon$ as a convolution on the Mellin transform side that $\rho^k \phi J(\tilde{a})$ is in fact in $H_{\mathrm{b}}^{m,l}$ for any $m < \frac{1}{2} - l - s + k$. Combining this with the asymptotic summation statement, using that b-conormal distributions of symbolic order $s-k$ lie in $H_{\mathrm{b}}^{m,l}$ for any $m<\frac{1}{2}-l-s+k$, we see that (modulo $H_{\mathrm{b}}^{\infty,l}$) multiplication by a $\mathcal{C}^\infty$ function indeed gives a distribution of the stated form.

On the other hand, under the stronger assumption that $Q\in\mathfrak{M}\subset\mathrm{Diff}_{\mathrm{b}}^1$, we have that Qw is of the same form, but now with the *same* s, i.e. it equals

$$\int \rho^{\mathrm{i}\sigma}\mathrm{e}^{\mathrm{i}v\gamma}|\gamma|^{\mathrm{i}\sigma-1}\tilde{c}(\sigma,v,y,\gamma)d\gamma d\sigma$$

modulo $H_{\mathrm{b}}^{\infty,l}$, where $\tilde{c}$ is order s. Again, this is readily seen by applying the generator vector fields ρD_ρ, ρD_v, vD_v and D_{y_j} to the oscillatory integral, with the argument for ρD_v having already been discussed above. In addition, if $Q=\rho D_\rho+vD_v$, then $\tilde{c}$ differs from $-\gamma D_\gamma\tilde{a}(\sigma,v,y,\gamma)$, hence from $\mathrm{i}s\tilde{a}$, by a classical symbol of order $s-1$; note that this says concretely that for $s=0$, the result is a classical symbol of order -1.

Taking this into account, for $\tilde{a}$ as above of order 0, one can iteratively solve the problem of constructing u of the form

$$J(\tilde{a}_\infty)=\int \rho^{\mathrm{i}\sigma}\mathrm{e}^{\mathrm{i}v\gamma}|\gamma|^{\mathrm{i}\sigma-1}\tilde{a}_\infty(\sigma,v,y,\gamma)d\gamma d\sigma$$

modulo $H_{\mathrm{b}}^{\infty,l}$, with $\tilde{a}_\infty-\tilde{a}$ classical of order -1, and with $Pu\in H_{\mathrm{b}}^{\infty,l}$. Indeed, take first $\tilde{a}_0=\tilde{a}$, so for $Q\in\mathfrak{M}^2$, the expression

$$PJ(\tilde{a}_0)=-4D_v(\rho D_\rho+vD_v)J(\tilde{a}_0)+QJ(\tilde{a}_0)$$

is of the form $J(\tilde{r}_0)$ with $\tilde{r}_0$ classical of order 0. Thus taking $\tilde{a}_1'$ of order -1, $PJ(\tilde{a}_1')$ is of the form $J(\tilde{r}_1')$ modulo $H_{\mathrm{b}}^{\infty,l}$ with $\tilde{r}_1'$ a symbol of order 0, equal to $-4\mathrm{i}\gamma\tilde{a}_1'$ modulo symbols of order -1. Thus, choosing $\tilde{a}_1'=-\frac{\mathrm{i}}{4}\tilde{r}_0$, $\tilde{a}_1=\tilde{a}_0+\tilde{a}_1'$, $PJ(\tilde{a}_1)$ is 0 modulo $J(\tilde{r}_1)+H_{\mathrm{b}}^{\infty,l}$, with $\tilde{r}_1$ symbol of order -1, i.e. it *is* of the form $J(\tilde{r}_1)+H_{\mathrm{b}}^{\infty,l}$, which is one order improvement over $\tilde{r}_0$ corresponding to $PJ(\tilde{a}_0)$. Similarly, we inductively construct $\tilde{a}_k=\tilde{a}_0+\sum_{j=1}^k\tilde{a}_j'$ such that $PJ(\tilde{a}_k)$ is of the form $J(\tilde{r}_k)+H_{\mathrm{b}}^{\infty,l}$, with $\tilde{r}_k$ classical of order $-k$. This can be done because for $\tilde{a}_k'$ classical of order $-k$, $P\tilde{J}(a_k')$ is of the form $J(\tilde{r}_k')$ modulo $H_{\mathrm{b}}^{\infty,l}$ with $\tilde{r}_k'$ a symbol of order $-k+1$, equal to $-4\mathrm{i}k\gamma\tilde{a}_k'$ modulo symbols of order $-k$; the point being that as $k\neq 0$, $-4\mathrm{i}k\gamma\tilde{a}_k'=-\tilde{r}_{k-1}$ (where $\tilde{r}_{k-1}$ corresponds to $PJ(\tilde{a}_{k-1})$) can be solved for $\tilde{a}_k'$. Finally asymptotically summing $\tilde{a}_\infty'\sim\sum_{j=1}^\infty\tilde{a}_j'$, we see that $\tilde{a}_\infty=\tilde{a}_0+\tilde{a}_\infty'$ satisfies the requirements of the lemma. □

We now define the *Poisson operator*

$$\mathcal{P}_I:=(P_I^{-1}-P_{I^{\mathrm{c}}}^{-1})P\tilde{\mathcal{P}}_I. \tag{5.7}$$

Let us analyze its mapping properties. First, $P\tilde{\mathcal{P}}_I$ maps $\tilde{\mathcal{I}}_I$ to $\mathrm{Ran}P_I$ directly from the definition as $\tilde{\mathcal{P}}_I$ maps into $\mathcal{X}_I$ by virtue of Lemma 5.2. Furthermore $P\tilde{\mathcal{P}}_I$ maps also to $\mathcal{Y}^{\infty,l}=\bigcup_m\mathcal{Y}^{m,l}$, which is a subset of $\mathcal{Y}_{I^{\mathrm{c}}}$. Since $P:\mathcal{X}_I\to\mathcal{Y}_I$ is Fredholm, the kernel of $P\tilde{\mathcal{P}}_I$ is finite dimensional and has thus a complement $\mathcal{K}_I\subset\tilde{\mathcal{I}}_I$. On $\mathcal{K}_I$, $P\tilde{\mathcal{P}}_I$ is injective, so the pre-image of $\mathcal{Z}_{I^{\mathrm{c}}}$ (where we recall that $\mathcal{Z}_{I^{\mathrm{c}}}$ is a complement of $\mathcal{Y}_{I^{\mathrm{c}}}$) is finite dimensional. Taking the pre-image of $\mathrm{Ran}P_{I^{\mathrm{c}}}$ and adding to it elements of $\mathrm{Ker}\,P\tilde{\mathcal{P}}_I$ we obtain a subspace of $\tilde{\mathcal{I}}_I$, denoted $\mathcal{I}_I$, which has a finite dimensional complement and such that $P\tilde{\mathcal{P}}_I\mathcal{I}_I\subset\mathrm{Ran}P_{I^{\mathrm{c}}}$. Thus, the Poisson operator (5.7) maps

$$\mathcal{P}_I:\mathcal{I}_I\to\mathrm{Sol}_I(P).$$

We will prove that ϱ_I maps $\mathrm{Sol}_I(P) \to \mathcal{I}_I$ and that it does so bijectively, with inverse $\mathcal{P}_I$. We will need the following lemma.

Lemma 5.3. *The operator $\tilde{\mathcal{P}}_I \circ \varrho_I$ acts on $\mathrm{Sol}(P)$ as a pseudodifferential operator that is microlocally the identity near $\mathcal{R}_I^+$ and microlocally vanishes near $\mathcal{R}_I^-$, modulo terms that map to $H_{\mathrm{b}}^{m',l}(M)$ for some $m' > \frac{1}{2} - l$.*

Proof. Let us introduce an analogue of the map $\tilde{\mathcal{P}}_I$ that acts on full symbols (rather than on principal symbols):

$$\tilde{\mathcal{P}}_0 a := \int \rho^{\mathrm{i}\sigma} \mathrm{e}^{\mathrm{i}v\gamma} \eta_+(v,\rho,y) a(\sigma,v,y,\gamma) d\gamma d\sigma, \tag{5.8}$$

and correspondingly

$$\varrho_0 u := (2\pi)^{-2} \int \rho^{-\mathrm{i}\sigma} \mathrm{e}^{-\mathrm{i}v\gamma} \eta_+(\rho,v,y) u(\rho,v,y)\, d\rho\, dv.$$

Now, the already discussed statement on the regularity of solutions is that they are of the form $\tilde{\mathcal{P}}_0 a$ for some symbol a as above (with the appropriate holomorphy properties) modulo $H_{\mathrm{b}}^{\infty,l}$. If they were actually of this form (and the difference in $H_{\mathrm{b}}^{\infty,l}$ is easy to deal with in any case), one would get

$$\tilde{\mathcal{P}}_0 \varrho_0 u = \tilde{\mathcal{P}}_0 \varrho_0 \tilde{\mathcal{P}}_0 a = \tilde{\mathcal{P}}_0 (\varrho_0 \tilde{\mathcal{P}}_0 a),$$

and hence one is done if $\varrho_0 \tilde{\mathcal{P}}_0$ is essentially the identity. Now,

$$\varrho_0 \tilde{\mathcal{P}}_0 a = \mathcal{F}_v \mathcal{M}_\rho \eta_+^2 \mathcal{M}^{-1} \mathcal{F}^{-1} a, \tag{5.9}$$

so the question is whether

$$\mathcal{F}_v \mathcal{M}_\rho (1-\eta_+^2) \mathcal{M}^{-1} \mathcal{F}^{-1} a$$

is trivial. But it indeed is, since $\mathcal{M}^{-1}\mathcal{F}^{-1}$ maps symbols to distributions which are in $H_{\mathrm{b}}^{\infty,l}$ away from $\{\rho = 0, v = 0\}$, thus on the support of $1 - \eta_+^2$, and then $\mathcal{F}\mathcal{M}$ sends these to symbols of order $-\infty$ in the required sense.

Given this, the map ϱ is simply a restriction of a rescaled version of ϱ_0 to $\pm\infty$ in γ; $\tilde{\mathcal{P}}$ is an analogous composition with extension from $\pm\infty$ (ignoring $\chi_\pm$ which just cuts everything in two), namely

$$\varrho = r_\infty |\gamma|^{-\mathrm{i}\sigma+1} \varrho_0, \ \tilde{\mathcal{P}} = \tilde{\mathcal{P}}_0 |\gamma|^{\mathrm{i}\sigma-1} E_\infty.$$

Thus,

$$\tilde{\mathcal{P}} \varrho = \tilde{\mathcal{P}}_0 \varrho_0 + \tilde{\mathcal{P}}_0 |\gamma|^{\mathrm{i}\sigma-1} (E_\infty r_\infty - \mathbf{1}) |\gamma|^{-\mathrm{i}\sigma+1} \varrho_0, \tag{5.10}$$

and the first term is microlocally the identity as we have seen before, while the second term maps to b-conormal distributions of one lower order because $E_\infty r_\infty - \mathbf{1}$ maps smooth functions on the compactified line (times various irrelevant factors) to functions vanishing to first order at $\pm\infty$. □

Now, since in the sense stated in the above lemma, $\tilde{\mathcal{P}}_I \varrho_I$ is microlocally the identity near $\mathcal{R}_I^+$ and microlocally vanishes near $\mathcal{R}_I^-$, arguing as in the paragraph below (4.2) we conclude that $P\tilde{\mathcal{P}}_I\varrho_I$ maps $\mathrm{Sol}_I(P)$ to $\mathrm{Ran}P_I\cap\mathrm{Ran}P_{I^c}$. This in turn implies that ϱ_I maps to $\mathcal{I}_I$. On the other hand using (4.3) we get

$$-(P_{I^c}^{-1}-P_I^{-1})P\tilde{\mathcal{P}}_I\varrho_I=\mathbf{1}\quad\text{on}\ \ \mathrm{Sol}_I(P), \tag{5.11}$$

that is $\mathcal{P}_I\varrho_I=\mathbf{1}$ on $\mathrm{Sol}_I(P)$. Thus, to deduce surjectivity of ϱ_I we need to show that $\mathcal{P}_I$ is injective.

First, we observe that $\mathcal{P}_I(a,a')=\tilde{\mathcal{P}}_I(a,a')$ at $\mathcal{R}_I^+$ modulo $H_{\mathrm{b}}^{m+1,l}$ terms with $\mathfrak{M}(M)$ regularity. Thus, it suffices to prove that $(a,a')\mapsto[w]=[\tilde{\mathcal{P}}_I(a,a')]$ is injective, with the equivalence class considered modulo $H_{\mathrm{b}}^{m+1,l}$, $-\frac{1}{2}+l<m<\frac{1}{2}+l$. This can be readily seen from the computation in (5.9) which gives injectivity of the auxiliary map $\tilde{\mathcal{P}}_0$, and hence the stated injectivity of $\tilde{\mathcal{P}}_I$ modulo $H_{\mathrm{b}}^{m+1,l}$.

We have thus proved:

Proposition 5.4. *The map* $\mathrm{Sol}_I(P)\ni u\mapsto\varrho_I u\in\mathcal{I}_I$ *defined in (5.5) is bijective with inverse* $\mathcal{P}_I$.

We now consider the pairing formula for smooth approximate solutions, i.e. for u satisfying

$$u\in H_{\mathrm{b}}^{-\infty,0}(M),\quad Pu\in H_{\mathrm{b}}^{\infty,0}(M),\quad \mathrm{WF}_{\mathrm{b}}^{\infty,0}(u)\subset\mathcal{R}; \tag{5.12}$$

the computations below are closely related to [52]. To this end we will need a family of operators $\mathcal{J}_r$ belonging to Ψ_{b}^{-N} for $r\in(0,1]$ (and N sufficiently large), uniformly bounded in Ψ_{b}^0 for $r\in(0,1]$ and tending to $\mathbf{1}$ as $r\to0$ in $\Psi_{\mathrm{b}}^\epsilon$ for any $\epsilon>0$, so that $[P,\mathcal{J}_r]\to0$ in $\Psi_{\mathrm{b}}^{1+\epsilon}$. Let us take concretely $\mathcal{J}_r$ to have principal symbol $j_r=(1+r|\gamma|)^{-N}$ near the radial sets. Then

$$\begin{aligned}\mathrm{i}^{-1}(\langle Pu_1,u_2\rangle_{\mathrm{b}}-\langle u_1,Pu_2\rangle_{\mathrm{b}})&=\mathrm{i}^{-1}\lim_{r\to0}(\langle\mathcal{J}_rPu_1,u_2\rangle_{\mathrm{b}}-\langle\mathcal{J}_ru_1,Pu_2\rangle_{\mathrm{b}})\\&=\lim_{r\to0}\langle\mathrm{i}[\mathcal{J}_r,P]u_1,u_2\rangle_{\mathrm{b}},\end{aligned} \tag{5.13}$$

for any u_1,u_2 satisfying (5.12) and the principal symbol of $\mathrm{i}[\mathcal{J}_r,P]$ is

$$-H_pj_r=(\mathrm{sgn}\gamma)Nr(1+r|\gamma|)^{-1}j_rH_p\gamma.$$

Moreover, $H_p|\gamma|=(\mathrm{sgn}\gamma)H_p\gamma$ is positive at sinks, negative at sources. Concretely, in our case, as p is given by $-4\gamma(v\gamma+\sigma)$ modulo terms that vanish quadratically at the radial set $\mathcal{R}$, $H_p\gamma$ is given by $4\gamma^2$ modulo terms vanishing at $\mathcal{R}$. Hence, $-H_pj_r$ equals $4\gamma^2(\mathrm{sgn}\gamma)Nr(1+r|\gamma|)^{-1}j_r$ modulo such terms, thus the sinks correspond to $\gamma>0$, whereas the sources to $\gamma<0$.

Now, u_1 and u_2 have module regularity of the same type as already discussed for $\mathrm{Sol}(P)$, so the result of the computation of (5.13) is unaffected if P is changed by terms in $\mathfrak{M}^2$ (provided they preserve the formal self-adjointness). Moreover, u_i can be replaced by distributions $\tilde{u}_i$ with $u_i-\tilde{u}_i\in H_{\mathrm{b}}^{m+1,l}$, $P\tilde{u}_i\in H_{\mathrm{b}}^{m,l}$ with wave front set in the radial sets. So in particular, for each i we may replace $u=u_i$ by $\tilde{\mathcal{P}}_\emptyset(a_-^+,a_-^-)+\tilde{\mathcal{P}}_{\{+,-\}}(a_+^+,a_+^-)$, where $a_\pm^\pm$ are the b-conormal principal symbols discussed before, with the superscript denoting the component of the characteristic set and the subscript the component of the radial set: $\mathcal{R}_\emptyset^-$ versus $\mathcal{R}_\emptyset^+$.

Therefore, as the Mellin transform and Fourier transform are isometries up to constant factors, we can reexpress (5.13) as

$$
\begin{aligned}
&= \lim_{r\to 0} 2\pi \sum_{\pm} \int 4\gamma^2 N r(1+r|\gamma|)^{-1} j_r |\gamma|^{\mathrm{i}\sigma-1}|\gamma|^{-\mathrm{i}\sigma-1} \\
&\qquad\qquad \times \Big(\chi^+(\gamma)^2 \sum_{\pm} \overline{a^{\pm}_{1,+}} a^{\pm}_{2,+} - \chi^-(\gamma)^2 \sum_{\pm} \overline{a^{\pm}_{1,-}} a^{\pm}_{2,-}\Big) |dh(y)| d\gamma d\sigma \\
&= \lim_{r\to 0} 2\pi \sum_{\pm} \Big(\int 4Nr(1+r|\gamma|)^{-1} j_r \chi^+(\gamma)^2 d\gamma\Big)\Big(\int \overline{a^{\pm}_{1,+}} a^{\pm}_{2,+} |dh(y)| d\sigma\Big) \\
&\qquad - \Big(\int 4Nr(1+r|\gamma|)^{-1} j_r \chi^-(\gamma)^2 d\gamma\Big)\Big(\int \overline{a^{\pm}_{1,-}} a^{\pm}_{2,-} |dh(y)| d\sigma\Big)
\end{aligned}
$$

where h is the metric on $S_\pm$ and the integral in σ is over $\operatorname{Im}\sigma = 0$. Integrating by parts and then applying the dominated convergence theorem gives

$$
\begin{aligned}
&= \lim_{r\to 0} 2\pi \sum_{\pm} \Big(\int -4\frac{d}{d\gamma}(j_r)\chi^+(\gamma)^2 d\gamma\Big)\Big(\int \overline{a^{\pm}_{1,+}} a^{\pm}_{2,+} |dh(y)| d\sigma\Big) \\
&\qquad - \Big(\int -4\frac{d}{d\gamma}(j_r)\chi^-(\gamma)^2 d\gamma\Big)\Big(\int \overline{a^{\pm}_{1,-}} a^{\pm}_{2,-} |dh(y)| d\sigma\Big) \\
&= \lim_{r\to 0} 2\pi \sum_{\pm} \Big(\int -4 j_r \frac{d}{d\gamma}\chi^+(\gamma)^2 d\gamma\Big)\Big(\int \overline{a^{\pm}_{1,+}} a^{\pm}_{2,+} |dh(y)| d\sigma\Big) \\
&\qquad - \Big(\int -4 j_r \frac{d}{d\gamma}\chi^-(\gamma)^2 d\gamma\Big)\Big(\int \overline{a^{\pm}_{1,-}} a^{\pm}_{2,-} |dh(y)| d\sigma\Big) \\
&= 8\pi \sum_{\pm} \Big(\int \overline{a^{\pm}_{1,+}} a^{\pm}_{2,+} |dh(y)| d\sigma - \int \overline{a^{\pm}_{1,-}} a^{\pm}_{2,-} |dh(y)| d\sigma\Big).
\end{aligned}
$$

This means that for $u_1 = \tilde{\mathcal{P}}_I(a_1^+, a_1^-)$, and $u_2 \in \mathrm{Sol}(P)$ with asymptotic data $\varrho_I u = (a_2^+, a_2^-)$ we have

$$
\langle P\tilde{\mathcal{P}}_I(a_1^+, a_1^-), u_2\rangle_{\mathrm{b}} = 8\pi\mathrm{i} \sum_{\pm} (-1)^{I(\pm)} \int \overline{a_1^{\pm}} a_2^{\pm} |dh(y)| d\sigma, \tag{5.14}
$$

where we have used the notation introduced before

$$
(-1)^{I(\pm)} = \begin{cases} 1 \text{ if } \ \pm \in I, \\ -1 \text{ otherwise.} \end{cases}
$$

If instead (a_2^+, a_2^-) are the asymptotics of u_2 at $\mathcal{R}_I^+ = \mathcal{R}_{I^{\mathrm{c}}}^-$ then

$$
\langle P\tilde{\mathcal{P}}_{I^{\mathrm{c}}}(a_1^+, a_1^-), u_2\rangle_{\mathrm{b}} = -8\pi\mathrm{i} \sum_{\pm} (-1)^{I(\pm)} \int \overline{a_1^{\pm}} a_2^{\pm} |dh(y)| d\sigma.
$$

This gives in the former case

$$
\varrho_I u_2 = 8\pi\mathrm{i} \begin{pmatrix} (-1)^{I(+)} & 0 \\ 0 & (-1)^{I(-)} \end{pmatrix} (P\tilde{\mathcal{P}}_I)^* u_2 \tag{5.15}
$$

and so if u_2 belongs to the restricted solution space $\mathrm{Sol}_I(P)$,

$$\begin{aligned} u_2 &= 8\pi\mathrm{i}\mathcal{P}_I \begin{pmatrix} (-1)^{I(+)} & 0 \\ 0 & (-1)^{I(-)} \end{pmatrix} (P\tilde{\mathcal{P}}_I)^* u_2 \\ &= 8\pi\mathrm{i}(P_I^{-1} - P_{I^{\mathrm{c}}}^{-1})P\tilde{\mathcal{P}}_I \begin{pmatrix} (-1)^{I(+)} & 0 \\ 0 & (-1)^{I(-)} \end{pmatrix} (P\tilde{\mathcal{P}}_I)^* u_2. \end{aligned}$$

In particular,

$$(P_I^{-1} - P_{I^{\mathrm{c}}}^{-1}) = 8\pi\mathrm{i}(P_I^{-1} - P_{I^{\mathrm{c}}}^{-1})P\tilde{\mathcal{P}}_I \begin{pmatrix} (-1)^{I(+)} & 0 \\ 0 & (-1)^{I(-)} \end{pmatrix} (P\tilde{\mathcal{P}}_I)^* (P_I^{-1} - P_{I^{\mathrm{c}}}^{-1}),$$

hence using (5.15) again,

$$(P_I^{-1} - P_{I^{\mathrm{c}}}^{-1}) = \mathrm{i}(8\pi)^{-1}(P_I^{-1} - P_{I^{\mathrm{c}}}^{-1})\varrho_I^* \begin{pmatrix} (-1)^{I(+)} & 0 \\ 0 & (-1)^{I(-)} \end{pmatrix} \varrho_I (P_I^{-1} - P_{I^{\mathrm{c}}}^{-1})$$

Denoting now

$$q_I := (8\pi)^{-1} \begin{pmatrix} (-1)^{I(+)} & 0 \\ 0 & (-1)^{I(-)} \end{pmatrix}, \tag{5.16}$$

and recalling that $G_I = P_I^{-1} - P_{I^{\mathrm{c}}}^{-1}$, this can be rewritten as $\mathrm{i}G_I = -G_I\varrho_I^* q_I \varrho_I G_I$. In the sense of sesquilinear forms on $\mathrm{Ran}P_I \cap \mathrm{Ran}P_{I^{\mathrm{c}}}$, $\mathrm{i}G_I$ is formally self-adjoint so this gives

$$\mathrm{i}G_I = G_I^* \varrho_I^* q_I \varrho_I G_I. \tag{5.17}$$

In summary:

Theorem 5.5. *Let $I \subset \{+,-\}$ and suppose $l = 0$ is not a resonance in the sense of Hypothesis 3.1. There are isomorphisms of symplectic spaces*

$$\frac{\mathrm{Ran}P_I \cap \mathrm{Ran}P_{I^{\mathrm{c}}}}{P(\mathcal{X}_I \cap \mathcal{X}_{I^{\mathrm{c}}})} \xrightarrow{[G_I]} \mathrm{Sol}_I(P) \xrightarrow{\varrho_I} \mathcal{I}_I, \tag{5.18}$$

where the symplectic form on the first one is given by $\langle \cdot, G_I \cdot \rangle$ and on the last one by (5.16).

As an aside, observe that if we get back to equation (5.14) specifically in the Feynman or anti-Feynman case, we obtain that for any approximate solution u with asymptotic data $\varrho_I u = (a^+, a^-)$, the quantity $\langle P\tilde{\mathcal{P}}_I(a^+, a^-), u \rangle_{\mathrm{b}}$ vanishes if and only if $(a^+, a^-) = 0$. In particular, if $u \in \mathrm{Ker}\, P_I$ (so that u is regular at $\mathcal{R}_I^-$) then

$$\langle P\tilde{\mathcal{P}}_I(a^+, a^-), u \rangle_{\mathrm{b}} = \langle \tilde{\mathcal{P}}_I(a^+, a^-), Pu \rangle_{\mathrm{b}} = 0$$

so $(a^+, a^-) = 0$. This implies u has above-threshold regularity at $\mathcal{R}_I^+$; it is also regular at $\mathcal{R}_I^-$ so in fact by above-threshold estimates we get:

Proposition 5.6. *In the Feynman ($I = \emptyset$) and anti-Feynman case ($I = \{+,-\}$), Hypothesis 3.2 is satisfied for $l = 0$, i.e.* $\mathrm{Ker}\, P_I \subset H_{\mathrm{b}}^{\infty,0}(M)$*.*

5.3. **Hadamard two-point functions.** The second arrow in (5.18) means that the symplectic space $\mathscr{V}_I$ is isomorphic to $\mathcal{I}_I$ equipped with the symplectic form $\mathrm{i}^{-1}q_I$, which is more tractable in applications.

Let us denote

$$\pi^+ = (8\pi)^{-1}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi^- = (8\pi)^{-1}\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

and for $I \subset \{+,-\}$ consider the pair of operators

$$\Lambda_I^{\pm} := G_I^* \varrho_I^* \pi^{\pm} \varrho_I G_I : H_{\mathrm{b}}^{\infty,0}(M) \to H_{\mathrm{b}}^{-\infty,0}(M). \tag{5.19}$$

They satisfy $P\Lambda_I^{\pm} = \Lambda_I^{\pm}P = 0$, $(-1)^{I(+)}\Lambda_I^+ + (-1)^{I(-)}\Lambda_I^- = \mathrm{i}G_I$ and $\Lambda_I^{\pm} \geq 0$ when identified with sesquilinear forms on $\mathrm{Ran}P_I \cap \mathrm{Ran}P_{I^c}$ via the product $\langle\cdot,\cdot\rangle_{\mathrm{b}}$. We will prove that they also satisfy the wave front set condition required from Hadamard two-point functions.

Theorem 5.7. *The pair of operators $\Lambda_I^{\pm}$ defined in (5.19) satisfy the Hadamard condition, and thus if $I = \{\pm\}$, $\Lambda_I^{\pm}$ are Hadamard two-point functions for P (cf. Definition 4.5).*

Proof. We assume for simplicity that all the operators P_I are invertible, otherwise one simply needs to use projections to the finite-dimensional spaces $\mathrm{Ker}\, P_I$ and $\mathcal{Z}_I$ to legitimize the arguments that follow. We consider the case $I = \{+\}$, the remaining ones being analogous, and we skip the subscript I for brevity of notation.

First observe that for any $v \in \mathcal{X}_+ \cap \mathcal{X}_-$, the distribution $f = \tilde{\mathcal{P}}\pi^+\varrho G v$ has above-threshold regularity at ${}^{\mathrm{b}}SN^{*+}S_-$, ${}^{\mathrm{b}}SN^{*-}S_-$ (due to the definition of $\tilde{\mathcal{P}}$) and also at ${}^{\mathrm{b}}SN^{*-}S_+$ (due to the presence of π^+). Now $\Lambda^+ v = (\mathbf{1} - P_+^{-1}P)f$ differs from f by a term regular at ${}^{\mathrm{b}}SN^*S_+$, thus $\Lambda^+ v$ is regular near ${}^{\mathrm{b}}SN^{*-}S_+$. It also solves the wave equation, so by propagation of singularities $\mathrm{WF}(\Lambda^+ v) \subset \Sigma^+$ in M°.

Applying this to $v = \delta_x$, this means on the level of the Schwartz kernel that $\mathrm{WF}'(\Lambda^+) \subset (\Sigma^+ \cup o) \times T^*M^\circ$, and in the same way one gets $\mathrm{WF}'(\Lambda^-) \subset (\Sigma^- \cup o) \times T^*M^\circ$. By Remark 4.6 this suffices to conclude that $\mathrm{WF}'(\Lambda^{\pm})$ equals $\cup_{t\in\mathbb{R}}\Phi_t(\mathrm{diag}_{T^*M^\circ}) \cap \pi^{-1}\Sigma^{\pm}$. □

As already outlined in the introduction, the two-point functions $\Lambda_+^{\pm}$ and $\Lambda_-^{\pm}$ constructed from asymptotic data ϱ_+ and ϱ_- can be thought as analogues of two-point functions constructed in other setups [44, 45, 21, 24] for the conformal wave equation and for the massive Klein-Gordon equation (rather than for the wave equation considered here).

5.4. **Blow-up of S.** In the setting of Definition 2.1, a convenient way to specify the asymptotic data of a solution of the wave equation is based on the radiation field blow-up proposed by Baskin, Vasy and Wunsch in [6] in the context of asymptotic expansions for the Friedlander radiation fields (much in the spirit of Friedlander's work [19]). In what follows we briefly discuss how this can be used in our situation to provide a more geometrical description of the data $\varrho_I u$ (for a restricted class of solutions), starting with the following example. Namely, on Minkowski space $\mathbb{R}^{1+d}$ with coordinates (t,x), a convenient choice of new coordinates is $s = t - |x|$, $y = x/|x|$, $\rho = (t^2 + |x|^2 + 1)^{-1/2}$. These make sense locally near the *front face* $\mathrm{ff} = \{\rho = 0\}$, and asymptotic properties of solutions can be described in terms of their restriction to ff, multiplied first by a $\rho^{-(n-2)/2}$ factor to make this restriction well-defined. The step that consists of multiplying a solution u by $\rho^{-(n-2)/2}$ can be interpreted as replacing the original metric by a conformally related one, which extends smoothly to $\{\rho = 0\}$, and then considering u as a solution for the conformally related wave operator.

In the general setting of Lorentzian scattering spaces, recalling that ρ is a boundary defining function of ∂M and (v, y) are coordinates on ∂M with $S = \{\rho = 0, v = 0\}$, the analogue of this construction consists of introducing coordinates (s, y) with $s = v/\rho$, valid near a boundary hypersurface 'ff' (the *front face*) of a new manifold that replaces M, constructed as the sum of $M \setminus S$ and the *inward-pointing spherical normal bundle* of S. More precisely, one replaces M with a manifold with corners $[M; S]$ (the *blow-up of M along S*, cf. [41]), equipped in particular with a smooth map $[M; S] \to M$ called the *blow-down* map which is a diffeomorphism between the interior of the two spaces. It is possible to canonically define $[M; S]$ in such way that 'polar coordinates' $R = (v^2 + \rho^2)^{1/2}$, $\vartheta = (\rho \cdot v)/R$ are smooth, and smooth functions on M lift to smooth ones on $[M; S]$ by the blow-down map. The boundary surface of interest ff is simply defined as the lift (i.e. inverse image) of S to $[M; S]$, and near its interior, (ρ, s, y) constitute a well-defined system of coordinates indeed.

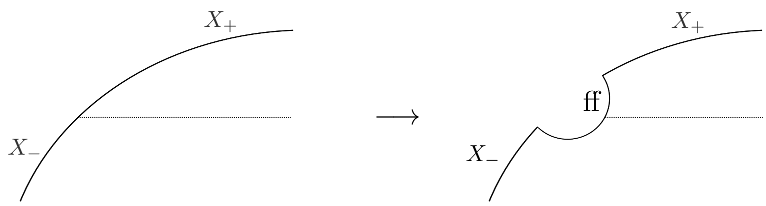

FIGURE 4. The radiation field blow-up of M along $S = S_+ \cup S_-$. The blow-down map goes in the reverse of the direction of the arrow.

Although the metric g (lifted using the blow-down map) is ill-behaved as ρ tends to 0, rescaling it by a conformal factor ρ^2 yields a Lorentzian metric $\rho^2 g$ which is smooth down to $\rho = 0$. Note that if $u(\rho, v, y)$ solves $Pu = f$, then $u(\rho, \rho s, y)$ is a solution of the inhomogeneous Klein-Gordon equation conformally related to $\Box_g$.

It can be argued that the restriction of u to the front face is well-defined for $u \in \mathrm{Sol}_I(P)$ at least if $l > 0$. Indeed, in that case, u can be (locally) expressed as $\tilde{\mathcal{P}}_0 a$ modulo some decaying terms, where we recall that $\tilde{\mathcal{P}}_0$ was defined in (5.8) and adapted to the present setup, it maps to distributions which are conormal to the front face (in particular we get decay in the $L^2_{\rm b}$ sense due to the assumption $l > 0$), thus the restriction to ff makes sense.

Now, recall that in our discussion of the asymptotic data ϱ_I, the starting point was the expression

$$\int \rho^{\mathrm{i}\sigma} \mathrm{e}^{\mathrm{i}v\gamma} |\gamma|^{\mathrm{i}\sigma - 1} a^{\pm}(\sigma, y) \chi^{\pm}(\gamma) d\gamma d\sigma \tag{5.20}$$

for elements of $\mathrm{Sol}(P)$, valid (near S) modulo terms in $H_{\mathrm{b}}^{m',l}(M)$ for some $m' > \frac{1}{2} - l$. Performing the σ integral first, one obtains (up to non-zero constant factors)

$$\int e^{iv\gamma} (\mathcal{M}^{-1} a^{\pm})(\rho|\gamma|, y) \chi^{\pm}(\gamma) |\gamma|^{-1} d\gamma,$$

where $\mathcal{M}^{-1}$ is the inverse Mellin transform in σ. Replacing γ by $\nu = \rho\gamma$, one has

$$\int e^{i\nu(v/\rho)} (\mathcal{M}^{-1} a^{\pm})(|\nu|, y) \chi^{\pm}(\rho^{-1}\gamma) |\nu|^{-1} d\nu.$$

As $\rho \to 0$ this becomes

$$\int_{\pm[0,\infty)} e^{i\nu(v/\rho)} (\mathcal{M}^{-1} a^{\pm})(|\nu|, y) |\nu|^{-1} d\nu,$$

which is the inverse Fourier transform in ν of $(\mathcal{M}^{-1} a^{\pm})(|\nu|, y) \mathbb{1}^{\pm}(\nu) |\nu|^{-1}$ ($\mathbb{1}^{\pm}$ being the characteristic function of $\pm[0, \infty)$) evaluated in the radiation face coordinate $s = v/\rho$:

$$\mathcal{F}^{-1}\left((\mathcal{M}^{-1} a^{\pm})(|\,.\,|, y) \mathbb{1}^{\pm}(\,.\,) |\,.\,|^{-1} \right)(v/\rho). \tag{5.21}$$

Note that the inverse Fourier transform above is well-defined because the product of $(\mathcal{M}^{-1} a^{\pm})(|\,.\,|, y)$ and $\mathbb{1}^{\pm}(\,.\,)|\,.\,|^{-1}$ is in L^1 by Lemma 5.1. As the inverse Fourier transform of a distribution conormal to the origin, (5.21) is a symbol, although it is difficult to make an exact statement for the exact class of symbols it is in since the superlogarithmic decay at the origin does not translate directly into nice estimates.

After performing the blow-up, we can view (5.21) as the restriction of a solution to the front face ff. Thus in the reverse direction, one takes $u\restriction_{\mathrm{ff}}$, one Fourier transforms it, then restricts to the positive or negative half-lines and then Mellin transforms the result to obtain the principal symbol of the solution in the respective half of ${}^{\mathrm{b}}SN^*S_{\pm} = {}^{\mathrm{b}}SN^{*+}S_{\pm} \cup {}^{\mathrm{b}}SN^{*-}S_{\pm}$. This means that ϱ_I can be expressed as

$$\mathrm{Sol}(P) \ni u \mapsto \varrho_I u := \left(\mathcal{M}(\mathcal{F}(\eta_{\pm} u\restriction_{\mathrm{ff}})|\gamma|\, \mathbb{1}^{\pm}), \mathcal{M}(\mathcal{F}(\eta_{\pm} u\restriction_{\mathrm{ff}})|\gamma|\, \mathbb{1}^{\pm}) \right) \in \tilde{\mathcal{I}}_I, \tag{5.22}$$

where the signs are chosen relatively to I, i.e. for each component the subscript indicates S_+ versus S_- and the sign in the superscript indicates ${}^{\mathrm{b}}SN^{*+}S$ versus ${}^{\mathrm{b}}SN^{*-}S$, provided that u is such that the restriction $u\restriction_{\mathrm{ff}}$ is well-defined.

We remark here that specifying $u\restriction_{\mathrm{ff}}$ is analogous to setting (part of) a characteristic Cauchy problem in the sense that the conormal of ff lies in the characteristic set of $\Box_{\rho^2 g}$, this bears thus some resemblance to the construction used in [44, 45, 21] in the case of the conformal wave equation.

6. Asymptotically de Sitter spacetimes

6.1. **Geometrical setup.** The proof of the Fredholm property of the rescaled wave operator P on asymptotically Minkowski spacetimes in [6, 30, 25] is based on a careful analysis of the Mellin transformed normal operator family $\widehat{N}(P)(\sigma)$, which is a holomorphic family of differential operators on the compact manifold ∂M. Recall also that we used results from [6] on module regularity of solutions of P, these in turn are based on the Mellin transformed version of the operator P. The relevant property is that for fixed σ one has an elliptic operator in the two connected components of the region $v > 0$ and a hyperbolic one in $v < 0$. Furthermore, in the respective regions they can be related to the Laplacian on an asymptotically hyperbolic space and to the wave operator on an asymptotically de Sitter space by conjugation with powers of the boundary-defining functions of $S_\pm$, with $S = S_+ \cup S_-$ playing the role of the asymptotically de Sitter conformal boundary. In this section we will be interested in the reverse construction, which extends a given asymptotically de Sitter space X_0 (conformally compactified, with conformal boundary $S = S_+ \cup S_-$) to a compact manifold X, and relates the Klein-Gordon operator on the asymptotically de Sitter region to a differential operator $\hat{P}_X$ defined on the whole 'extended' manifold X. The main merit of this construction is that $\hat{P}_X$ acts on a manifold without boundary and more importantly it fits into the framework of [53, 30], with bicharacteristics beginning and ending at the radial sets located above S_+ and S_-.

These various relations are explained in more detail in [50, 56], here as an illustration we start with the special case of actual $n = 1 + d$-dimensional Minkowski space $\mathbb{R}^{1+d}$ with metric $g_{\mathbb{R}^{1,d}} = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$. Its radial compactification is a compact manifold M with boundary $\partial M = \mathbb{S}^{d-1}$, and with $\rho = (z_0^2 + \cdots + z_d^2)^{-1/2}$ the boundary defining function, Mellin transforming the rescaled wave operator $P = \rho^{-(d-1)/2}\rho^{-2}\Box_g \rho^{(d-1)/2}$ yields a (σ-dependent) differential operator $\hat{P}_{\partial M}$ on the boundary ∂M

$$\hat{P}_{\partial M}(\sigma) := \mathcal{M}_\rho \rho^{-(d-1)/2}\rho^{-2}\Box_g \rho^{(d-1)/2}\mathcal{M}_\rho^{-1} \in \mathrm{Diff}^2(\partial M).$$

Now the crucial observation is that the region in the boundary $\mathbb{S}^d$ corresponding to $z_1^2 + \cdots + z_d^2 > z_0^2$ in the interior can be identified with the de Sitter hyperboloid $z_0^2 - (z_1^2 + \cdots + z_d^2) = -1$. The latter is a manifold that we denote X_0 and which is equipped with the de Sitter metric g_{X_0}, related to the Minkowski metric by

$$g_{\mathbb{R}^{1,d}} = -dr_{X_0}^2 + r_{X_0}^2 g_{X_0} = \frac{1}{\rho^2}\bigg(- x_{X_0}^2 \left(-\frac{d\rho}{\rho} + \frac{dx_{X_0}}{x_{X_0}} \right)^2 + x_{X_0}^2 g_{X_0} \bigg),$$

where $r_{X_0} = (z_1^2 + \cdots + z_d^2 - z_0^2)^{1/2}$ is the space-like Lorentzian distance function and

$$x_{X_0} = \left(\frac{z_1^2 + \cdots + z_d^2 - z_0^2}{z_1^2 + \cdots + z_d^2 + z_0^2} \right)^{\frac{1}{2}} = r_{X_0}\rho.$$

Here we consider the de Sitter space X_0 as a manifold with boundary $S = S_+ \cup S_-$ (this is the so-called *conformal boundary* of X_0) and boundary-defining function x_{X_0}.

Remarkably, as shown in [56], $\hat{P}_{\partial M}(\sigma)$ is related to the (Laplace-Beltrami) wave operator on X_0 by[15]

$$\hat{P}_{\partial M}(\sigma)\restriction_{X_0} = x_{X_0}^{-\mathrm{i}\sigma-(d-1)/2-2}(\Box_{X_0} - \sigma^2 - (d-1)^2/4)x_{X_0}^{\mathrm{i}\sigma+(d-1)/2}.$$

In turn, the two connected regions on the boundary $\mathbb{S}^{d-1}$ that correspond to $|z_0| > z_1^2 + \cdots + z_d^2$ and respectively $\pm z_0 > 0$ in the interior of M can be identified with the two hyperboloids

$$z_0^2 - (z_1^2 + \cdots + z_d^2) = 1, \quad \pm z_0 > 0.$$

These hyperboloids are in fact two copies of hyperbolic space, here in the compactified setting we consider them as two manifolds $X_\pm$with boundary $\partial X_\pm = S_\pm$, with metric

[15]Note that this differs from the formulas in [56] by a sign in front of σ, because there the Mellin transform is taken with respect to ρ^{-1} instead of ρ.

$g_{X_\pm}$ satisfying

$$g_{\mathbb{R}^{1,d}} = dr_{X_\pm}^2 - r_{X_\pm}^2 g_{X_\pm} = -\frac{1}{\rho^2}\bigg(- x_{X_\pm}^2 \left(-\frac{d\rho}{\rho} + \frac{dx_{X_\pm}}{x_{X_\pm}}\right)^2 + x_{X_\pm}^2 g_{X_\pm}\bigg),$$

with $r_{X_+} = r_{X_-} = (z_0^2 - z_1^2 + \dots + z_d^2)^{1/2}$ the time-like Lorentzian distance function and $x_{X_\pm} = r_{X_\pm}\rho$; note that the pull-back of the Minkowski metric to the hyperboloid is the negative of the Riemannian metric. Similarly as in the case of X_0, one has an identity relating $\hat{P}_{\partial M}$ to the Laplace-Beltrami operator on $X_\pm$:

$$\hat{P}_{\partial M}(\sigma)\restriction_{X_\pm} = x_{X_\pm}^{-\mathrm{i}\sigma-(d-1)/2-2}(\Delta_{X_\pm} - \sigma^2 - (d-1)^2/4)x_{X_\pm}^{\mathrm{i}\sigma+(d-1)/2}.$$

We now consider the more general setup of asymptotically hyperbolic and asymptotically de Sitter spacetimes (note that the latter have to be thought as a generalization of 'global' de Sitter space, as opposed for instance to the static or cosmological de Sitter patch), following [50, 56].

Definition 6.1. *Let $X_\bullet$ be a compact d-dimensional manifold with boundary, equipped with a metric g on $X_\bullet^\circ$, and let x be a boundary defining function. One says that $(X_\bullet, g)$ is:*

- asymptotically hyperbolic *if $g = x^{-2}\hat{g}$, where $\hat{g}$ is a smooth Riemannian metric on $X_\bullet$ with $\hat{g}(dx, dx)\restriction_{x=0} = 1$;*
- asymptotically de Sitter *if $g = x^{-2}\hat{g}$, where $\hat{g}$ is a smooth Lorentzian metric on $X_\bullet$ of signature $(1, d-1)$, with $\hat{g}(dx, dx)\restriction_{x=0} = 1$, and the boundary is the union $\partial X_\bullet = S_+ \cup S_-$ of two connected components, with all null geodesics in $X_\bullet^\circ$ parametrized by $t \in \mathbb{R}$ tending either to S_+ as $t \to \infty$ and to S_- as $t \to -\infty$, or vice versa.*

An argument from [30] (discussed therein for a class of asymptotically Minkowski spacetimes) can be used to show that (X_0, g_{X_0}) is asymptotically de Sitter then (X_0°, g_{X_0}) is globally hyperbolic. Moreover, it is well-known that X_0 diffeomorphic to $[-1,1] \times S_+$ (and to $[-1,1] \times S_-$).

Furthermore, one says that an asymptotically de Sitter space (X_0, g_{X_0}) is *even* if it admits a product decomposition $[0,\epsilon)_x \times (\partial X_0)_y$ near ∂X_0 such that

$$g_{X_0} = \frac{dx_{X_0}^2 - h(x_{X_0}^2, y, dy)}{x_{X_0}^2} \tag{6.1}$$

with $h(x_{X_0}^2, y, dy)$ smooth. In a similar way (but with different sign in front of h) one defines even asymptotically hyperbolic spaces [50, 56], cf. also the work of Guillarmou [27] for the original definition. It can be shown that the product decomposition (6.1) is a general feature of asymptotically de Sitter spacetimes (this is analogous to the Riemannian case treated in [26]), so the essential property in the definition of even spaces is smoothness of $h(x_{X_0}^2, y, dy)$. For us what matters the most is that this amounts to requiring that h is smooth with respect to a $\mathcal{C}^\infty$ structure on X, modified with respect to the original one in such way that $v = -x_{X_0}^2$ is a valid boundary-defining function (we call it the *even $\mathcal{C}^\infty$ structure* on X_0).

Now, given an even asymptotically de Sitter space (X_0, g_{X_0}), as explained in [56], one can construct two even asymptotically hyperbolic spaces $(X_\pm, g_{X_\pm})$ with boundary defining functions $x_{X_\pm}$, and a compact manifold X (without boundary) of the form

$$X = X_+ \cup X_0 \cup X_-,$$

where $\partial X_\pm$ is smoothly identified with the component $S_\pm$ of the boundary $\partial X_0 = S$ of X_0. Strictly speaking, in general one needs to replace X_0 with two copies of it for topological reasons (although this is not necessary in the case of exact de Sitter space). Next, equipping X with the even $\mathcal{C}^\infty$ structure on the respective components allows one to construct an asymptotically Minkowski space (M, g) with $M = \mathbb{R}^+_\rho \times X$ (so that $\partial M = X$) and g a smooth metric of the form

$$g = \frac{1}{\rho^2}\left(v\frac{d\rho^2}{\rho^2} - \frac{1}{2}\left(\frac{d\rho}{\rho}\otimes dv + dv \otimes \frac{d\rho}{\rho}\right) - h(-v, y, dy)\right)$$

with $v = -x_{X_0}^2$ on X_0 and $v = x_{X_\pm}^2$ on $X_\pm$. The Mellin transformed (rescaled) wave operator on M defines a family of differential operators $\hat{P}_X(\sigma) \in \mathrm{Diff}^2(X)$ which is related to the Laplace-Beltrami (wave) operator on $X_\pm$ and X_0 by

$$\hat{P}_X(\sigma)\restriction_{X_\pm^\circ} = x_{X_0}^{\mathrm{i}\tilde{\sigma}-2}\hat{P}_{X_0}(\sigma)x_{X_0}^{-\mathrm{i}\tilde{\sigma}}, \quad \hat{P}_X(\sigma)\restriction_{X_\pm^\circ} = x_{X_\pm}^{\mathrm{i}\tilde{\sigma}-2}\hat{P}_{X_\pm}(\sigma)x_{X_\pm}^{-\mathrm{i}\tilde{\sigma}}, \tag{6.2}$$

where we have set $\tilde{\sigma} = -\sigma + \mathrm{i}(d-1)/2$ and

$$\hat{P}_{X_0}(\sigma) = \Box_{X_0} - \sigma^2 - (d-1)^2/4, \quad \hat{P}_{X_\pm}(\sigma) = -\Delta_{X_\pm} + \sigma^2 + (d-1)^2/4. \tag{6.3}$$

Denoting $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_{X_0}$, $\langle\cdot,\cdot\rangle_{X_\pm}$ the pairings induced from the respective metrics, we have that $\hat{P}_X(\overline{\sigma})$ is the formal adjoint of $\hat{P}_X(\sigma)$ with respect to $\langle\cdot,\cdot\rangle_X$, similarly $\hat{P}_{X_\bullet}(\overline{\sigma})$ is the formal adjoint of $\hat{P}_{X_\bullet}(\sigma)$ with respect to $\langle\cdot,\cdot\rangle_{X_\bullet}$, note also the relation

$$\langle\cdot,\cdot\rangle_X = \langle\cdot, x_{X_\bullet}^2\cdot\rangle_{X_\bullet} \text{ on } X_\bullet.$$

Turning our attention to inverses, by global hyperbolicity of (X_0, g_0), it is well known that $\hat{P}_{X_0}(\sigma)$ has retarded and advanced propagators[16] $\hat{P}_{X_0,\pm}(\sigma)^{-1}$ for any value of σ. The two operators $\hat{P}_{X_\pm}(\sigma)$ possess inverses $\hat{P}_{X_\pm}(\sigma)^{-1}$ for sufficiently large values of $|\operatorname{Im}\sigma|$ in the sense of the resolvent of the positive operator $-\Delta_{X_\pm}$ (on the closure of its natural domain in L^2), and moreover it was shown in [27, 40, 50] that $\hat{P}_{X_\pm}(\sigma)^{-1}$ continues from say $\operatorname{Im}\sigma \gg 0$ to $\mathbb{C}$ a meromorphic family of operators (cf. also [60] for a recent, more concise account).

On the other hand, $\hat{P}_X(\sigma)$ fits into the framework of [53], which allows to set up a Fredholm problem in the spaces

$$\mathcal{X}^s = \{u \in H^s(X) : \ \hat{P}_X(\sigma)u \in \mathcal{Y}^{s-1}\}, \quad \mathcal{Y}^{s-1} = H^{s-1}(X),$$

with the conclusion that $\hat{P}_X(\sigma) : \mathcal{X}^s \to \mathcal{Y}^{s-1}$ possess in particular two inverses $\hat{P}_{X,\pm}(\sigma)^{-1}$ in the sense of meromorphic families of operators, where the sign $+$ corresponds to requiring above-threshold regularity $s > \frac{1}{2} - \operatorname{Im}\sigma$ near N^*S_+ and below-threshold regularity $s < \frac{1}{2} - \operatorname{Im}\sigma$ near N^*S_-, while the sign $-$ corresponds to the same conditions with N^*S_+ and N^*S_- interchanged. In a similar vein one can define Feynman and anti-Feynman inverses (as pointed out in [52]), we have thus four inverses $\hat{P}_{X,I}(\sigma)^{-1}$. Focusing our attention on retarded and advanced ones, it is proved in [56]

[16] This means here that $\hat{P}_{X_0,\pm}(\sigma)^{-1}$ are the inverses of $\hat{P}_{X_0,\pm}(\sigma)$ that solve respectively the advanced, retarded inhomogeneous problem.

that just as the identities (6.2) suggest, with additional subtleties in the sign of σ (corresponding to whether the inverse is defined by analytic continuation from Im $\sigma \gg 0$ or from Im $\sigma \ll 0$), it holds that

$$\begin{aligned} \hat{P}_{X,\pm}(\sigma)^{-1}\restriction_{X_0^\circ\to X_0^\circ} &= x_{X_0}^{\mathrm{i}\tilde{\sigma}}\hat{P}_{X_0,\pm}(\sigma)^{-1}x_{X_0}^{-\mathrm{i}\tilde{\sigma}+2}, \\ \hat{P}_{X,+}(\sigma)^{-1}\restriction_{X_\pm^\circ\to X_\pm^\circ} &= x_{X_\pm}^{\mathrm{i}\tilde{\sigma}}\hat{P}_{X_\pm}(\sigma)^{-1}x_{X_\pm}^{-\mathrm{i}\tilde{\sigma}+2}, \\ \hat{P}_{X,-}(\sigma)^{-1}\restriction_{X_\pm^\circ\to X_\pm^\circ} &= x_{X_\pm}^{\mathrm{i}\tilde{\sigma}}\hat{P}_{X_\pm}(-\sigma)^{-1}x_{X_\pm}^{-\mathrm{i}\tilde{\sigma}+2}, \end{aligned} \tag{6.4}$$

away from poles of $\hat{P}_{X,\pm}(\sigma)^{-1}$ and $\hat{P}_{X_\pm}(\sigma)^{-1}$. Here the subscript $\restriction_{X_\bullet^\circ\to X_\bullet^\circ}$ means that we act with $\hat{P}_{X,\pm}(\sigma)^{-1}$ on $\mathcal{C}^\infty(X_\bullet)$ and restrict the result to the interior of $X_\bullet$, so (6.4) contains no direct information on how $\hat{P}_{X,\pm}(\sigma)^{-1}$ acts between different components of X.

To derive a more precise relation, [56] makes use of asymptotic data of solutions at the common boundaries of X_0 and $X_\pm$. Here we will discuss the corresponding symplectic spaces in a similar way as in Subsect. 5.2, starting first with the analogues of the space of solutions smooth away from the radial set (we focus here mainly on the spaces defined using the retarded and advanced propagator).

6.2. **Symplectic spaces of solutions.** Assuming $\sigma \in \mathbb{R}$, the symplectic spaces associated to $\hat{P}_X(\sigma)$ and the various isomorphisms between them can in fact be introduced in a very similar fashion as in the asymptotically Minkowski case. We denote $\mathrm{Sol}(\hat{P}_X(\sigma))$ the space of solutions of $\hat{P}_X(\sigma)u = 0$ with $\mathrm{WF}(u) \subset N^*S$ and set

$$\hat{G}_X(\sigma) := \hat{P}_{X,+}(\sigma)^{-1} - \hat{P}_{X,-}(\sigma)^{-1}. \tag{6.5}$$

From now on the dependence on σ will often be skipped in the notation, we stress however that we always make the implicit assumption that σ is not a pole of the two operators $\hat{P}_{X,+}(\sigma)^{-1}$, $\hat{P}_{X,-}(\sigma)^{-1}$. Using essentially the same arguments as before (this is even in many ways simpler due to $\hat{P}_{X,\pm}^{-1}$ being exact inverses of $\hat{P}_X$) we get a bijection

$$\frac{\mathcal{C}^\infty(X)}{\hat{P}_X\mathcal{C}^\infty(X)} \xrightarrow{[\hat{G}_X]} \mathrm{Sol}(\hat{P}_X). \tag{6.6}$$

Furthermore, the sesquilinear form $\langle\cdot, \hat{G}_X\cdot\rangle$ induces a well-defined symplectic form on $\mathcal{C}^\infty(X)/\hat{P}_X\mathcal{C}^\infty(X)$, and since $(\hat{P}_{X,+}^{-1})^* = \hat{P}_{X,-}^{-1}$ by [52], $\hat{G}_X$ is anti-hermitian. Although the method of proof of (6.6) is fully analogous to the case of asymptotically Minkowski spacetimes, we stress that the physical outcome is much more unusual, as it allows to build a non-interacting quantum field theory governed by a differential operator that is not everywhere hyperbolic. Note also that one can obtain an analogue of (6.5) in the 'Feynman minus anti-Feynman' case.

In turn, the discussion of symplectic spaces on X_0 is rather standard due to global hyperbolicity of the interior. Let $\mathrm{Sol}(\hat{P}_{X_0})$ be the space of solutions of $\hat{P}_{X_0}u = 0$ that are smooth in the interior X_0°. Setting $\hat{G}_{X_0} := \hat{P}_{X_0,+}^{-1} - \hat{P}_{X_0,-}^{-1}$, one gets isomorphisms

$$\frac{\mathcal{C}_{\mathrm{c}}^\infty(X_0^\circ)}{\hat{P}_{X_0}\mathcal{C}_{\mathrm{c}}^\infty(X_0^\circ)} \xrightarrow{[\hat{G}_{X_0}]} \mathrm{Sol}(\hat{P}_{X_0}), \tag{6.7}$$

either by using well-known results (cf. for instance [3]) or by repeating the proof of the asymptotically Minkowski case.

The next proposition shows that the symplectic spaces (6.6) and (6.7) are in fact isomorphic, so the content of a QFT on X is induced by a QFT in the asymptotically de Sitter region.

Proposition 6.2. *We have isomorphisms*

$$\frac{\mathcal{C}^\infty(X)}{\hat{P}_X\mathcal{C}^\infty(X)} \xrightarrow{[\hat{G}_X]} \hat{G}_X\mathcal{C}^\infty_{\rm c}(X_0^\circ) \xrightarrow{\restriction_{X_0}} (\hat{G}_X\mathcal{C}^\infty_{\rm c}(X_0^\circ))\restriction_{X_0} \xrightarrow{x_{X_0}^{-{\rm i}\tilde\sigma}} {\rm Sol}(\hat{P}_{X_0}). \tag{6.8}$$

Proof. Bijectivity of the first arrow is proved in analogy to Propositions 4.2 and 4.7 (the time-slice property).

To prove that the second arrow is bijective, we use the expression for $\hat{G}_X$ resulting from (6.4). Specifically, if $f\in\mathcal{C}^\infty_{\rm c}(X_0^\circ)$ then

$$(\hat{G}_X f)\restriction_{X_0} = x_{X_0}^{{\rm i}\tilde\sigma}\hat{G}_{X_0}x_{X_0}^{-{\rm i}\tilde\sigma+2}f. \tag{6.9}$$

By the isomorphism (6.7) this entails that $(\hat{G}_X f)\restriction_{X_0}$ determines f modulo $\hat{P}_X\mathcal{C}^\infty_{\rm c}(X_0^\circ)$, and therefore determines $\hat{G}_X f$ uniquely.

Bijectivity of the third arrow follows immediately from $\hat{G}_{X_0}\mathcal{C}^\infty_{\rm c}(X_0^\circ)={\rm Sol}(\hat{P}_{X_0})$ (this is surjectivity of the first arrow in (6.7)) and (6.9). □

6.3. **Hadamard states.** We now discuss how the relation between symplectic spaces on X_0 and X translates to the level of two-point functions. We denote $\hat\Sigma$ the characteristic set of $\hat{P}_X$ and $\hat\Sigma^\pm$ its two connected components.

In the region X_0 it is quite clear what a Hadamard two-point function is, we can adopt Definition 4.5 quite directly indeed and say that $\Lambda^\pm_{X_0}:\mathcal{C}^\infty_{\rm c}(X_0^\circ)\to\mathcal{C}^\infty(X_0^\circ)$ are Hadamard two-point function for $\hat{P}_{X_0}$ if

$$\hat{P}_{X_0}\Lambda^\pm_{X_0}=\Lambda^\pm_{X_0}\hat{P}_{X_0}=0,\quad \Lambda^+_{X_0}-\Lambda^-_{X_0}={\rm i}\hat{G}_{X_0},\quad \Lambda^\pm_{X_0}\geq 0 \tag{6.10}$$

and ${\rm WF}'(\Lambda^\pm_{X_0})=\cup_{t\in\mathbb{R}}\hat\Phi_t({\rm diag}_{T^*X_0^\circ})\cap\pi^{-1}\hat\Sigma^\pm$, where $\hat\Phi_t$ is the bicharacteristic flow of $\hat{P}_{X_0}$ and $\pi:\hat\Sigma\times\hat\Sigma\to\hat\Sigma$ projects to the left component. This ensures that $\Lambda^\pm_{X_0}$ induce well-defined hermitian forms on $\mathcal{C}^\infty_{\rm c}(X_0^\circ)/\hat{P}_{X_0}\mathcal{C}^\infty_{\rm c}(X_0^\circ)$, and agrees with the standard definition of Hadamard two-point functions on globally hyperbolic spacetimes [46].

Essentially the same definition can be used on X, with the same form of the wave front set since the regions $X_\pm$ are irrelevant for the bicharacteristic flow.

Definition 6.3. *We say that* $\Lambda^\pm_X:\mathcal{C}^\infty(X)\to\mathcal{C}^{-\infty}(X)$ *are Hadamard two-point functions for* $\hat{P}_X(\sigma)$ *if* $\hat{P}_X\Lambda^\pm_X=\Lambda^\pm_X\hat{P}_X=0$, $\Lambda^+_X-\Lambda^-_X={\rm i}\hat{G}_X$, $\Lambda^\pm_X\geq 0$, *and*

$${\rm WF}'(\Lambda^\pm_X)=\cup_{t\in\mathbb{R}}\hat\Phi_t({\rm diag}_{T^*X^\circ})\cap\pi^{-1}\hat\Sigma^\pm, \tag{6.11}$$

where $\hat\Phi_t$ *is the bicharacteristic flow of* $\hat{P}_X$ *and* $\pi:\hat\Sigma\times\hat\Sigma\to\hat\Sigma$ *is the projection to the left component.*

As a consequence of Proposition 6.2, Hadamard states on X_0 extend to Hadamard states on X in the following sense:

Theorem 6.4. *Let (X_0, g_{X_0}) be an even asymptotically de Sitter space and let $\Lambda^{\pm}_{X_0}$ be Hadamard two-point functions for $\hat{P}_{X_0}(\sigma)$. If σ is not a pole of $\hat{P}_{X_+}(\sigma)^{-1}$ nor of $\hat{P}_{X_-}(\sigma)^{-1}$ then $\Lambda_{X_0\pm}$ induce canonically two-point functions $\Lambda^{\pm}_X$ of a Hadamard state for $\hat{P}_X(\sigma)$ via the isomorphisms (6.7) and (6.8).*

Proof. It is easy to see that the isomorphisms (6.7) and (6.8) induce a pair of operators $\Lambda^{\pm}_X : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ with the properties $\hat{P}_X \Lambda^{\pm}_X = \Lambda^{\pm}_X \hat{P}_X = 0$, $\Lambda_{X,+} - \Lambda_{X,-} = \mathrm{i}\hat{G}_X$ and $\Lambda^{\pm}_X \geq 0$. Furthermore, $\Lambda^{\pm}_X \restriction_{X_0^\circ \to X_0^\circ} = x^{\mathrm{i}\tilde{\sigma}}_{X_0} \Lambda^{\pm}_{X_0} x^{-\mathrm{i}\tilde{\sigma}+2}_{X_0}$ so by propagation of singularities the wave front set condition (6.11) is satisfied modulo possible terms in $o \times S^*X$ and $S^*X \times o$. These can however be excluded using positivity of $\Lambda^{\pm}_X$ and the Cauchy-Schwarz argument from the proof of Proposition 3.1. □

6.4. **Asymptotic data.** We now discuss symplectic spaces of asymptotic data using the results from [50, 56, 57, 6].

To start with, assuming $\mathrm{i}\sigma \notin \mathbb{Z}$, any $u \in \mathrm{Sol}(\hat{P}_X)$, i.e. any solution of $\hat{P}_X u = 0$ with $\mathrm{WF}(u) \subset N^*S$, is of the form

$$u = (\mu + \mathrm{i}0)^{-\mathrm{i}\sigma} \tilde{a}^+_X + (\mu - \mathrm{i}0)^{-\mathrm{i}\sigma} \tilde{a}^-_X + \tilde{a}_X, \tag{6.12}$$

for some $\tilde{a}^{\pm}_X, \tilde{a}_X \in \mathcal{C}^\infty(X)$. Furthermore, the restriction of $\tilde{a}^+_X$ and $\tilde{a}^-_X$ to either S_+ or S_- defines a pair of smooth functions on X that determine u uniquely [56, Prop. 4.11]. We have thus two maps $\varrho_{X,\pm}$ assigning data one at S_+ and the other one at S_-, defined on $\mathrm{Sol}(\hat{P}_X)$ by

$$\varrho_{X,\pm} u = (\varrho^+_{X,\pm} u, \varrho^-_{X,\pm} u) := (a^+_X \restriction_{S_\pm}, a^-_X \restriction_{S_\pm}) \in \mathcal{C}^\infty(S_\pm) \oplus \mathcal{C}^\infty(S_\pm).$$

We can construct an approximate Poisson operator $\tilde{\mathcal{P}}_{X,\pm}$ by setting

$$\tilde{\mathcal{P}}_{X,\pm}(a^+, a^-) = (\mu + \mathrm{i}0)^{-\mathrm{i}\sigma} a^+(y) + (\mu - \mathrm{i}0)^{-\mathrm{i}\sigma} a^-(y), \quad a^+, a^- \in \mathcal{C}^\infty(S_\pm) \tag{6.13}$$

Note that this is a very rough approximation, in the sense that $P\tilde{\mathcal{P}}_{X,\pm}(a^+, a^-)$ needs not even be smooth (though more precise approximate solutions can be easily constructed as asymptotic series, cf. [6, Lem. 6.4]), all that matters here is that it has above-threshold regularity. In fact

$$\mathcal{P}_{X,\pm} := \tilde{\mathcal{P}}_{X,\pm} - \hat{P}^{-1}_{X,\mp} P \tilde{\mathcal{P}}_{X,\pm}$$

is the corresponding Poisson operator, i.e. the inverse of $\varrho_{X,\pm} : \mathrm{Sol}(\hat{P}_X) \to \mathcal{C}^\infty(S_\pm) \oplus \mathcal{C}^\infty(S_\pm)$. We can now adapt the arguments of Subsect. 5.2 and using an analogous commutator argument show the identity

$$\mathrm{i}\hat{G}_X = \hat{G}^*_X \varrho^*_{X,\pm} q_X \varrho_{X,\pm} \hat{G}_X, \quad \text{where } q_X = \alpha \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \ \alpha \in \mathbb{R} \setminus \{0\}. \tag{6.14}$$

We now turn our attention to asymptotic data for solutions of $\hat{P}_{X_0}$ and $\hat{P}_{X_\pm}$, assuming $\sigma \in \mathbb{R}$. Recall that $\mathrm{Sol}(\hat{P}_{X_0})$ is the space of solutions of $\hat{P}_{X_0} u = 0$ that are smooth in the interior of X_0. By the results of [56, 57], each solution $u \in \mathrm{Sol}(\hat{P}_{X_0})$ can be written in the form

$$u = \tilde{a}^+_{X_0} x^{-\mathrm{i}\sigma + (d-1)/2}_{X_0} + \tilde{a}^-_{X_0} x^{-\mathrm{i}\sigma + (d-1)/2}_{X_0}, \quad a^{\pm}_{X_0} \in \mathcal{C}^\infty(X_0).$$

In order to have a similar structure on the two asymptotically hyperbolic spaces $X_{\pm}$, we define $\mathrm{Sol}(\hat{P}_{X_{\pm}})$ to be the space of solutions of $\hat{P}_{X_{\pm}}u=0$ that can be written as

$$u=\tilde{a}^{+}_{X_{\pm}}x_{X_{\pm}}^{-\mathrm{i}\sigma+(d-1)/2}+\tilde{a}^{-}_{X_{\pm}}x_{X_{\pm}}^{\mathrm{i}\sigma+(d-1)/2},\quad a^{\pm}_{X_0}\in\mathcal{C}^{\infty}(X_{\pm}).$$

In the case $u\in\mathrm{Sol}(\hat{P}_{X_0})$, u is uniquely determined by its asymptotic data $\varrho_{X_0,+}u$ at S_+, and the same is true for the data at S_-, where

$$\varrho_{X_0,\pm}u=(\varrho^{+}_{X_0,\pm}u,\varrho^{-}_{X_0,\pm}u):=(a^{+}_{X_0}\restriction_{S_{\pm}},a^{-}_{X}\restriction_{S_{\pm}})\in\mathcal{C}^{\infty}(S_{\pm})\oplus\mathcal{C}^{\infty}(S_{\pm}).$$

On the other hand, as follows from the results in [40, 36, 56], in each of the cases $u\in\mathrm{Sol}(\hat{P}_{X_{\pm}})$, there are two maps $\varrho^{+}_{X_{\pm}}$ and $\varrho^{-}_{X_{\pm}}$ defined by

$$\varrho^{+}_{X_0,\pm}u:=a^{+}_{X_{\pm}}\restriction_{\partial X_{\pm}},\quad \varrho^{-}_{X_0,\pm}u:=a^{-}_{X_{\pm}}\restriction_{\partial X_{\pm}}.$$

Here, *any* of the two possible data $\varrho^{+}_{X_{\pm}}u$ or $\varrho^{-}_{X_{\pm}}u$ determines u uniquely. The inverse of $\varrho_{X_0,\pm}$, resp. $\varrho^{+}_{X,\pm}$, $\varrho^{-}_{X,\pm}$ is the Poisson operator denoted $\mathcal{P}_{X_0,\pm}$, resp. $\mathcal{P}^{+}_{X,\pm}$, $\mathcal{P}^{-}_{X,\pm}$, note that changing the sign of σ inverses one type of data with the other, so

$$\varrho^{-}_{X,\pm}(\sigma)=\varrho^{+}_{X,\pm}(-\sigma),\quad \mathcal{P}^{-}_{X,\pm}(\sigma)=\mathcal{P}^{+}_{X,\pm}(-\sigma).$$

More details on the construction of the various Poisson operators and the relation between them can be found in [56] and references therein.

We now have all the necessary ingredients to state the result from [56] that describes how $\hat{P}^{-1}_{X,\pm}$ acts on different components of X.

Theorem 6.5. *The inverse $\hat{P}_{X,-}(\sigma)^{-1}$ exists as a meromorphic family in σ, and its poles in $\mathbb{C}\setminus\mathrm{i}\mathbb{Z}$ are precisely the union of the poles of $\hat{P}_{X_-}(\sigma)^{-1}$ and $\hat{P}_{X_-}(-\sigma)^{-1}$. Furthermore,*

$$\hat{P}_{X,-}(\sigma)^{-1}=\begin{pmatrix} x^{\mathrm{i}\tilde{\sigma}}_{X_+}\hat{P}_{X_+}(\sigma)^{-1}x^{-\mathrm{i}\tilde{\sigma}+2}_{X_+} & 0 & 0\\ x^{\mathrm{i}\tilde{\sigma}}_{X_0}c_{0,+}(\sigma)x^{-\mathrm{i}\tilde{\sigma}+2}_{X_+} & x^{\mathrm{i}\tilde{\sigma}}_{X_0}\hat{P}^{-1}_{X_0,-}(\sigma)x^{-\mathrm{i}\tilde{\sigma}+2}_{X_0} & 0\\ x^{\mathrm{i}\tilde{\sigma}}_{X_-}c_{-,+}(\sigma)x^{-\mathrm{i}\tilde{\sigma}+2}_{X_+} & x^{\mathrm{i}\tilde{\sigma}}_{X_-}c_{-,0}(\sigma)x^{-\mathrm{i}\tilde{\sigma}+2}_{X_0} & x^{\mathrm{i}\tilde{\sigma}}_{X_-}\hat{P}_{X_-}(-\sigma)^{-1}x^{-\mathrm{i}\tilde{\sigma}+2}_{X_-}\end{pmatrix}$$

where

$$\begin{aligned} c_{0,+}(\sigma)&=\mathcal{P}_{X_0,+}(\sigma)\imath^{-}\varrho_{X_+}(-\sigma)\hat{P}_{X_+}(\sigma)^{-1},\\ c_{-,+}(\sigma)&=\mathcal{P}_{X_-}(-\sigma)(\imath^{-})^{*}\varrho_{X_0,-}(\sigma)c_{0,+}(\sigma),\\ c_{-,0}(\sigma)&=\mathcal{P}_{X_-}(-\sigma)(\imath^{-})^{*}\varrho_{X_0,+}(\sigma)\hat{P}_{X_0,-}(\sigma)^{-1},\end{aligned}$$

and $\imath^{\pm}:\mathcal{C}^{\infty}(\partial_{\bullet}X_0)\to\mathcal{C}^{\infty}(\partial_{\bullet}X_0)\oplus\mathcal{C}^{\infty}(\partial_{\bullet}X_0)$ is the left/right embedding. The matrix notation above means that given $f\in\mathcal{C}^{\infty}(X)$ there is a unique distribution u with $\hat{P}_{X,-}(\sigma)^{-1}f=u$ and such that $(u\restriction_{X_+},u\restriction_{X_0},u\restriction_{X_-})$ equals the matrix of $\hat{P}_{X,-}(\sigma)^{-1}$ applied to $(f\restriction_{X_+},f\restriction_{X_0},f\restriction_{X_-})$.

There is an analogous statement for $\hat{P}^{-1}_{X,+}(\sigma)$, namely, it is a meromorphic family whose poles in $\mathbb{C}\setminus\mathrm{i}\mathbb{Z}$ are precisely the union of the poles of $\hat{P}_{X_-}(\sigma)^{-1}$ and $\hat{P}_{X_-}(-\sigma)^{-1}$, and

$$\hat{P}_{X,+}(\sigma)^{-1}=\begin{pmatrix} x^{\mathrm{i}\tilde{\sigma}}_{X_+}\hat{P}_{X_+}(-\sigma)^{-1}x^{-\mathrm{i}\tilde{\sigma}+2}_{X_+} & x^{\mathrm{i}\tilde{\sigma}}_{X_+}c_{+,0}(\sigma)x^{-\mathrm{i}\tilde{\sigma}+2}_{X_0} & x^{\mathrm{i}\tilde{\sigma}}_{X_+}c_{+,-}(\sigma)x^{-\mathrm{i}\tilde{\sigma}+2}_{X_-}\\ 0 & x^{\mathrm{i}\tilde{\sigma}}_{X_0}\hat{P}^{-1}_{X_0,+}(\sigma)x^{-\mathrm{i}\tilde{\sigma}+2}_{X_0} & x^{\mathrm{i}\tilde{\sigma}}_{X_0}c_{0,-}(\sigma)x^{-\mathrm{i}\tilde{\sigma}+2}_{X_-}\\ 0 & 0 & x^{\mathrm{i}\tilde{\sigma}}_{X_-}\hat{P}_{X_-}(\sigma)^{-1}x^{-\mathrm{i}\tilde{\sigma}+2}_{X_-}\end{pmatrix}$$

using the same matrix notation, where

$$\begin{aligned} c_{0,-}(\sigma) &= \mathcal{P}_{X_0,-}(\sigma)\imath^+ \varrho_{X_-}(\sigma)\hat{P}_{X_-}(\sigma)^{-1}, \\ c_{+,-}(\sigma) &= \mathcal{P}_{X_+}(-\sigma)(\imath^+)^* \varrho_{X_0,+}(\sigma)c_{0,-}(\sigma), \\ c_{+,0}(\sigma) &= \mathcal{P}_{X_+}(-\sigma)(\imath^+)^* \varrho_{X_0,-}(\sigma)\hat{P}_{X_0,+}(\sigma)^{-1}. \end{aligned}$$

In particular, $\hat{P}^{-1}_{X,\pm} f$ is supported in $X_\pm$ if f is supported in $X_\pm$, and $\hat{P}^{-1}_{X,\pm} f$ is supported in $X_\pm \cup X_0$ if f is supported in $X_\pm \cup X_0$ (this weaker statement was already proved in [6]).

Recall also that if $\sigma \in \mathbb{R}$ then $\hat{P}^*_{X,+} = \hat{P}_{X,-}$ with respect to $\langle \cdot, \cdot \rangle_X$, so we conclude immediately

$$c^*_{0,-} = c_{-,0}, \quad c^*_{+,-} = c_{-,+}, \quad c^*_{+,0} = c_{-,0},$$

where the adjoints are taken using the respective the scalar products $\langle \cdot, \cdot \rangle_{X_\bullet}$.

6.5. **QFT in the hyperbolic caps $X_\pm$.** Despite the elliptic character of $\hat{P}_{X_\pm}$, various similarities between the structure of the solutions of $\hat{P}_{X_\pm}$ and $\hat{P}_{X_0}$ suggest that $\mathrm{Sol}(\hat{P}_{X_\pm})$ could be characterized as the range of the operator

$$\hat{G}_{X_\pm}(\sigma) := (\hat{P}^{-1}_{X_\pm}(\sigma) - \hat{P}^{-1}_{X_\pm}(-\sigma)) = x_{X_\pm}^{-\mathrm{i}\tilde{\sigma}} (\hat{G}_X(\sigma)^{-1}\restriction_{X^\circ_\pm \to X^\circ_\pm}) x_{X_\pm}^{\mathrm{i}\tilde{\sigma}-2}$$

on a suitable class of functions. We prove that this is indeed the case for $\hat{G}_{X_\pm}$ acting on $\dot{\mathcal{C}}^\infty(X_\pm)$ — the space of smooth functions that vanish with all derivatives at the boundary $\partial X_\pm = S_\pm$.

Proposition 6.6. *We have bijections*

$$\frac{\dot{\mathcal{C}}^\infty(X_\pm)}{\hat{P}_{X_\pm}\dot{\mathcal{C}}^\infty(X_\pm)} \xrightarrow{[\hat{G}_{X_\pm}]} \mathrm{Sol}(\hat{P}_{X_\pm}) \xrightarrow{\varrho^+_{X_\pm}} \mathcal{C}^\infty(\partial X_\pm). \tag{6.15}$$

Moreover, $\langle \cdot, \hat{G}_{X_\pm} \cdot \rangle_{X_\pm}$ induces a well-defined symplectic form on the quotient space $\dot{\mathcal{C}}^\infty(X_\pm)/\hat{P}_{X_\pm}\dot{\mathcal{C}}^\infty(X_\pm)$.

Proof. We consider the case X_+, the other one being analogous, and prove bijectivity of the first arrow.

The inclusion $\hat{G}_{X_+}\dot{\mathcal{C}}^\infty(X_+) \subset \mathrm{Sol}(\hat{P}_{X_+})$ is proved using the identity (6.2) that relates $\hat{P}_{X_+}$ with $\hat{P}_X$, and the asymptotics (6.12) for solutions of $\hat{P}_X$. We now show the reverse inclusion (this then gives surjectivity of the first arrow). Recall that any $u \in \mathrm{Sol}(\hat{P}_{X_+})$ can be written as $v^+ + v^-$, where $v^\pm = \tilde{a}^\pm_{X_+} x_{X_+}^{\mp \mathrm{i}\sigma + (d-1)/2}$ and $\tilde{a}^\pm_{X_+} \in \mathcal{C}^\infty(X_+)$. Observe that $\hat{P}_{X_+} v^+$ and $-\hat{P}_{X_+} v^-$ are equal, but with Taylor expansions of different type at the boundary, so in fact $\hat{P}_{X_+} v^\pm \in \dot{\mathcal{C}}^\infty(X_+)$. We will now use the fact that $\hat{P}^{-1}_{X_+}(\pm\sigma)$ maps $\dot{\mathcal{C}}^\infty(X_\pm)$ to distributions with asymptotic behavior of the same type as $v^\mp$, cf. [40]. This implies that $w^\pm = \hat{P}^{-1}_{X_+}(\mp\sigma)\hat{P}_{X_+} v^\pm - v^\pm$ has asymptotic behavior of the type $v^\pm$, so $w^\pm$ is a solution of $\hat{P}_{X_+} w^\pm = 0$ with data $\varrho^\mp_{X_+} w^\pm = 0$ and therefore vanishes. We conclude

$$u = v^+ + v^- = \hat{P}^{-1}_{X_+}(\sigma)\hat{P}_{X_+} v^- + \hat{P}^{-1}_{X_+}(-\sigma)\hat{P}_{X_+} v^+ = (\hat{P}^{-1}_{X_+}(\sigma) - \hat{P}^{-1}_{X_+}(-\sigma))\hat{P}_{X_+} v^-,$$

that is $u = \hat{G}_{X_+} f$ with $f = \hat{P}_{X_+} v^- \in \dot{\mathcal{C}}^\infty(X_+)$ as claimed.

To prove injectivity of the first arrow, observe that if $f \in \dot{\mathcal{C}}^\infty(X_+)$ is in the kernel of $\hat{G}_{X_+}$ then $\hat{P}^{-1}_{X_+}(\sigma) f$ equals $\hat{P}^{-1}_{X_+}(-\sigma) f$, with asymptotic behavior of the two distinct types at the same time, so in fact $\hat{P}^{-1}_{X_+}(\sigma) f \in \dot{\mathcal{C}}^\infty(X_+)$. This means that $f = \hat{P}_{X_+} g$ with $g = \hat{P}^{-1}_{X_+}(\sigma) f \in \dot{\mathcal{C}}^\infty(X_+)$. □

Proposition 6.6 allows to set up a QFT on X_+ and X_-, in particular it is natural to define Hadamard two-point functions for say, $\hat{P}_{X_+}$, to be operators $\Lambda_{X_+,\pm} : \dot{\mathcal{C}}^\infty(X_+) \to \mathcal{C}^{-\infty}(X_+)$ such that $\hat{P}_{X_+} \Lambda_{X_+,\pm} = \Lambda_{X_+,\pm} \hat{P}_{X_+} = 0$, $\Lambda_{X_+,+} - \Lambda_{X_+,-} = \mathrm{i} \hat{G}_{X_+}$, $\Lambda_{X_+,\pm} \geq 0$ with respect to $\langle \cdot, \cdot \rangle_{X_+}$, and the Schwartz kernel of $\Lambda_{X_+,\pm}$ is smooth in $X_+^\circ \times X_+^\circ$.

Recall that by Theorem 6.4, Hadamard two-point functions for the asymptotically de Sitter Klein-Gordon operator $\hat{P}_{X_0}$ induce two-point functions for the 'extended' operator $\hat{P}_X$. In turn, since there is a monomorphism

$$\mathrm{Sol}(\hat{P}_X) \xrightarrow{x_{X_+}^{-\mathrm{i}\tilde{\sigma}} \circ \restriction_{X_+}} \mathrm{Sol}(\hat{P}_{X_+}),$$

two-point functions $\Lambda_{X,\pm}$ for $\hat{P}_X$ induce two-point functions for $\hat{P}_{X_+}$ by

$$\Lambda_{X_+,\pm} := x_{X_+}^{-\mathrm{i}\tilde{\sigma}} (\Lambda_{X,\pm} \restriction_{X_\pm^\circ \to X_\pm^\circ}) x_{X_\pm}^{\mathrm{i}\tilde{\sigma}-2}. \tag{6.16}$$

The two-point functions $\Lambda_{X_+,\pm}$ obtained this way can be given an explicit formula in terms of asymptotic data $\lambda_+^\pm$ at S_+ of the asymptotically de Sitter two-point function we started with.

Theorem 6.7. *Let $\Lambda^\pm_{X_0}$ be two-point functions for $\hat{P}_{X_0}$ of the form*

$$\Lambda^\pm_{X_0} = \hat{G}^*_{X_0} \varrho^*_{X_0,+} \lambda^\pm_+ \varrho_{X_0,+} \hat{G}_{X_0}. \tag{6.17}$$

Then the two-point functions for $\hat{P}_{X_+}$ induced via Theorem 6.4 and (6.16) are given by

$$\Lambda^\pm_{X_+} = \hat{G}^*_{X_+} (\varrho^-_{X_+})^* (\imath^-)^* \lambda^\pm_+ \imath^- \varrho^-_{X_+} \hat{G}_{X_+}, \tag{6.18}$$

and the induced two-point functions for $\hat{P}_{X_-}$ are equal

$$\Lambda^\pm_{X_-} = \hat{G}^*_{X_-} (\varrho^-_{X_-})^* (\imath^-)^* \mathcal{S}^*_{X_0} \lambda^\pm_+ \mathcal{S}_{X_0} \imath^- \varrho^-_{X_-} \hat{G}_{X_-}, \tag{6.19}$$

where $\mathcal{S}_{X_0}(\sigma) = \varrho_{X_0,+}(\sigma) \circ \mathcal{P}_{X_0,-}(\sigma)$ is the asymptotically de Sitter scattering operator.

Proof. Let $Q \in \mathcal{C}^\infty(X)$ be equal 0 in a neighborhood of X_+ and 1 in a neighborhood of X_-. The two-point functions for $\hat{P}_X$ induced by $\Lambda^\pm_{X_0}$ are given by $\Lambda^\pm_X = A^* \Lambda^\pm_{X_0} A$ for

$$\begin{aligned} A &= -x_{X_0}^{-\mathrm{i}\tilde{\sigma}+2} [\hat{P}_X, Q] \hat{G}_X = -x_{X_0}^{-\mathrm{i}\tilde{\sigma}+2} \hat{P}_X Q x_{X_0}^{\mathrm{i}\tilde{\sigma}} x_{X_0}^{-\mathrm{i}\tilde{\sigma}} (\restriction_{X_0^\circ} \circ \hat{G}_X) \\ &= -[\hat{P}_{X_0}, Q_1] x_{X_0}^{-\mathrm{i}\tilde{\sigma}} (\restriction_{X_0^\circ} \circ \hat{G}_X), \end{aligned}$$

where we have denoted $Q_1 = x_{X_0}^{-\mathrm{i}\tilde{\sigma}} Q x_{X_0}^{-\mathrm{i}\tilde{\sigma}}$. Using (6.17) we get that $\Lambda^\pm_X = B^* \lambda^\pm_+ B$, where

$$\begin{aligned} B &= \varrho_{X_0,+} \hat{G}_{X_0} A = -\varrho_{X_0,+} x_{X_0}^{-\mathrm{i}\tilde{\sigma}} (\restriction_{X_0^\circ} \circ \hat{G}_X) \\ &= -\varrho_{X_0,+} (c_{0,+} x_{X_+}^{-\mathrm{i}\tilde{\sigma}+2}, \hat{G}_{X_0} x_{X_0}^{-\mathrm{i}\tilde{\sigma}+2}, c_{0,+} x_{X_-}^{-\mathrm{i}\tilde{\sigma}+2}), \end{aligned}$$

here in the last identity we used the formula from Theorem 6.5. More specifically, the first component in the above expression equals

$$\begin{aligned}\varrho_{X_0,+}c_{0,+}x_{X_+}^{-\mathrm{i}\tilde\sigma+2} &= \varrho_{X_0,+}\mathcal{P}_{X_0,+}(\sigma)\imath^-\varrho_{X_+}(-\sigma)\hat P_{X_+}(\sigma)^{-1}x_{X_+}^{-\mathrm{i}\tilde\sigma+2}\\ &= \imath^-\varrho_{X_+}(-\sigma)\hat P_{X_+}(\sigma)^{-1}x_{X_+}^{-\mathrm{i}\tilde\sigma+2} = \imath^-\varrho_{X_+}(-\sigma)\hat G_{X_+}^{-1}x_{X_+}^{-\mathrm{i}\tilde\sigma+2}\end{aligned}$$

when applied to $\dot{\mathcal{C}}^\infty(X_+)$. This yields

$$\begin{aligned}\Lambda^\pm_{X_+} &= x_{X_+}^{-\mathrm{i}\tilde\sigma}(\Lambda^\pm_X\restriction_{X_+^\circ\to X_+^\circ})x_{X_+}^{\mathrm{i}\tilde\sigma-2} = x_{X_+}^{-\mathrm{i}\tilde\sigma}(\restriction_{X_+^\circ}\circ B)^*\lambda^\pm_+(\restriction_{X_+^\circ}\circ B)x_{X_+}^{\mathrm{i}\tilde\sigma-2}\\ &= \hat G^*_{X_+}(\varrho^-_{X_+})^*(\imath^-)^*\lambda^\pm_+\imath^-\varrho^-_{X_+}\hat G_{X_+}\end{aligned}$$

The proof of (6.16) is similar. □

As a special case of (6.17) we can take $\lambda^\pm_+$ to be equal $\pi^\pm$ (just as in the case of Minkowski space) and get this way a distinguished pair of Hadamard two-point functions

$$\Lambda^\pm_{X_0} = \hat G^*_{X_0}\varrho^*_{X_0,+}\pi^\pm\varrho_{X_0,+}\hat G_{X_0}, \tag{6.20}$$

the proof of the Hadamard condition being fully analogous to that of Theorem 5.7.

At the present stage it is worth mentioning that beside abstract existence arguments, there is a relatively simple construction named after Bunch and Davies that gives a 'maximally symmetric' Hadamard two-point function on exact de Sitter space [1, 8, 9]. Furthermore, the work of Dappiaggi, Moretti and Pinamonti [13] provides a distinguished Hadamard two-point function for a class of cosmological spacetimes that asymptotically resemble the de Sitter cosmological chart.

Here the main novelty, beside working on 'global' asymptotically de Sitter spacetimes, is the extension across the boundary, with a quite surprising outcome for the two-point functions $\Lambda^\pm_{X_+}$ on $X_\pm$ induced from (6.20). In fact, since $(\imath^-)^*\pi^+\imath^- = 0$ we obtain $\Lambda^+_{X_+} = 0$ and $\Lambda^-_{X_+} = \mathrm{i}\hat G_{X_+}$! This illustrate that the resulting field theory has quite unusual features, yet to be explored.

Appendix A

A.1. **Quasi-free states and their two-point functions.** In this appendix we briefly recall the relation between quantum fields, quantum states and two-point functions in the framework of algebraic QFT. Although this is standard material which can be found in many books and review articles, see e.g. [15, 28, 39], it is worth stressing that there exist several equivalent formalisms — here we follow [22, 23] and use the complex formalism (used to describe charged fields) as opposed to the real one (used for neutral fields). The advantage of the complex formalism is that one works with sesquilinear forms, so the positivity condition for two-point functions has a very neat formulation. On the other hand, the real formalism is particularly useful if one wants to work with C^*-algebras rather than mere *-algebras.

Let $\mathscr{V}$ be a complex vector space $\mathscr{V}$ equipped with an anti-hermitian form G. It is slightly more convenient to have a hermitian form, so we set $q := \mathrm{i}^{-1}G$. The polynomial CCR $*$-algebra $\mathrm{CCR}^{\mathrm{pol}}(\mathscr{V}, q)$ (see e.g. [15, Sect. 8.3.1]) is defined as the algebra generated by the identity $\mathbf{1}$ and all abstract elements of the form $\psi(v)$, $\psi^*(v)$, $v \in \mathscr{V}$, with $v \mapsto \psi(v)$ anti-linear, $v \mapsto \psi^*(v)$ linear, and subject to the canonical commutation relations

$$(A.1) \qquad [\psi(v), \psi(w)] = [\psi^*(v), \psi^*(w)] = 0, \quad [\psi(v), \psi^*(w)] = \overline{v} q w \mathbf{1}, \quad v, w \in \mathscr{V}.$$

A state ω is a linear functional on $\mathrm{CCR}^{\mathrm{pol}}(\mathscr{V}, q)$ such that $\omega(a^* a) \geq 0$ for all a in $\mathrm{CCR}^{\mathrm{pol}}(\mathscr{V}, q)$ and $\omega(\mathbf{1}) = 1$.

The *bosonic two-point functions* (or complex covariances) $\Lambda^{\pm}$ of a state ω on the polynomial CCR $*$-algebra are the two hermitian forms $\Lambda^{\pm}$ defined by

$$(A.2) \qquad \overline{v} \Lambda^{+} w = \omega\big(\psi(v)\psi^*(w)\big), \quad \overline{v} \Lambda^{-} w = \omega\big(\psi^*(w)\psi(v)\big), \quad v, w \in \mathscr{V}$$

Note that both $\Lambda^{\pm}$ are positive and by the canonical commutation relations one has always $\Lambda^{+} - \Lambda^{-} = q$. Crucially, there is reverse construction, namely if one has a pair of hermitian forms $\Lambda^{\pm}$ such that $\Lambda^{+} - \Lambda^{-} = q$ and $\Lambda^{\pm} \geq 0$ then there exists a state ω such that (A.2) holds, and this assignment is one-to-one for the class of *bosonic quasi-free states*, see e.g. [2, 15].

Once a state ω is fixed, the *GNS construction* provides: a Hilbert space $\mathfrak{H}$, unbounded operators $\hat{\psi}(v)$, $v \in \mathscr{V}$, such that $v \mapsto \hat{\psi}(v)$ is anti-linear (on a common dense domain in $\mathfrak{H}$), and a vector $\Omega \in \mathfrak{H}$ in the common domain of $\hat{\psi}(v)$ such that

$$(A.3) \qquad \overline{v} \Lambda^{+} w = \langle \Omega, \hat{\psi}(v) \hat{\psi}^*(w) \Omega \rangle_{\mathfrak{H}}, \quad \overline{v} \Lambda^{-} w = \langle \Omega, \hat{\psi}^*(w) \hat{\psi}(v) \Omega \rangle_{\mathfrak{H}}, \quad v, w \in \mathscr{V},$$

and

$$(A.4) \qquad [\hat{\psi}(v), \hat{\psi}(w)] = [\hat{\psi}^*(v), \hat{\psi}^*(w)] = 0, \quad [\hat{\psi}(v), \hat{\psi}^*(w)] = \overline{v} q w \mathbf{1}, \quad v, w \in \mathscr{V}$$

on a suitable dense domain. In the case when $\mathscr{V}$ is a quotient space of the form $\mathcal{C}^\infty_{\mathrm{c}}(M)/P\mathcal{C}^\infty_{\mathrm{c}}(M)$ for some $P \in \mathrm{Diff}(M)$ (or a similar quotient, such as the space $H^{\infty,0}_{\mathrm{b}}(M)/PH^{\infty,0}_{\mathrm{b}}(M)$ considered in the main part of the text), then, disregarding issues due to unboundedness of $\hat{\psi}(v)$, $\mathcal{C}^\infty_{\mathrm{c}}(M) \ni v \mapsto \hat{\psi}(\overline{v})$ can be interpreted as an operator-valued distribution that solves $P\hat{\psi} = 0$. The distributions $\hat{\psi}$ are the (non-interacting) *quantum fields* and are the main object of interest from the physical point of view. Note that although they are solutions of a differential equation, their analysis differs from usual PDE techniques, as $\hat{\psi}$ take values in operators on a Hilbert space $\mathfrak{H}$ that is not given a priori, but is constructed simultaneously with $\hat{\psi}$.

There is also a *fermionic* version of the above construction, relevant for instance for Dirac fields. Namely, if q is in addition positive then the polynomial CAR $*$-algebra $\mathrm{CAR}^{\mathrm{pol}}(\mathscr{V}, q)$ is defined similarly as the CCR one, except that (A.1) is replaced by the canonical anti-commutation relations

$$(A.5) \qquad \{\psi(v), \psi(w)\} = \{\psi^*(v), \psi^*(w)\} = 0, \quad \{\psi(v), \psi^*(w)\} = \overline{v} q w \mathbf{1}, \quad v, w \in \mathscr{V},$$

where $\{a, b\} = ab + ba$. Fermionic two-point functions of a state ω on $\mathrm{CAR}^{\mathrm{pol}}(\mathscr{V}, q)$ are defined in analogy to (A.2). This always gives $\Lambda^{\pm} \geq 0$ and in view of the canonical anti-commutation relations $\Lambda^{+} + \Lambda^{-} = q$. Again, there is a construction that assigns to two hermitian forms $\Lambda^{\pm}$ a *fermionic quasi-free state* provided that $\Lambda^{\pm} \geq 0$ and $\Lambda^{+} + \Lambda^{-} = q$. The GNS construction provides then field operators $\hat{\psi}(v)$ that satisfy (A.3) and the anti-commutator version of (A.4), with the key technical difference that in the fermionic case $\hat{\psi}(v)$ are bounded operators.

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References

[1] B. Allen, *Vacuum States in de Sitter Space*, Phys. Rev. D 32, 3136 (1985).

[2] H. Araki, M. Shiraishi, *On quasi-free states of canonical commutation relations I*, Publ. RIMS Kyoto Univ. 7 (1971/72), 105-120.

[3] C. Bär, N. Ginoux, F. Pfäffle, *Wave equation on Lorentzian manifolds and quantization*, ESI Lectures in Mathematics and Physics, EMS (2007).

[4] C. Bär, A. Strohmaier, *An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary*, preprint `arXiv:1506.00959`, (2015).

[5] C. Bär, A. Strohmaier, *A rigorous geometric derivation of the chiral anomaly in curved backgrounds*, preprint `arXiv:1508.05345`, (2015).

[6] D. Baskin, A. Vasy, J. Wunsch, *Asymptotics of radiation fields in asymptotically Minkowski space*, to appear in Am. J. Math., `arXiv:1212.5141` (2012).

[7] D. Baskin, F. Wang, *Radiation fields on Schwarzschild spacetime*, Comm. Math. Phys. 331:477–506 (2014).

[8] J. Bros, U. Moschella, *Two point functions and quantum fields in de Sitter universe*, Rev. Math. Phys. 8, 327 (1996)

[9] J. Bros, U. Moschella, J. P. Gazeau, *Quantum field theory in the de Sitter universe*, Phys. Rev. Lett. 73, 1746 (1994)

[10] M. Brum, S. E. Jorás, *Hadamard state in Schwarzschild-de Sitter spacetime*, Class. Quantum Grav. 32, no. 1 (2014).

[11] N.V. Dang, *Renormalization of quantum field theory on curved spacetimes, a causal approach*, Ph.D. thesis, Paris Diderot University, (2013).

[12] C. Dappiaggi, N. Drago, *A new deformation argument for Hadamard states via an extended Møller operator*, preprint `arXiv:1506.09122` (2015)

[13] C. Dappiaggi, V. Moretti and N. Pinamonti, *Distinguished quantum states in a class of cosmological spacetimes and their Hadamard property*, J. Math. Phys. 50 (2009) 062304.

[14] C. Dappiaggi, V. Moretti and N. Pinamonti, *Rigorous construction and Hadamard property of the Unruh state in Schwarzschild spacetime*, Adv. Theor. Math. Phys. 15 (2011) 355.

[15] J. Dereziński, C. Gérard, *Mathematics of Quantization and Quantum Fields*, Cambridge Monographs in Mathematical Physics, Cambridge University Press (2013).

[16] J.J. Duistermaat, L. Hörmander, *Fourier integral operators II*, Acta Math. **128** (1972), 183–269.

[17] S. Dyatlov, *Asymptotics of linear waves and resonances with applications to black holes*, Comm. Math. Phys. 335 (2015), 14451485.

[18] C. J. Fewster, R. Verch, *A quantum weak energy inequality for Dirac fields in curved spacetime*, Comm. Math. Phys. 225, (2002), 331.

[19] F.G. Friedlander, *Radiation fields and hyperbolic scattering theory*, Math. Proc. Cambridge Philos. Soc., (1980), 88:483-515.

[20] S.A. Fulling, F.J. Narcowich, R.M. Wald, *Singularity structure of the two-point function in quantum field theory in curved space-time, II*, Annals of Physics, **136** (1981), 243-272.

[21] C. Gérard, M. Wrochna, *Construction of Hadamard states by characteristic Cauchy problem*, preprint `arXiv:1409.6691` (2014).

[22] C. Gérard, M. Wrochna, *Construction of Hadamard states by pseudo-differential calculus*, Comm. Math. Phys. **325** (2) (2014), 713–755.
[23] C. Gérard, M. Wrochna, *Hadamard states for the linearized Yang-Mills equation on curved spacetime*, Comm. Math. Phys. 337 (1) (2015), 253–320.
[24] C. Gérard, M. Wrochna, *Feynman propagators and Hadamard states from scattering data for the massive Klein-Gordon equation on asymptotically Minkowski spacetimes*, work in progress.
[25] J. Gell-Redman, N. Haber, A. Vasy, *The Feynman propagator on perturbations of Minkowski space*, to appear in Comm. Math. Phys., `arXiv:1410.7113` (2014)
[26] C. R. Graham, J. M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math., 87(2), (1991), 186–225.
[27] C. Guillarmou, *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds*, Duke Math. J., 129(1), (2005), 1–37.
[28] R. Haag, *Local quantum physics: Fields, particles, algebras*, Texts and Monographs in Physics, Springer (1992).
[29] N. Haber, A. Vasy, *Propagation of singularities around a Lagrangian submanifold of radial points*, Microlocal Methods in Mathematical Physics and Global Analysis. Springer Basel, (2013), 113-116.
[30] P. Hintz, A. Vasy, *Semilinear wave equations on asymptotically de Sitter, Kerr-de Sitter and Minkowski spacetimes*, to appear in Anal. PDE, `arXiv:1306.4705` (2013).
[31] S. Hollands, R.M. Wald, *Quantum fields in curved spacetime*, in: General Relativity and Gravitation: A Centennial Perspective, Cambridge University Press (2015).
[32] L. Hörmander, *The analysis of linear partial differential operators I-IV*, Classics in Mathematics, Springer (2007).
[33] A. Jaffe, C. Jäkel, R.E. Martinez *Complex classical fields: a framework for reflection positivity*, Comm. Math. Phys. 329 (2014), 1-28.
[34] A. Jaffe, G. Ritter, *Quantum Field Theory on curved backgrounds. I. The Euclidean functional integral*, Comm. Math. Phys. 270 (2007), 545572.
[35] A. Jaffe, G. Ritter, *Reflection positivity and monotonicity*, J. Math. Phys. 49 (2008), 052301, 1-10.
[36] M. Joshi, A. Sá Barreto, *Inverse scattering on asymptotically hyperbolic manifolds*. Acta Math., 184(1):41–86, (2000).
[37] B.S. Kay, *The principle of locality and quantum field theory on (non globally hyperbolic) curved spacetimes*, Rev. Math. Phys. (Special Issue), (1992), 167195.
[38] B.S. Kay, U. Lupo, *Non-existence of isometry-invariant Hadamard states for a Kruskal black hole in a box and for massless fields on 1+1 Minkowski spacetime with a uniformly accelerating mirror*, preprint `arXiv:1502.06582` (2015).
[39] I. Khavkine, V. Moretti, *Algebraic QFT in curved spacetime and quasifree Hadamard states: an introduction*, in: Advances in Algebraic Quantum Field Theory, Springer (2015).
[40] R. Mazzeo, R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal., 75(2):260-310, (1987).
[41] R. Melrose, *The Atiyah-Patodi-singer index theorem*, Vol. 4. Wellesley: AK Peters, (1993).
[42] R. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidean spaces*, Lecture Notes in Pure and Appl. Math., vol. 161, Dekker, New York (1994), 85–130.
[43] R. Melrose, *Geometric scattering theory*, Vol. 1, Cambridge University Press (1995).
[44] V. Moretti, *Uniqueness theorem for BMS-invariant states of scalar QFT on the null boundary of asymptotically flat spacetimes and bulk-boundary observable algebra correspondence*, Comm. Math. Phys. **268** (2006), 727-756.
[45] V. Moretti, *Quantum out-states holographically induced by asymptotic flatness: invariance under space-time symmetries, energy positivity and Hadamard property*, Comm. Math. Phys. **279** (2008), 31-75.
[46] M. Radzikowski, *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, Comm. Math. Phys. **179** (1996), 529–553.
[47] K. Sanders, *Equivalence of the (generalized) Hadamard and microlocal spectrum condition for (generalized) free fields in curved space-time*, Comm. Math. Phys. **295** (2010), 485–501.

[48] K. Sanders, *On the Construction of Hartle-Hawking-Israel States Across a Static Bifurcate Killing Horizon*, Lett. Math. Phys. 105, 4 (2015), 575–640.
[49] H. Sahlmann, R. Verch, *Microlocal spectrum condition and Hadamard form for vector-valued quantum fields in curved spacetime*, Rev. Math. Phys., 13(10) (2001), 1203-1246.
[50] A. Vasy, *Microlocal analysis of asymptotically hyperbolic spaces and high energy resolvent estimates*, MSRI Publications, vol. 60, Cambridge University Press, (2012).
[51] A. Vasy, *Analytic continuation and high energy estimates for the resolvent of the Laplacian on forms on asymptotically hyperbolic spaces*, preprint `arxiv:1206.5454` (2012).
[52] A. Vasy, *On the positivity of propagator differences*, preprint `arxiv:1411.7242` (2014).
[53] A. Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces, (With an appendix by S. Dyatlov)*, Inventiones Math., 194:381513, (2013), 194.2: 381-513.
[54] A. Vasy, *Propagation of singularities for the wave equation on manifolds with corners*, Ann. of Math. (2) 168(3), (2008), 749–812.
[55] A. Vasy, *Propagation phenomena*, Lecture Notes, Stanford University (2014).
[56] A. Vasy, *Resolvents, Poisson operators and scattering matrices on asymptotically hyperbolic and de Sitter spaces*, J. Spect. Theory, vol. 4 (4), (2014), 643–673.
[57] A. Vasy, *The wave equation on asymptotically de Sitter-like spaces*, Adv. Math., 223(1):49–97, (2010).
[58] R.M. Wald, *Dynamics in nonglobally hyperbolic, static spacetimes*, J. Math. Phys., 21, 2802-2805 (1980).
[59] J. Zahn, *Generalized Wentzell boundary conditions and holography*, preprint `arXiv:1512.05512` (2015).
[60] M. Zworski, *Resonances for asymptotically hyperbolic manifolds: Vasy's method revisited*, preprint `arXiv:1511.03352` (2015).

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