PROPAGATION OF SINGULARITIES FOR THE WAVE EQUATION ON MANIFOLDS WITH CORNERS

ANDRÁS VASY

Abstract. In this talk we describe the propagation of $\mathcal{C}^\infty$ and Sobolev singularities for the wave equation on $\mathcal{C}^\infty$ manifolds with corners M equipped with a Riemannian metric g. That is, for $X = M \times \mathbb{R}_t$, $P = D_t^2 - \Delta_M$, and $u \in H^1_{\mathrm{loc}}(X)$ solving $Pu = 0$ with homogeneous Dirichlet or Neumann boundary conditions, we show that $\mathrm{WF}_{\mathrm{b}}(u)$ is a union of maximally extended generalized broken bicharacteristics. This result is a $\mathcal{C}^\infty$ counterpart of Lebeau's results for the propagation of analytic singularities on real analytic manifolds with appropriately stratified boundary, [7]. Our methods rely on b-microlocal positive commutator estimates, thus providing a new proof for the propagation of singularities at hyperbolic points even if M has a smooth boundary (and no corners).

These notes are a summary of [17], where the detailed proofs appear.

1. Introduction

In this talk we describe the propagation of singularities for the wave equation on manifolds with corners. Physically, this relates geometric optics, namely that light moves along geodesics, or straight lines in vacuum, reflecting/refracting at surfaces so that the tangential component of the momentum and the kinetic energy are conserved, and the singularities of solutions of the wave equation $D_t^2 u = \Delta u$ (with appropriate boundary conditions), the electromagnetic field being one example of this. Due to its relevance, this problem has a long history, and has been studied extensively by Keller in the 1940s and 1950s in various special settings. Our main results, stated below, have been obtained by Lebeau in the real analytic setting, for the analytic wave front set [7]. Our result is thus a $\mathcal{C}^\infty$ and Sobolev counterpart of Lebeau's theorem.

This problem is closely related to N-body scattering, and indeed the approach is motivated by my previous work in that setting. In spite of the similarities, due to which I have stated in lectures on N-body scattering a number of years ago that it should be possible to prove the result presented here, there are many differences. The main complication in the N-body setting is that there are 'bound states in subsystems', which have no analogue for the wave equation. Their presence complicates the geometry of phase space significantly, in particular the bicharacteristics have to be defined rather differently from the approach taken by Lebeau [7] as there are no 'hyperbolic' or 'glancing' points (there is mixed behavior). On the other hand, for the wave equation in domain with corners, as explained below, there are two phase

Date: April 18, 2005.

1991 *Mathematics Subject Classification.* 58J47, 35L20.

This work is partially supported by NSF grant DMS-0201092, a Clay Research Fellowship and a Fellowship from the Alfred P. Sloan Foundation.

spaces one needs to consider: T^*X and ${}^{\mathrm{b}}T^*X$. This causes numerous technical complications.

First, recall the form of the propagation results if X is a manifold without boundary, e.g. $X = M \times \mathbb{R}_t$ where M is a manifold without boundary, and P is a differential or pseudodifferential operator on X of order m: $P \in \Psi^m(X)$. The standard setting for microlocal analysis is then T^*X. For any coordinate system z_j on X, we write canonical coordinates on T^*X as (z_j, ζ_j). Now, T^*X, as a vector bundle, is equipped with an $\mathbb{R}^+$-action given by dilations in the fibers: $\mathbb{R}^+_s \times T^*X \ni (s, z, \zeta) \mapsto (z, s\zeta)$. It is also a symplectic manifold, equipped with a canonical symplectic form ω, $\omega = \sum d\zeta_j \wedge dz_j$ in local coordinates.

If $P \in \Psi^m(X)$, then its principal symbol $p = \sigma_m(P)$ is a $\mathcal{C}^\infty$ homogeneous degree m function on $T^*X \setminus o$, o denoting the zero section. The symplectic form also associates a vector field to it, namely its Hamilton vector field, defined by $\omega(V, H_p) = Vp$ for all vector fields V on $T^*X \setminus o$. This is homogeneous of degree $m-1$ with respect to the $\mathbb{R}^+$-action. Bicharacteristics are then integral curves of H_p inside the characteristic set $\Sigma = p^{-1}(\{0\})$.

Finally, recall that for distributions $u \in \mathcal{C}^{-\infty}(X)$, their wave front set $\mathrm{WF}(u)$ is a conic (i.e. invariant under the $\mathbb{R}^+$-action) subset of $T^*X \setminus o$. A convenient definition, which generalizes immediately to the corners setting, is that $q \notin \mathrm{WF}(u)$ if there is $A \in \Psi^0(X)$ of compact support such that $\sigma_0(A)(q) \neq 0$ (i.e. A is elliptic at q) and $Au \in \mathcal{C}^\infty(X)$, or equivalently $LAu \in L^2_{\mathrm{loc}}(X)$ for all $L \in \mathrm{Diff}(X)$.

The main facts about the analysis of P, in this form due to Duistermaat and Hörmander [5, 2], are:

(i) Microlocal elliptic regularity. Let $\Sigma(P) = p^{-1}(\{0\})$ be the characteristic set of P. If $u \in \mathcal{C}^{-\infty}(X)$ then $\mathrm{WF}(u) \subset \mathrm{WF}(Pu) \cup \Sigma(P)$. In particular, if $Pu \in \mathcal{C}^\infty(X)$ then $\mathrm{WF}(u) \subset \Sigma(P)$.

(ii) Propagation of singularities. Suppose that p is real, $Pu \in \mathcal{C}^\infty(X)$. Then $\mathrm{WF}(u)$ is a union of maximally extended bicharacteristics in $\Sigma(P)$. That is, if $q \in \mathrm{WF}(u)$ (hence in $\Sigma(P)$) then so is the whole bicharacteristic through q.

Note that (ii) may be vacuous; indeed, if H_p is radial, i.e. tangent to the orbits of the $\mathbb{R}^+$-action, then it does not give any information on $\mathrm{WF}(u)$, as the latter is already known to be conic. Such points are called radial points, and in recent work with Hassell and Melrose [3], they have been extensively analyzed under non-degeneracy assumptions. If P is the wave operator, there are no radial points in $\Sigma = \Sigma(P)$, but such points are very important in scattering theory (where the $\mathbb{R}^+$-action, or its remnants, are in the base variables z) – indeed, this was the subject of a seminar here at École Polytechnique a few years ago [4].

If $P = D_t^2 - \Delta$ is the wave operator on $X = M \times \mathbb{R}$, then the projection of the bicharacteristics to M are geodesics (or, projected to $M \times \mathbb{R}$, time parameterized geodesics), and (ii) is a precise version of the relationship between geometric optics (light rays) and solutions of the wave equation (electromagnetic field).

We can now turn to boundaries and corners. So suppose X is a manifold with corners. Locally this means that X is diffeomorphic to an open subset of $[0,\infty)^k \times \mathbb{R}^{n-k}$; we denote the corresponding coordinates by (x, y). Note that this is a restriction as compared to what one might want when considering domains in Euclidean space (or manifolds without boundary), for it implies that all corners are 'interior', i.e. have angles $\leq \pi$, so the domain is 'almost convex' in the sense of

[14][1]. Also, note that planar domains with angles $> \pi$ at a corner contain open line segments through the corner, not tangent to either side; but there can be no such curve segments in a manifold with corners under our definition.) Nonetheless, it is fully expected that the propagation theorems are valid in this more general setting. The reason for the more restrictive definition is that it is in this setting that totally characteristic, or b-, pseudodifferential operators have been defined (and indeed are well-behaved). The best way to approach the more general setting is to generalize it further by blowing up the corners, effectively introducing polar coordinates around them, and using b-ps.d.o's on this new space. This aspect is already present in an ongoing project with Melrose and Wunsch [10].

Roughly, in this corner setting, the results have the same form as in the boundaryless case, but the definitions of wave front set and the bicharacteristics change significantly. In particular, the relevant wave front set is $\mathrm{WF}_{\mathrm{b}}(u)$, introduced by Melrose. Both $\mathrm{WF}_{\mathrm{b}}(u)$ and the image of the (generalized broken) bicharacteristics are subsets of the b-cotangent bundle ${}^{\mathrm{b}}T^*X$.

The reason for this is that one cannot microlocalize in T^*X (naively defined ps.d.o's do not act on functions on X in general, and even when they do, they do not preserve boundary conditions – see the discussion in [6, Section 18.2-3]). In fact, this is exactly the origin of the algebra of totally characteristic pseudodifferential operators, denoted by $\Psi_{\mathrm{b}}(X)$, in the $\mathcal{C}^\infty$ boundary setting [12]. The presence of two phase spaces causes technical complications, for we are interested in the wave operator, $P = D_t^2 - \Delta$, whose principal symbol is a $\mathcal{C}^\infty$ function on T^*X, *not* on ${}^{\mathrm{b}}T^*X$ where we microlocalize. Indeed, from a PDE point of view, this discrepancy is what causes the diffractive phenomena. The interaction of these two algebras, $\mathrm{Diff}(X)$ and $\Psi_{\mathrm{b}}(X)$, also explains why we prove even microlocal elliptic regularity via the quadratic form of P (the Dirichlet form), rather than by standard arguments, valid if one studies microlocal elliptic regularity for an element of an algebra (such as $\Psi_{\mathrm{b}}(X)$) with respect to the same algebra. It is worth remarking that while ${}^{\mathrm{b}}T^*X$, or at least the image of T^*X in it under the natural map π described below, as a topological space, has been used to study the propagation of singularities, see the work of Melrose and Sjöstrand in the setting of smooth boundaries [8, 9], and Lebeau's paper [7] for corners, the proof presented here is the first occasion when it is fully used for this purpose as the space carrying symbols of ps.d.o's (in the same sense that T^*X° is used for standard microlocal analysis).

We can now define our new phase space, ${}^{\mathrm{b}}T^*X$. First, we let $\mathcal{V}(X)$ be the Lie algebra of $\mathcal{C}^\infty$ vector fields on X, and $\mathcal{V}_{\mathrm{b}}(X)$ be the Lie algebra of $\mathcal{C}^\infty$ vector fields on X tangent to every boundary face of X. Thus, in local coordinates as above, such vector fields have the form

$$\sum a_j(x,y) x_j \partial_{x_j} + \sum_j b_j(x,y)\partial_{y_j}$$

wit a_j, b_j smooth. Correspondingly, $\mathcal{V}_{\mathrm{b}}(X)$ is the set of all $\mathcal{C}^\infty$ sections of a vector bundle ${}^{\mathrm{b}}TX$ over X: locally $x_j\partial_{x_j}$ and ∂_{y_j} generate $\mathcal{V}_{\mathrm{b}}(X)$ (over $\mathcal{C}^\infty(X)$), and thus (x, y, a, b) are local coordinates on ${}^{\mathrm{b}}TX$. The dual bundle of ${}^{\mathrm{b}}TX$ is ${}^{\mathrm{b}}T^*X$; this is

[1]Roughly, almost convexity means that approximating the domain from inside by smooth domains, the second fundamental form of the boundaries of the approximations is uniformly bounded below; if it were non-negative, the domain would be convex.

the phase space in our setting. Sections of these have the form

$$\sum \sigma_j(x,y)\,\frac{dx_j}{x_j}+\sum_j \zeta_j(x,y)\,dy_j, \tag{1.1}$$

and correspondingly (x,y,σ,ζ) are local coordinates on it. Again, ${}^{\mathrm{b}}T^*X\setminus o$ is equipped with an $\mathbb{R}^+$-action (fiberwise multiplication) which has no fixed points. It is often natural to take the quotient with the $\mathbb{R}^+$-action, and work on the b-cosphere bundle, ${}^{\mathrm{b}}S^*X$.

We still need to relate the two bundles T^*X and ${}^{\mathrm{b}}T^*X$. There is a natural map $\pi: T^*X\to{}^{\mathrm{b}}T^*X$, induced by identifying ${}^{\mathrm{b}}TX$ with TX in the interior of X, where the condition on tangency to boundary faces is vacuous. In view of (1.1), in the canonical local coordinates (x,y,ξ,ζ) on T^*X (so one forms are $\sum\xi_j\,dx_j+\sum\zeta_j\,dy_j$), π takes the form

$$\pi(x,y,\xi,\zeta)=(x,y,x\xi,\zeta),\ \text{with}\ x\xi=(x_1\xi_1,\ldots,x_k\xi_k).$$

Thus, π is a $\mathcal{C}^\infty$ map, but at ∂X, it is not a diffeomorphism. In fact, if $\mathcal{F}_i$ are the closed boundary faces of X, and $\mathcal{F}_{i,\mathrm{reg}}$ is the interior ('regular part') of $\mathcal{F}_i$, then at $\mathcal{F}_{i,\mathrm{reg}}$ the kernel of π is $N^*\mathcal{F}_{i,\mathrm{reg}}$, and the range is $T^*\mathcal{F}_{i,\mathrm{reg}}$, the latter being a well-defined subset of ${}^{\mathrm{b}}T^*X$.

The differential operator algebra generated by $\mathcal{V}_{\mathrm{b}}(X)$ is denoted by $\mathrm{Diff}_{\mathrm{b}}(X)$, and its microlocalization is $\Psi_{\mathrm{b}}(X)$, the algebra of b-, or totally characteristic, pseudodifferential operators. For $A\in\Psi^m_{\mathrm{b}}(X)$, $\sigma_{b,m}(A)$ is a homogeneous degree m function on ${}^{\mathrm{b}}T^*X\setminus o$. Since X is not compact, even if M is, we always understand that $\Psi^m_{\mathrm{b}}(X)$ stands for properly supported ps.d.o's, so its elements define continuous maps $\dot{\mathcal{C}}^\infty(X)\to\dot{\mathcal{C}}^\infty(X)$ as well as $\mathcal{C}^{-\infty}(X)\to\mathcal{C}^{-\infty}(X)$.

We are now ready to define the wave front set $\mathrm{WF}_{\mathrm{b}}(u)$ for $u\in H^1_{\mathrm{loc}}(X)$. The standard wave front set measures whether distributions are microlocally smooth, i.e. $\mathcal{C}^\infty$. On manifolds with corners, $\mathcal{C}^\infty(X)$ is too small a space. Indeed, even solutions of elliptic equations need not be smooth up to the corners. An example is eigenfunctions of the flat Laplacian in a planar sector with Dirichlet boundary conditions, whose asymptotics at the corner given by non-integral powers of the the distance from the corner for most values of the angle at the corner. Thus, we replace $\mathcal{C}^\infty$ by a larger space: our space of 'very nice' (or 'trivial') functions consists of functions that are conormal to all boundary faces. In view of our methods, we consider L^2-based conormal spaces, and the relevant class of conormal functions then consists of $u\in L^2_{\mathrm{loc}}(X)$ such that $Lu\in L^2_{\mathrm{loc}}(X)$ for all $L\in\mathrm{Diff}_{\mathrm{b}}(X)$ (of any order).

It is now straightforward to microlocalize: for $u\in L^2(X)$, $q\in{}^{\mathrm{b}}T^*X\setminus o$, we say that $q\notin\mathrm{WF}_{\mathrm{b}}(u)$ if there is $A\in\Psi^0_{\mathrm{b}}(X)$ such that $\sigma_{b,0}(A)(q)\neq 0$ and $LAu\in L^2(X)$ for all $L\in\mathrm{Diff}_{\mathrm{b}}(X)$. Then $\mathrm{WF}_{\mathrm{b}}(u)$ is a conic subset of ${}^{\mathrm{b}}T^*X\setminus o$; hence it is natural to identify it with a subset of ${}^{\mathrm{b}}S^*X$. Over X°, ${}^{\mathrm{b}}T^*X$ and T^*X are naturally identified via π, and

$$\mathrm{WF}_{\mathrm{b}}(u)\cap{}^{\mathrm{b}}T^*_{X^\circ}X=\pi(\mathrm{WF}(u)\cap T^*_{X^\circ}X).$$

Thus, in the interior of X, $\mathrm{WF}_{\mathrm{b}}(u)$ measures if u is microlocally in $\mathcal{C}^\infty$, and near the boundary, it measures if u is microlocally conormal.

In fact, in the proofs we also need spaces of functions possessing finite regularity, and will need to measure regularity relative to $H^1(X)$, but this makes no difference for the statement of the theorem below (see Lemma 2.5). Nevertheless, we indicate the flavor of this by noting that if Δ is the Laplacian of a smooth metric on X, and $u \in H^1_0(X)$ solves $(\Delta - \lambda)u = 0$ for some real λ, then $Lu \in H^1_{\mathrm{loc}}(X)$ for all $L \in \mathrm{Diff}_{\mathrm{b}}(X)$. In other words, u has one full derivative (corresponding to H^1) and infinitely many b-derivatives (or tangential derivatives, corresponding to L) in $L^2_{\mathrm{loc}}(X)$.

If $P \in \mathrm{Diff}^m(X)$, then $p = \sigma_m(P)$ is a homogenous degree m function on $T^*X \setminus o$, the characteristic set $\Sigma(P) = p^{-1}(\{0\})$ is a subset of $T^*X \setminus o$, while its Hamilton vector field H_p is a vector field on T^*X. Let $\dot\Sigma = \pi(\Sigma(P)) \subset {}^{\mathrm{b}}T^*X$, equipped with the subspace topology. Indeed, if P is non-characteristic for any boundary face $\mathcal{F}$, e.g. is the wave operator, then $\dot\Sigma \subset {}^{\mathrm{b}}T^*X \setminus o$ since at $\mathcal{F}_{\mathrm{reg}}$ the kernel of π is $N^*\mathcal{F}_{\mathrm{reg}}$, and P non-characteristic means that p does not vanish on this (away from o). Generalized broken bicharacteristics are curves inside $\dot\Sigma$, namely they are continuous maps $\gamma : I \to \dot\Sigma$, where I is an interval, satisfying a Hamilton vector field condition, given below in (i), which is the minimal requirement one would want any notion of bicharacteristic to satisfy. Note that we need to use π to relate γ (which is a curve in ${}^{\mathrm{b}}T^*X$) and H_p (which is a vector field on T^*X). More precisely, we make the following definition.

Definition 1.1. A generalized broken bicharacteristic of P is a continuous map $\gamma : I \to \dot\Sigma$, where $I \subset \mathbb{R}$ is an interval, satisfying the following requirements:

(i)
$$\liminf_{s \to s_0} \frac{(f \circ \gamma)(s) - (f \circ \gamma)(s_0)}{s - s_0} \geq \inf\{H_p(\pi^* f)(q) :\ q \in \pi^{-1}(\gamma(s_0)) \cap \Sigma(P)\}$$
for all real-valued $f \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X)$.

(ii) If $q_0 = \gamma(t_0) \in T^*\mathcal{F}_{i,\mathrm{reg}}$, and $\mathcal{F}_i$ is a boundary hypersurface (i.e. has codimension 1), then in a neighborhood of t_0, γ is a generalized broken bicharacteristic in the sense of Melrose-Sjöstrand [8], see also [6, Definition 24.3.7].

Some comments are in order about this definition. First, in the interior of X, where ${}^{\mathrm{b}}T^*X$ can be identified with T^*X via π, the inf is taken over a single point, namely at $\gamma(s_0)$ (we are using the identification), and using (i) for both f and $-f$ gives that $f \circ \gamma$ is differetiable at s_0, with derivative given by $H_p f(\gamma(s_0))$, which is the standard definition of a bicharacteristic.

At ∂X, we need to apply H_p to $\pi^* f$. As π is not one-to-one, there are several points where this could be evaluated, which is the reason for the inf in (i). We thus define the glancing set $\mathcal{G}$ as the set of points in $\dot\Sigma$ whose preimage under $\hat\pi = \pi|_\Sigma$ consists of a single point, and define the hyperbolic set $\mathcal{H}$ as its complement in $\dot\Sigma$. Then for $\gamma(s_0) \in \mathcal{G}$ one concludes, as above, that $f \circ \gamma$ is differentiable, with derivative given by $H_p \pi^* f(q_0)$, q_0 being the unique point in the preimage of $\gamma(s_0)$ under $\hat\pi$. Indeed, this is the route that Lebeau takes in his definition [7], and was also the route taken in [17]. Moreover, for the wave operator, at $\mathcal{H} \cap \mathcal{F}_{i,\mathrm{reg}}$ in local coordinates (x, y) in which $\mathcal{F}_{i,\mathrm{reg}}$ is given by $x = 0$ and the metric is in a model form discussed below around (2.3) with $C|_{x=0} = 0$, taking $f = \sum_j \sigma_j$ (in terms of the coordinates induced by (1.1)) we see that $\pi^* f = \sum_j x_j \xi_j$ in canonical coordinates on T^*X. A simple calculation, see (2.7), shows that this is positive (bounded below by a positive constant) at all points in $\hat\pi^{-1}(\gamma(s_0))$ if $\gamma(s_0) \in \mathcal{H}$, which implies that $\pi^* f$ is nonzero for s near s_0 but $s \neq s_0$, and in particular, in a neighborhood of

s_0, the bicharacteristic is at the corner $\mathcal{F}_{i,\mathrm{reg}}$ only at s_0, giving the other part of the definition used in [7] and [17]. (The other direction of the equivalence of (i) with Lebeau's definition is essentially proved by Lebeau in [7].) One advantage of the present definition that it can be used also in N-body scattering – indeed, it appeared there originally [16].

Moreover, (ii) is a strengthening of (i) at diffractive points on boundary hypersurfaces (near any other kind of point on a boundary hypersurface (i) implies (ii)), see [6, Section 24.3]. It reflects that, unlike analytic singularities, $\mathcal{C}^\infty$ singularities cannot move into the shadow of a convex obstacle. The propagation of analytic singularities, as in Lebeau's case, does not distinguish between gliding and diffractive points, hence (ii) can be dropped to define what we may call analytic generalized broken bicharacteristics. It is an interesting question whether in the $\mathcal{C}^\infty$ setting there are also analogous diffractive phenomena at higher codimension boundary faces, i.e. whether the following theorem can be strengthened at certain points.

If $X = M \times \mathbb{R}$, M a manifold with corners, g a Riemannian metric on M, $P = D_t^2 - \Delta_g$ is the wave operator, then Snell's law is encoded in the statement that γ is continuous. Thus, any (locally defined) smooth functions on ${}^{\mathrm{b}}T^*X$, such as $x, y, t, \sigma, \zeta, \tau$, are continuous along γ. However, $\xi_j = x_j^{-1}\sigma_j$ is *not* continuous, so the normal momentum may jump.

We are now ready to state the main theorem. Recall that $H_0^1(X)$ is the completion of $\mathcal{C}_c^\infty(X^\circ)$ in the norm

$$\|u\|_{H^1(X)}^2 = \|du\|_{L^2(X)}^2 + \|u\|_{L^2(X)}^2,$$

and that elements of $H_0^1(X)$ restrict as 0 to ∂X, i.e. $u \in H_{0,\mathrm{loc}}^1(X)$ means that u satisfies the Dirichlet boundary conditions.

Theorem 1.2. *(cf. the main theorem in* [17]*.) Suppose $Pu = 0$, $u \in H_{0,loc}^1(X)$. Then $\mathrm{WF}_b(u) \subset \dot\Sigma$, and it is a union of maximally extended generalized broken bicharacteristics of P.*

This theorem can be stated in a completely microlocal manner, and one can also measure the regularity modulo Sobolev spaces. In addition, it also holds for Neumann boundary conditions. We refer to [17] for these results.

It was proved in the real analytic setting by Lebeau, and in the $\mathcal{C}^\infty$ setting with $\mathcal{C}^\infty$ boundaries (and no corners) by Melrose, Sjöstrand and Taylor. This result is thus the $\mathcal{C}^\infty$ version of Lebeau's theorem: the geometry is similar in the real analytic vs. $\mathcal{C}^\infty$ settings, but the analysis is quite different.

It is expected that these results will generalize to iterated edge-type structures (under suitable hypotheses), whose simplest example is given by conic points, recently analyzed by Melrose and Wunsch [11], extending the product cone analysis of Cheeger and Taylor [1]. This is subject of an ongoing project with Richard Melrose and Jared Wunsch [10].

It is an interesting question whether this propagation theorem can be improved in the following sense. Suppose that there is a bicharacteristic segment $\gamma_0 : (0, t_0) \to \dot\Sigma$ in X° which emanates from a corner (i.e. $q_0 = \lim_{t\to 0}\gamma_0(t)$ lies over the corner), and suppose that u is a solution of the wave equation. Let Γ denote the set of all generalized broken bicharacteristics extending γ_0 (extending backwards through the corner is the interesting part here). In many cases (indeed, usually), if u is singular along any one of the bicharacteristic segments $\gamma|_{(-\epsilon,0)}$, $\gamma \in \Gamma$, then $\gamma_0 \subset \mathrm{WF}_\mathrm{b}(u)$.

(Theorem 1.2 states if all the $\gamma|_{(-\epsilon,0)}$ as above are disjoint from $\mathrm{WF}_{\mathrm{b}}(u)$, then so is γ_0.) However, it is reasonable to expect that (under certain non-focussing assumptions, which excludes spherical waves collapsing onto the corner) the strength of the singularity along γ_0 will depend on *along which* $\gamma|_{(-\epsilon,0)}$ the solution u was singular. In fact, $\gamma|_{(-\epsilon,0)}$ geometrically related to γ_0, i.e. γ which are the limits of bicharacteristics disjoint from the corner, could be expected to propagate a stronger singularity into γ_0 than the (in this sense) geometrically unrelated segments $\gamma|_{(-\epsilon,0)}$, $\gamma \in \Gamma$. While this is unknown in the present context, the corresponding statement was proved by Cheeger and Taylor [1] for product cones, and by Melrose and Wunsch [11] for asymptotically conic metrics in general.

One can see the role of the conic metrics by blowing up a corner which has dimension 0; the front face is then the cross section of the cone, although it is now a manifold with boundary unlike in [1, 11]. In [11], the metric is put in a model form, $dx^2+x^{-2}h(x,y,dy)$, h a metric on the cross-section Y of the cone (with the tip given by $x=0$), and state the non-focussing assumption as a requirement involving the fiber Laplacian Δ_Y, i.e. in the corner setting this would be the Laplacian on the fibers of the blow-down map. Namely they assume that for some N, the rays $\gamma|_{(-\epsilon,0)}$, $\gamma \in \Gamma$, are disjoint from the wave front set of $(1+\Delta_Y)^{-N}u$, measured relative to a Sobolev space $H^{r'}$. Their conclusion (see [11, Theorem I.3]) is that if $R < r'$, and one merely assumes that the geometrically related rays $\gamma|_{(-\epsilon,0)}$ are disjoint from the wave front set of u, measured relative to a Sobolev space H^R, i.e. from $\mathrm{WF}^R(u)$, then one can still conclude that $\gamma_0 \cap \mathrm{WF}^R(u) = \emptyset$. Note that for conic manifolds, rays hitting the cone tip at $t=0$ are disjoint from it for small non-zero t, which explains why all these statements can be phrased in terms of the usual wave front set.

While the analogous result (including its precise statement) for manifolds with corners is still some time away, significant progress has been made on analyzing edge-type metrics (on manifolds with boundaries) in the project [10]. For manifolds with corners, these correspond to blowing up corners of arbitrary dimension (not necessarily 0), although again the fibers of the blow-down map are manifolds with corners themselves in this setting. For edge-type metrics, one cannot put the metric into a model form as for conic metrics, so the non-focussing condition must be phrased microlocally. In addition, one expects the geometric improvement only for rays not tangential to the edge, i.e. using the manifolds with corners terminology, for rays that would lie in $\mathcal{H}$ over the corner. For such rays, one way to state the non-focussing assumption is by considering the backward flow-out $\mathcal{F}$ of the edge microlocally near q_0. Away from the edge, this is a smooth coisotropic submanifold of T^*X°, and indeed it extends to a smooth submanifold of the edge cotangent bundle, ${}^eT^*X$, which we do not define here. Since non-tangential rays hitting the edge at $t=0$ are disjoint from it for t near 0 but non-zero, we can again phrase our assumptions using the standard wave front set and the standard ps.d.o's. So let $\mathcal{M}$ be the set of first order ps.d.o's with symbol vanishing along $\mathcal{F}$, and let $\mathcal{M}^j$ be the set of finite sums of products of at most j factors, each of which is in $\mathcal{M}$. The non-focussing condition is that microlocally near $\Gamma|_{(-\epsilon,0)} = \{\gamma|_{(-\epsilon,0)} : \gamma \in \Gamma\}$, for some N, $u = \sum A_j v_j$, where $A_j \in \mathcal{M}^N$ and $v_j \in H^{r'}$. Note that in the conic setting, $\Delta_Y \in \mathcal{M}^2$, so this is an analogue of the non-focussing assumption there: $u = (\Delta_Y+1)^N v$, with v microlocally in $H^{r'}$. The conclusion of the theorem for edge metrics will be that under the non-focussing hypothesis, if $R < r'$ and geometrically related bicharacteristics $\gamma_{(-\epsilon,0)}$ are disjoint from $\mathrm{WF}^R(u)$, then so is γ_0. The detailed proof is currently being written up; for details see [10].

We remark that this statement is quite natural: the non-focussing condition, in this form, states that while $u \in H^{r'-N}$ only, it is in a better space, $H^{r'}$, 'to finite order along Γ' (rather than in any neighborhood of Γ), as reflected by the presence of $\mathcal{M}^N$ in the condition. (This 'finite order' corresponds to saying that an operator in $\mathcal{M}$, while first order, is in fact zeroth order to 'first order along $\mathcal{F}$'.) Thus, modulo $H^{r'}$, one can expect singularities to follow limits of integral curves of H_p, i.e. geometrically related broken bicharacteristics. One of the main applications is to (microlocal) Lagrangian distributions, such as the fundamental solution with initial condition a delta distribution near, but not at, the edge, which satisfy the non-focussing condition by virtue of the Lagrangian Λ intersecting the coisotropic manifold $\mathcal{F}$ transversally inside the characteristic set. In fact, inside Λ, the codimension of this intersection is the dimension f of the fibers (i.e. the codimension of the corner before the blow up, minus 1), which implies that u satisfies the non-focussing condition with an improvement of $f/2$: roughly speaking, a Lagrangian distribution u associated to Λ is smooth along Λ, so one can divide u by some first order factors vanishing at $\mathcal{F} \cap \Lambda$ and still improve Sobolev regularity.

2. Ideas of the proof

The basic idea is to use positive commutator estimates to gain b-regularity relative to the Dirichlet form. Since perhaps the most interesting place is the set $\mathcal{H}$ of hyperbolic points, we concentrate on giving at least a rough description of the relevant estimates there. In addition, as it is technically easier, we sketch the elliptic estimates in somewhat more detail. But as the very first step, we make some remarks regarding the relationship between $\Psi_{\mathrm{b}}(X)$ and $\mathrm{Diff}(X)$, which is crucial in the discussion below. A good reference for the basic properties of $\Psi_{\mathrm{b}}(X)$ is [13].

The key point in analyzing smooth vector fields on X, and thereby differential operators such as the wave operator, is that while $D_{x_j} \notin \mathcal{V}_{\mathrm{b}}(X)$, for any $A \in \Psi_{\mathrm{b}}^m(X)$ there is an operator $\tilde{A} \in \Psi_{\mathrm{b}}^m(X)$ such that

$$D_{x_j} A - \tilde{A} D_{x_j} \in \Psi_{\mathrm{b}}^m(X). \tag{2.1}$$

This can be seen by writing

$$D_{x_j} A = x_j^{-1}(x_j D_{x_j})A = x_j^{-1}[x_j D_{x_j}, A] + x_j^{-1} A x_j D_{x_j}.$$

Now, as $A \in \Psi_{\mathrm{b}}^m(X)$, so is $\tilde{A} = x_j^{-1} A x_j$. On the other hand, the commutator $[x_j D_{x_j}, A]$ is that of elements of $\Psi_{\mathrm{b}}(X)$, in fact of an element of $\mathrm{Diff}_{\mathrm{b}}^1(X)$ and $\Psi_{\mathrm{b}}^m(X)$.

In general, such a commutator lies in $\Psi_{\mathrm{b}}^m(X)$, i.e. while there is a gain in the differential order, relative to the products (which are order $m+1$), there is no gain (i.e. vanishing) at the boundary. However, for $B \in \Psi_{\mathrm{b}}(X)$, there is a family $\hat{N}_j(B)(\sigma_j)$ of indicial operators which measure vanishing of B at the boundary hypersurfaces H_j; in particular if the indicial family vanishes identically at H_j, then $B \in x_j \Psi_{\mathrm{b}}(X)$. For each $\sigma_j \in \mathbb{R}$, $\hat{N}_j(B)(\sigma_j)$ is an operator on functions on H_j, and the indicial family should be considered a non-commutative analogue of the principal symbol, characterized by the following. If $f \in \mathcal{C}^\infty(H_j)$ and $u \in \mathcal{C}^\infty(X)$ is any extension of f, i.e. $u|_{H_j} = f$, then

$$\hat{N}_j(B)(\sigma_j)f = (x_j^{-i\sigma_j} B x_j^{i\sigma_j} u)|_{H_j}.$$

Now, as already mentioned above for σ_j replaced by $-i$, $x_j^{-i\sigma_j} B x_j^{i\sigma_j} \in \Psi_{\mathrm{b}}(X)$, and $\Psi_{\mathrm{b}}(X)$ preserves $\mathcal{C}^\infty(X)$, so $x_j^{-i\sigma_j} B x_j^{i\sigma_j} u \in \mathcal{C}^\infty(X)$, and indeed the restriction to H_j is independent of the choice of u. (To be precise, we need to fix the defining function x_j of H_j up to $x_j^2\mathcal{C}^\infty(X)$ for $\hat{N}_j(B)$ to be well-defined.)

Now, the relevance of this discussion to (2.1) is that $\hat{N}_j(x_j D_{x_j})(\sigma_j) = \sigma_j$ (i.e. is a multiplication operator by a constant for each $\sigma_j \in \mathbb{R}$). Thus, for each σ_j, $\hat{N}_j(x_j D_{x_j})(\sigma_j)$ commutes with $\hat{N}_j(A)(\sigma_j)$. Since $\hat{N}_j$ is multiplicative, $\hat{N}_j([x_j D_{x_j}, A]) = [\hat{N}_j(x_j D_{x_j}), \hat{N}_j(A)] = 0$. Correspondingly, $[x_j D_{x_j}, A] \in x_j\, \Psi_{\mathrm{b}}^m(X)$, so $x_j^{-1}[x_j D_{x_j}, A] \in \Psi_{\mathrm{b}}^m(X)$, proving (2.1).

It is worth remarking that $\sigma_{b,m}(A) = \sigma_{b,m}(x_j^{-1} A x_j)$, so at the principal symbol level the fact that $\tilde{A}$ is usually not equal to A is irrelevant. Thus $A - \tilde{A} \in \Psi_{\mathrm{b}}^{m-1}(X)$, and as the tangential vector fields, such as D_{y_j}, cause no trouble (they are already in $\Psi_{\mathrm{b}}(X)$ by virtue of being in $\mathcal{V}_{\mathrm{b}}(X)$), we have the following lemma.

Lemma 2.1. *Suppose $V \in \mathcal{V}(X)$, $A \in \Psi_b^m(X)$. Then $[V, A] = \sum A_j V_j + B$ with $A_j \in \Psi_b^{m-1}(X)$, $V_j \in \mathcal{V}(X)$, $B \in \Psi_b^m(X)$.*

Similarly, $[V, A] = \sum V_j A'_j + B'$ with $A'_j \in \Psi_b^{m-1}(X)$, $V_j \in \mathcal{V}(X)$, $B' \in \Psi_b^m(X)$.

This means in particular that one can define $\mathrm{Diff}^k \Psi_{\mathrm{b}}^s(X)$ is the vector space of operators of the form

$$\sum_j P_j A_j,\ P_j \in \mathrm{Diff}^k(X),\ A_j \in \Psi_{\mathrm{b}}^s(X), \tag{2.2}$$

where the sum is locally finite in X, and show that this is a ring. It is also possible to define a principal symbol as a function on T^*X by pulling back $\sigma_{b,s}(A_j)$ via π, but it does not seem easy to use it (in particular, to give the usual symbol short exact sequence). Thus, in practice it is easier to write elements if $\mathrm{Diff}^k \Psi_{\mathrm{b}}^s(X)$ explicitly in a form (2.2), even making P_j into coordinate vector fields or constants if $k = 1$, and use the symbol calculus on $\Psi_{\mathrm{b}}(X)$ for the A_j. This is the main explanation for why the formal version of the commutator estimates, etc, is relatively straightforward, while the technically correct version requires much more elaboration.

That $\Psi_{\mathrm{b}}(X)$ is suitable for analyzing the actual boundary value problem follows from the following simple lemma.

Lemma 2.2. *Any $A \in \Psi_b^0(X)$ with compact support defines a continuous linear maps $A : H^1(X) \to H^1(X)$, $A : H_0^1(X) \to H_0^1(X)$.*

Proof. We may assume that A is supported in a coordinate chart. First, by (2.1), for $u \in \mathcal{C}^\infty(X)$,

$$D_{x_j} A u = \tilde{A} D_{x_j} u + Bu$$

with $\tilde{A}, B \in \Psi_{\mathrm{b}}^0(X)$. Since $\tilde{A}, B$ are bounded on $L^2(X)$, we deduce that

$$\|D_{x_j} A u\|_{L^2(X)} \leq C\|u\|_{H^1(X)}.$$

Similar estimates hold (even more easily) for vector fields tangent to the boundary, so we deduce that $\|d_X A u\|_{L^2(X)} \leq C\|u\|_{H^1(X)}$. Since A is bounded on $L^2(X)$, we also have $\|Au\|_{L^2(X)} \leq C\|u\|_{H^1(X)}$, so by the density of $\mathcal{C}^\infty(X)$ in $H^1(X)$, we deduce that $\|Au\|_{H^1(X)} \leq C\|u\|_{H^1(X)}$, and hence A extends to a continuous linear map from $H^1(X)$ to itself. As A also preserves $\dot{\mathcal{C}}^\infty(X)$, which is dense in $H^1_0(X)$, it is immediate that A has analogous mapping properties on $H^1_0(X)$. □

We already indicated in the introduction that in the proofs we need to measure b-regularity relative to $H^1(X)$. Thus, we make the following definition.

Definition 2.3. Suppose $u \in H^1_{\mathrm{loc}}(X)$, $m \geq 0$. We say that $q \in {}^{\mathrm{b}}T^*X \setminus o$ is not in $\mathrm{WF}_{\mathrm{b}}^{1,m}(u)$ if there exists $A \in \Psi_{\mathrm{b}}^m(X)$ such that $\sigma_{b,m}(A)(q) \neq 0$ and $Au \in H^1(X)$.

For $m = \infty$, we say that $q \in {}^{\mathrm{b}}T^*X \setminus o$ is not in $\mathrm{WF}_{\mathrm{b}}^{1,m}(u)$ if there exists $A \in \Psi_{\mathrm{b}}^0(X)$ such that $\sigma_{b,0}(A)(q) \neq 0$ and $LAu \in H^1(X)$ for all $L \in \mathrm{Diff}_{\mathrm{b}}(X)$.

With this definition, $\Psi_{\mathrm{b}}(X)$ acts microlocally on $H^1_{\mathrm{loc}}(X)$ so that for $B \in \Psi_{\mathrm{b}}^k(X)$,

$$\mathrm{WF}_{\mathrm{b}}^{1,m-k}(Bu) \subset \mathrm{WF}_{\mathrm{b}}^{1,m}(u) \cap \mathrm{WF}'_{\mathrm{b}}(B).$$

For the actual proofs below, we need a quantitative version of this inclusion, giving estimates on the H^1-norm of $A(Bu)$, with some A as in the definition of $\mathrm{WF}_{\mathrm{b}}^{1,m-k}(Bu)$. Since we do not give full details here, we refer to [17] for further comments.

We can now turn to the specific case of the wave operator on $X = M \times \mathbb{R}_t$. *In a slight change of convention, we reserve y for tangential coordinates on M, so (y,t) are the tangential coordinates on X that have so far been denoted by y.*

First, we describe the form of $p = \sigma_2(P)$, $P = D_t^2 - \Delta$ in some detail. So let $(p, t_0) \in \mathcal{F}_{i,\mathrm{reg}}$ and let (x,y) be local coordinates on M near p so that $\mathcal{F}_{i,\mathrm{reg}}$ is given by $x = 0$. In corresponding canonical coordinates (x,y,ξ,ζ) on T^*M, the metric function on T^*M has the form

$$g(x,y,\xi,\zeta) = \sum_{i,j} A_{ij}(x,y)\xi_i\xi_j + \sum_{i,j} 2C_{ij}(x,y)\xi_i\zeta_j + \sum_{i,j} B_{ij}(x,y)\zeta_i\zeta_j \tag{2.3}$$

with A, B, C smooth. Moreover, the coordinates on M can be chosen (i.e. the y_j can be adjusted) so that $C(0,y) = 0$. Now, if $\mathcal{U} = {}^{\mathrm{b}}T^*_U X$, U a neighborhood of (p, t_0) in $\mathcal{F}_{i,\mathrm{reg}}$ then

$$p|_{x=0} = \tau^2 - \xi \cdot A(y)\xi - \zeta \cdot B(y)\zeta,$$

with A, B positive definite matrices depending smoothly on y, so

$$\mathcal{G} \cap \mathcal{U} = \{(0,y,t,0,\zeta,\tau) : \ \tau^2 = \zeta \cdot B(y)\zeta, \ (\zeta,\tau) \neq 0\},$$
$$\mathcal{H} \cap \mathcal{U} = \{(0,y,t,0,\zeta,\tau) : \ \tau^2 > \zeta \cdot B(y)\zeta, \ (\zeta,\tau) \neq 0\}.$$

It is also convenient to break up the set of elliptic points (i.e. points in ${}^{\mathrm{b}}T^*X$ which do not lie in $\dot\Sigma$) into two parts, namely those which lie in the range of π, but not in that of $\hat\pi = \pi|_\Sigma$, and those which do not lie in the range of π. So we let ${}^{\mathrm{b}}\dot{T}^*X \subset {}^{\mathrm{b}}T^*X$ be the image of T^*X under π, and we let $\mathcal{E}$ be the set of points in ${}^{\mathrm{b}}\dot{T}^*X$ which are disjoint from the image of Σ under π. Thus, in local coordinates,

$$\mathcal{E} \cap \mathcal{U} = \{(0,y,t,0,\zeta,\tau) : \ \tau^2 < \zeta \cdot B(y)\zeta, \ (\zeta,\tau) \neq 0\}.$$

Proposition 2.4. *(Microlocal elliptic regularity; cf.* [17, Proposition 4.6]*.) If $u \in H^1_{0,loc}(X)$, $Pu = 0$ then for all m,*

$$\mathrm{WF}_b^{1,m}(u) \subset {}^b\dot{T}^*X, \ \textit{and} \ \mathrm{WF}_b^{1,m}(u) \cap \mathcal{E} = \emptyset.$$

Proof. (Sketch.) Suppose that either $q \in {}^{\mathrm{b}}T^*X \setminus {}^{\mathrm{b}}\dot{T}^*X$ or $q \in \mathcal{E}$. We may assume iteratively that $q \notin \mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u)$; we need to prove then that $q \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$ (note that the inductive hypothesis holds for $s = 1/2$ since $u \in H^1_{\mathrm{loc}}(X)$). Let $A \in \Psi^s_{\mathrm{b}}(X)$ be such that $\mathrm{WF}'_{\mathrm{b}}(A) \cap \mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u) = \emptyset$, and have $\mathrm{WF}'_{\mathrm{b}}(A)$ in a small conic neighborhood U of q so that for a suitable $C > 0$ or $\epsilon > 0$, in U

(i) $\tau^2 < C\sum_j \sigma_j^2$ if $q \in {}^{\mathrm{b}}T^*X \setminus {}^{\mathrm{b}}\dot{T}^*X$,

(ii) $|\sigma_j| < \epsilon(\tau^2 + |\zeta|^2)^{1/2}$ for all j, and $\frac{|\zeta|}{|\tau|} > 1 + \epsilon$, if $q \in \mathcal{E}$.

We need a regularization argument, and for this we need the space $\Psi_{\mathrm{bc}}(X)$ of b-ps.d.o's which arise by quantization of symbols satisfying symbol estimates, rather than the 'classical' symbols used in defining $\Psi_{\mathrm{b}}(X)$. So let $\Lambda_r \in \Psi_{\mathrm{b}}^{-2}(X)$ for $r > 0$, such that $\mathcal{L} = \{\Lambda_r : \ r \in (0,1]\}$ is a bounded family in $\Psi^0_{\mathrm{bc}}(X)$, and $\Lambda_r \to \mathrm{Id}$ as $r \to 0$ in $\Psi^{\tilde{\epsilon}}_{\mathrm{b}}(X)$, $\tilde{\epsilon} > 0$, e.g. the symbol of Λ_r could be taken as $(1 + r(\tau^2 + |\zeta|^2 + |\sigma|^2))^{-1}$. Let $A_r = \Lambda_r A$. Let a be the symbol of A, and let A_r have symbol $(1 + r(\tau^2 + |\zeta|^2 + |\sigma|^2))^{-1}a$, $r > 0$, so $A_r \in \Psi_{\mathrm{b}}^{s-2}(X)$ for $r > 0$, and A_r is uniformly bounded in $\Psi^s_{\mathrm{bc}}(X)$, $A_r \to A$ in $\Psi^{s+\tilde{\epsilon}}_{\mathrm{bc}}(X)$.

Now, it is not hard to see that

$$\int_X \big(|d_M A_r u|^2 - |D_t A_r u|^2\big)$$

is uniformly bounded for $r \in (0,1]$. Indeed, for $r \in (0,1]$, $A_r u \in H^1_0(X)$, so

$$\int_X (|d_M A_r u|^2 - |D_t A_r u|^2) = -\int_X P A_r u\, \overline{A_r u}.$$

Here the right hand side is the pairing of $H^{-1}(X)$ with $H^1_0(X)$. Writing $PA_r = A_r P + [P, A_r]$, we see that the right hand side can be estimated by

$$|\int_X A_r P u\, \overline{A_r u}| + |\int_X [P, A_r] u\, \overline{A_r u}|, \tag{2.4}$$

and the first term vanishes as $Pu = 0$. The second term can be estimated using (2.1): for elliptic estimates the commutators are 'negligible' (can be easily estimated inductively). This gives the uniform boundedness of $\int_X \big(|d_M A_r u|^2 - |D_t A_r u|^2\big)$; we refer to [17, Lemma 4.2] for details.

On the other hand, we can expand $\int_X |d_M A_r u|^2$ using the form (2.3) of the metric, with $C|_{x=0} = 0$, and freezing the coefficients of A, B at $x = 0$. Thus, there exist $c > 0$, $\tilde{C} > 0$ and $\delta_0 > 0$ such that if $\delta < \delta_0$ and A is supported in $|x| < \delta$ then

$$\begin{aligned} c\int_X \sum_j |D_{x_j} A_r u|^2 + \int_X ((1 - \tilde{C}\delta) \sum_j |D_{y_j} A_r u|_h^2 - |D_t A_r u|^2) \\ \leq \int_X (|d_M A_r u|^2 - |D_t A_r u|^2), \end{aligned} \tag{2.5}$$

where we used the notation

$$|D_{y_j} A_r u|_h^2 = \sum_{ij} B_{ij}(0, y) D_{y_i} A_r u\, \overline{D_{y_j} A_r u},$$

i.e. h is the metric g restricted to the span of the ∂_{y_j}, $j = 1, \ldots, l$.

Now, if $q \in \mathcal{E}$, then for $\delta > 0$ sufficiently small (i.e. A supported near the corner) the second term on the left hand side of (2.5) can be written as $\|BA_r u\|^2$

for an operator $B \in \Psi^1_{\mathrm{b}}(X)$, modulo a term of the form $\langle FA_r u, A_r u\rangle_{L^2(X)}$ with $F \in \Psi^1_{\mathrm{b}}(X)$ (which is controlled by the inductive hypothesis). Letting $r \to 0$, we deduce that $c\sum_j \|D_{x_j}A_r u\|^2 + \|BA_r u\|^2$ is uniformly bounded, hence $D_{x_j}Au$, BAu are in $L^2(X)$, proving the proposition for $q \in \mathcal{E}$.

For $q \in {}^{\mathrm{b}}T^*X \setminus {}^{\mathrm{b}}\dot{T}^*X$, A supported in $|x| < \delta$, we use

$$\int_X \delta^{-2}|x_j D_{x_j}A_r u|^2 \leq \int_X |D_{x_j}A_r u|^2$$

to modify the first term on the left hand side in (2.5). Then the left hand side of (2.5) can be rewritten as $\|BA_r u\|^2$, modulo a term of the form $\langle FA_r u, A_r u\rangle_{L^2(X)}$ with $F \in \Psi^1_{\mathrm{b}}(X)$, and the proof is finished as above. $\square$

In fact, related estimates give the equivalence of the L^2- and H^1-based wave front sets for solutions of the wave equation:

Lemma 2.5. *(cf. [17, Lemma 6.2]) Suppose $u \in H^1_{0,loc}(X)$, $Pu = 0$. Then $\mathrm{WF}^{1,\infty}_b(u) = \mathrm{WF}_b(u)$. Moreover,*

$$\mathrm{WF}^{1,m}_b(u)^c = \{q \in {}^bT^*X \setminus o :\ \exists A \in \Psi^{m+1}_b(X),\ \sigma_{b,m+1}(A)(q) \neq 0,\ Au \in L^2(X)\},$$

where the right hand side is, by definition, $\mathrm{WF}^{m+1}_b(u)^c$.

After these preliminary discussions, we turn to the propagation estimate at $q \in \mathcal{H}$. As usual, the key ingredient is to find a $\mathcal{C}^\infty$ function f on ${}^bT^*X$ such that, at least near q, $H_p\pi^* f$ has a fixed sign. In our setting, we can take $f = \eta$ where $\eta = -\frac{x\cdot\xi}{|\tau|} = -\frac{\sum\sigma_j}{|\tau|}$. Indeed, the Hamilton vector field H_p of p is given by

(2.6)
$$\begin{aligned}H_p = 2\tau\partial_t - H_g = 2\tau\partial_t - 2A\xi\cdot\partial_x - 2B\zeta\cdot\partial_y - 2\sum C_{ij}\zeta_j\partial_{x_i} - 2\sum C_{ij}\xi_i\partial_{y_j}\\ + 2\sum(\partial_{x_k}A_{ij})\xi_i\xi_j\partial_{\xi_k} + 2\sum(\partial_{x_k}C_{ij})\xi_i\zeta_j\partial_{\xi_k}\\ + 2\sum(\partial_{x_k}B_{ij})\zeta_i\zeta_j\partial_{\xi_k}\\ + 2\sum(\partial_{y_k}A_{ij})\xi_i\xi_j\partial_{\zeta_k} + 2\sum(\partial_{y_k}C_{ij})\xi_i\zeta_j\partial_{\zeta_k}\\ + 2\sum(\partial_{y_k}B_{ij})\zeta_i\zeta_j\partial_{\zeta_k}.\end{aligned}$$

Thus,

$$\begin{aligned}|\tau|H_p\eta = 2\xi\cdot A\xi + 2\sum C_{ij}\xi_i\zeta_j - 2\sum(\partial_{x_k}A_{ij})\xi_i\xi_j x_k\\ - 2\sum(\partial_{x_k}C_{ij})\xi_i\zeta_j x_k - 2\sum(\partial_{x_k}B_{ij})\zeta_i\zeta_j x_k,\end{aligned}$$

so at $x = 0$, where C vanishes,

$$|\tau|H_p\eta = 2\xi\cdot A\xi = 2\tau^2 - 2\zeta\cdot B\zeta - 2p = 2\tau^2 - 2|\zeta|^2_y - 2p. \tag{2.7}$$

Thus, $H_p\eta > 0$ at $\pi^{-1}(\mathcal{H}) \cap \Sigma(P) = \hat{\pi}^{-1}(H)$.

We only state the following propagation result for propagation in the forward direction along the generalized broken bicharacteristics. A similar result holds in the backward direction, i.e. if we replace $\eta(\xi) < 0$ by $\eta(\xi) > 0$ in (2.8); the proof in this case only requires changes in some signs in the argument given below. The construction of a positive commutator below closely mirrors that of [15] in the N-body setting.

Proposition 2.6. *(cf.* [17, Proposition 6.3].*) Let* $q_0 = (0, y_0, t_0, 0, \zeta_0, \tau_0) \in \mathcal{H} \cap {}^{\mathrm{b}}T^*_{\mathcal{F}_{reg}}X$ *and let* $\eta = -\frac{x\cdot\xi}{|\tau|}$ *be the* π*-invariant function defined in the local coordinates discussed above, and suppose that* $u \in H^1_{0,loc}(X)$*,* $Pu = 0$*. If there exists a conic neighborhood* U *of* q_0 *in* $\dot{\Sigma}$ *such that*

$$q \in U \text{ and } \eta(q) < 0 \Rightarrow q \notin \mathrm{WF}_b^{1,\infty}(u) \tag{2.8}$$

then $q_0 \notin \mathrm{WF}_b^{1,\infty}(u)$*.*

In fact, if the wave front set assumption is relaxed to the existence of a conic neighborhood U *of* q_0 *in* $\dot{\Sigma}$ *such that*

$$q \in U \text{ and } \eta(q) < 0 \Rightarrow q \notin \mathrm{WF}_b^{1,s}(u), \tag{2.9}$$

then we can still conclude that $q_0 \notin \mathrm{WF}_b^{1,s}(u)$*.*

We remark that for $q \in \dot{\Sigma}$, $\eta(q) < 0$ implies $x \neq 0$, so $q \notin {}^{\mathrm{b}}T^*_{\mathcal{F}}X$. Thus, the hypotheses on u are made away from the corner.

Proof. (Sketch.) As in Proposition 2.4 we use an inductive argument to show that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$, provided that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u)$; again the inductive hypothesis holds for $s = 1/2$ since $u \in H^1_{\mathrm{loc}}(X)$. By an estimate closely related to the proof of Lemma 2.5, which roughly says that for solutions of the wave equation, $D_t A_r u$ controls $d_M A_r u$, modulo terms we control *a priori*, we only need to show that for some $B \in \Psi_{\mathrm{b}}^{s+1}(X)$ with $\sigma_{b,s+1}(B)(q_0) \neq 0$, $Bu \in L^2(X)$.

Below we fix a small neighborhood U_0 of q_0 such that U_0 is inside a coordinate neighborhood of q_0.

The key is to construct an operator A with $\mathrm{WF}'_{\mathrm{b}}(u) \subset U$ and $i[A^*A, P]$ positive, modulo terms that we can estimate either by the a priori assumptions, namely those on Pu and those on $\mathrm{WF}_{\mathrm{b}}(u)$, summarized in (2.8) above. Thus, we do not need to make the commutator positive in $\eta < 0$, and also 'away from $\Sigma(P)$', although the latter is a moral statement as the locus of the microlocalization is ${}^{\mathrm{b}}T^*X \setminus o$, not $T^*X \setminus o$. Our A will in fact be formally self-adjoint modulo lower order operators, and we only take A^*A to avoid having to comment on the subprincipal terms.

As mentioned several times already, the main technical problem below is that P does not lie in $\Psi_{\mathrm{b}}(X)$, so we cannot simply use the symbol calculus on $\Psi_{\mathrm{b}}(X)$ – we need to write out various expressions semi-explicitly as elements of $\mathrm{Diff}\,\Psi_{\mathrm{b}}(X)$. On the other hand, at least formally, we can use the symbol caluclus of $\mathrm{Diff}\,\Psi_{\mathrm{b}}(X)$, regarding symbols as functions on $T^*X \setminus o$, in case of $\Psi_{\mathrm{b}}(X)$ via pull-back by π. This has the advantage that p is a function on T^*X, as is the pull-back of symbols on ${}^{\mathrm{b}}T^*X$ via π, so one can calculate their Poisson bracket, etc. However, it is not trivial (although it is relatively straightforward) to make this into a technically useful computation, since we need to control various expression in $\mathrm{Diff}\,\Psi_{\mathrm{b}}(X)$. Here we merely give the formal argument to show why the constructed symbol *should* be useful; the actual proof is given in [17, Proposition 6.3].

To set the topology straight, we note that every conic neighborhood U of $q_0 = (0, y_0, t_0, 0, \zeta_0, \tau_0) \in \mathcal{H} \cap {}^{\mathrm{b}}T^*_{\mathcal{F}_{\mathrm{reg}}}X$ in $\dot{\Sigma}$ contains an open set of the form

$$\{q \in \dot{\Sigma} : \ |x(q)|^2 + |y(q) - y_0|^2 + |t(q) - t_0|^2 + |\hat{\zeta}(q) - \hat{\zeta}_0|^2 < \delta\}, \tag{2.10}$$

$\hat{\zeta} = \frac{\zeta}{\tau}$. Indeed, as $\dot{\Sigma}$ is equipped with the subspace topology, this would be standard if we included a term $|\hat{\sigma}(q)|^2$, with $\hat{\sigma} = \frac{\sigma}{\tau}$, in the sum. However, in $\dot{\Sigma}$, $\hat{\sigma}$ can be estimated by $C|x|$ (using that $|\hat{\sigma}_j| = x_j|\xi_j| \leq x_j|\tau|$ on $\dot{\Sigma}$ whenever $x_j \neq 0$, and $\hat{\sigma}_j = 0$ if $x_j = 0$), proving the claim surrounding (2.10).

Note also that (2.8) implies the same statement with U replaced by any smaller neighborhood of q_0; in particular, for the set (2.10), provided that δ is sufficiently small.

We construct the symbol of A in a few steps. The two main ingredients are a homogeneous degree zero function that is increasing along the Hamilton flow, which will be η, and a homogeneous degree zero function ω on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X \setminus o$ that roughly measures the square of the distance from q_0 in ${}^{\mathrm{b}}\dot{T}^*X$. Note that ω can also be regarded as a function on a subset of ${}^{\mathrm{b}}S^*X$, if desired. Thus, we let

$$\omega(q) = |x(q)|^2 + |y(q) - y_0|^2 + |t(q) - t_0|^2 + |\hat{\zeta}(q) - \hat{\zeta}_0|^2, \tag{2.11}$$

$|.|$ denoting the Euclidean norm, and $\hat{\zeta} = \frac{\zeta}{\tau}$ as above. Then ω vanishes quadratically at q_0, in fact is a sum of squares, so $|d\omega| \leq C_1'\omega^{1/2}$, and in particular

$$|\tau^{-1}H_p\omega| \leq C_1''\omega^{1/2}. \tag{2.12}$$

Next, we use the variable $\eta = -\frac{x\cdot\xi}{|\tau|}$ to measure propagation. Since

$$\eta = -\frac{x\cdot\xi}{|\tau|} = -\sum_j \sigma_j|\tau|^{-1},$$

η is a homogeneous degree zero $\mathcal{C}^\infty$ function on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X \setminus o$, hence its pullback by π is a $\mathcal{C}^\infty$ function on T^*X. This function indeed measures the flow along bicharacteristics near q_0 since at points $\tilde{q}_0$ in $\hat{\pi}^{-1}(\{q_0\})$, where thus $p = 0$,

$$|\tau|H_p\eta(\tilde{q}_0) = \tau_0^2 - |\zeta_0|^2_{y_0} = c_0\tau_0^2 > 0, \tag{2.13}$$

due to (2.7), where we used that $q_0 \in \mathcal{H}$. Thus, for U_0 sufficiently small, $|\tau|H_p\eta > c_0\tau^2/2 > 0$ on U_0.

We are now ready to define the symbol a of A. For $\epsilon > 0$, $\delta > 0$, with other restrictions to be imposed later on, let

$$\phi = \eta + \frac{1}{\epsilon^2\delta}\omega, \tag{2.14}$$

so ϕ is a homogeneous degree zero $\mathcal{C}^\infty$ function on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X \setminus o$ – we can again regard it as a function on $T^*X \setminus o$ via pull-back by π. (Here ϵ^{-2} plays the role of β in the analogous – normal – propagation estimate of [15].)

Let $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$ be equal to 0 on $(-\infty, 0]$ and $\chi_0(t) = \exp(-1/t)$ for $t > 0$. Thus, $\chi_0'(t) = t^{-2}\chi_0(t)$. Let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ be 0 on $(-\infty, 0]$, 1 on $[1, \infty)$, with $\chi_1' \geq 0$ satisfying $\chi_1' \in \mathcal{C}_c^\infty((0,1))$. Finally, let $\chi_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ be supported in $[-2c_1, 2c_1]$, identically 1 on $[-c_1, c_1]$, where c_1 is such that if $|\sigma|^2/\tau^2 < c_1/2$ in $\dot{\Sigma}\cap U_0$. Thus, $\chi_2(|\sigma|^2/\tau^2)$ is a cutoff in $|\sigma|/|\tau|$, with its support properties ensuring that $d\chi_2(|\sigma|^2/\tau^2)$ is supported in $|\sigma|^2/\tau^2 \in [c_1, 2c_1]$ hence outside $\dot{\Sigma}$ – it should be thought of as a factor that microlocalizes near the characteristic set but effectively commutes with P. Then, for $A_0 > 0$ large, to be determined, let

$$a = \chi_0(A_0^{-1}(2 - \phi/\delta))\chi_1(\eta/\delta + 2)\chi_2(|\sigma|^2/\tau^2); \tag{2.15}$$

so a is a homogeneous degree zero $\mathcal{C}^\infty$ function on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X$. Indeed, as we see momentarily, for any $\epsilon > 0$, a has compact support inside this neighborhood (regarded as a subset of ${}^{\mathrm{b}}S^*X$, i.e. quotienting out by the $\mathbb{R}^+$-action) for δ sufficiently small, so in fact it is globally well-defined. In fact, on $\operatorname{supp} a$ we have $\phi \leq 2\delta$ and $\eta \geq -2\delta$. Since $\omega \geq 0$, the first of these inequalities implies that $\eta \leq 2\delta$, so on $\operatorname{supp} a$

$$|\eta| \leq 2\delta. \tag{2.16}$$

Hence,

$$\omega \leq \epsilon^2\delta(2\delta - \eta) \leq 4\delta^2\epsilon^2. \tag{2.17}$$

In view of (2.11) and (2.10), this shows that for any $\epsilon > 0$, a is supported in U, provided $\delta > 0$ is sufficiently small. The role that A_0 large plays is that it increases the size of the first derivatives of a relative to the size of a, hence it allows us to give a bound for a in terms of a small multiple of its derivative along the Hamilton vector field. This is crucial as we need to deal with weight factors, such as $|\tau|^{s+1/2}$ in the next paragraph, if the weight factors do not commute with P. In this case, they can be arranged to commute (at least microlocally, which suffices), so we could eliminate A_0, but its presence is helpful if one is to weaken the assumptions on the structure of P.

Thus, using (2.12), (2.17),

$$|\tau|^{-1}H_p\phi = H_p\eta + \frac{1}{\epsilon^2\delta}H_p\omega \geq c_0/2 - \frac{1}{\epsilon^2\delta}C_1''\omega^{1/2} \geq c_0/2 - 2C_1''\epsilon^{-1} \geq c_0/4 > 0$$

provided that $\epsilon > \frac{8C_1''}{c_0}$, i.e. that ϵ is not too small. We fix some such ϵ for the rest of the arguments below, and then we will take $\delta > 0$ sufficiently small. With this,

$$H_p a^2 = -b^2 + e,\ b = |\tau|^{1/2}(2|\tau|^{-1}H_p\phi)^{1/2}(A_0\delta)^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2,$$

with e arising from the derivative of $\chi_1\chi_2$. Here χ_0 stands for $\chi_0(A_0^{-1}(2 - \frac{\phi}{\delta}))$, etc. We let $A \in \Psi^0_{\mathrm{b}}(X)$ be such that $\sigma_{b,0}(A) = a$. Since $\eta < 0$ on $d\chi_1$ while $d\chi_2$ is disjoint from the characteristic set, both being regions disjoint from $\mathrm{WF}_{\mathrm{b}}(u)$, $i[A^*A, P]$ is positive modulo terms that we can a priori control, so the standard positive commutator argument gives an estimate for Bu, where B has symbol b. Replacing a by $a|\tau|^{s+1/2}$, we still have a positive commutator (in this case τ, or rather D_t, actually commutes with P, but in any case we could use A_0 to bound the additional commutator term), which now gives (with the new B) that $Bu \in L^2(X)$, which means in particular that $q_0 \notin \mathrm{WF}^{1,s}_{\mathrm{b}}(u)$.

This argument is of course very imprecise; the commutator calculation needs to be written up in $\operatorname{Diff}\Psi_{\mathrm{b}}(X)$. We refer to [17, Proposition 6.3] for full details. □

Following Melrose and Sjöstrand [8, 9], see also Lebeau's paper [7], it is not hard to prove Theorem 1.2 using this proposition and its companion for glancing points. As usual, we refer to [17] for details.

We remark that the symbol $\sum \sigma_j = \sum x_j\xi_j$ played a major role, via η, in the positive commutator estimate yielding Proposition 2.6. This is the symbol of the vector field that is also used in N-body scattering for the Mourre estimates – although in the latter case the region of interest is where x is large, while now where x is small. In fact, the microlocal constructions in N-body scattering are similar to the one presented above, at least at the formal level, which was the original reason for my interest in the present topic.

Finally, I would like to thank Ecole Polytechnique for the invitation and Johannes Sjöstrand for his hospitality. I would also like to use this occasion to thank Mikhail Shubin for inviting me to Northeastern University to give a 'Basic Notions' seminar talk on the propagation of singularities (without boundaries) in November 2003: preparing for that talk I decided I should finally try to see if all my previous claims on the relationship between N-body scattering and the wave equation in domains with corners were well founded. This turned out to be a very fruitful project, also aided by many conversations with Richard Melrose which I happily acknowledge. Finally, I am very grateful to Richard Melrose and Jared Wunsch for the discussions on our joint project [10], on which the remarks at the end of the introduction are based.

References

[1] J. Cheeger and M. Taylor. Diffraction by conical singularities, I, II. *Comm. Pure Applied Math.*, 35:275–331, 487–529, 1982.

[2] J. J. Duistermaat and L. Hörmander. Fourier integral operators, II. *Acta Mathematica*, 128:183–269, 1972.

[3] A. Hassell, R. B. Melrose, and A. Vasy. Scattering for symbolic potentials of order zero and microlocal propagation near radial points. *Preprint*, 2005.

[4] A. Hassell, R. B. Melrose, and A. Vasy. Spectral and scattering theory for symbolic potentials of order zero. In *Séminaire: Équations aux Dérivées Partielles, 2000–2001*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XIII, 21. École Polytech., Palaiseau, 2001.

[5] L. Hörmander. Fourier integral operators, I. *Acta Mathematica*, 127:79–183, 1971.

[6] L. Hörmander. *The analysis of linear partial differential operators,* vol. 1-4. Springer-Verlag, 1983.

[7] G. Lebeau. Propagation des ondes dans les variétés à coins. *Ann. Scient. Éc. Norm. Sup.*, 30:429–497, 1997.

[8] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math*, 31:593–617, 1978.

[9] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math*, 35:129–168, 1982.

[10] R. B. Melrose, A. Vasy, and J. Wunsch. Propagation of singularities for the wave equation on manifolds with edges. In preparation.

[11] R. B. Melrose and J. Wunsch. Propagation of singularities for the wave equation on conic manifolds. *Invent. Math.*, 156(2):235–299, 2004.

[12] R. B. Melrose. Transformation of boundary problems. *Acta Math.*, 147(3-4):149–236, 1981.

[13] R. B. Melrose and P. Piazza. Analytic K-theory on manifolds with corners. *Adv. Math.*, 92(1):1–26, 1992.

[14] M. Mitrea, M. Taylor, and A. Vasy. Lipschitz domains, domains with corners and the Hodge Laplacian. *Preprint*, 2004.

[15] A. Vasy. Propagation of singularities in many-body scattering. *Ann. Sci. École Norm. Sup. (4)*, 34:313–402, 2001.

[16] A. Vasy. Propagation of singularities in many-body scattering in the presence of bound states. *J. Func. Anal.*, 184:177–272, 2001.

[17] A. Vasy. Propagation of singularities for the wave equation on manifolds with corners. *Preprint*, 2004.

Department of Mathematics, MIT and Northwestern University
E-mail address: andras@math.mit.edu