A SEMICLASSICAL APPROACH TO GEOMETRIC X-RAY TRANSFORMS IN THE PRESENCE OF CONVEXITY

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Abstract. In this short paper we introduce a variant of the approach to inverting the X-ray transform that originated in the author's work with Uhlmann. The new method is based on semiclassical analysis and eliminates the need for using sufficiently small domains and layer stripping for obtaining the injectivity and stability results, assuming natural geometric conditions are satisfied.

1. Introduction

In this short paper we introduce a variant of the approach to inverting the X-ray transform that originated in the author's work with Uhlmann [14]. Here recall that on a compact Riemannian manifold with boundary M the X-ray transform is the map I that assigns to each $f \in C^\infty(M)$ the function If on SM, the unit sphere bundle, defined by

$$(If)(\beta) = \int_{\gamma_\beta} f(\gamma_\beta(t))\, dt,$$

where γ_β is the geodesic whose lift to SM goes through β. (Other similar families of curves work equally well, as observed by H. Zhou in the appendix to [14]. There is also no need to consider SM the *unit* sphere bundle; indeed it is convenient to consider If defined on $TM \setminus o$ as a homogeneous function of degree -1. Also, compactness can be relaxed.) Here the geodesics are assumed to be sufficiently well-behaved so that the integrals are over finite intervals, i.e. the geodesics reach the boundary in finite affine parameter; the more strict requirements later on make this automatic. It is also useful to consider M as a smooth domain in a manifold with boundary $\tilde{M}$; in this case we regard f as a function supported in M (via extension by 0). The inverse problem is to recover f from If, i.e. to construct a left inverse, or at least show that I is injective with suitable stability estimates. Typically one approaches such problems by considering the normal operator I^*I, or some modification. In the present context (using the above over-parameterization, in that many β correspond to the same geodesic) I^* is replaced by a closely related operator of the form

$$(Lv)(z) = \int_{S_z M} v(\beta)\, d\nu(\beta)$$

of integration along the geodesics through z; ideally one would like LI invertible, at least up to 'trivial' errors.

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The key idea of [14] was to introduce an *artificial boundary*, which is a hypersurface in M, which meant that rather than working on all of M, one initially attempts to recover f from If in the region on one side of this hypersurface. More precisely, one is working with a family of hypersurfaces which are level sets of a function $\tilde{x}$. This function $\tilde{x}$ is required to be strictly concave from the side of the super-level sets, i.e. $\frac{d(\tilde{x}\circ\gamma_\beta)}{dt}(t_0) = 0$ implies $\frac{d^2(\tilde{x}\circ\gamma_\beta)}{dt^2}(t_0) > 0$. If $\tilde{x}$ is normalized so that M is contained in $\tilde{x} \leq 0$, then the main result of [14] was invertibility, in the above sense, in a region $\tilde{x} \geq -c$, where $c > 0$ was sufficiently small, depending on various analytic quantities. Here $\tilde{x} = -c$ is the artificial boundary, and the strict concavity is in fact only required in $\tilde{x} \geq -c$. This could be repeated in a layer stripping argument, allowing a global result after a multi-step process. The analytic heart of this artificial boundary argument involves Melrose's *algebra of scattering pseudodifferential operators* [7]; while the artificial boundary is at a geometrically finite place, the analytic way of obtaining a modified 'normal operator' effectively pushes it to infinity. This is done by, most crucially (another modification is also needed), inserting a localizer $\chi(\beta)$ into the formula for L that concentrates on geodesics almost tangential to the level sets of $\tilde{x}$, with the approximate tangency becoming more strict as $\tilde{x}$ approaches the artificial boundary, i.e. as $x = \tilde{x} + c \to 0+$; the precise way this happens determines the analytic structure:

$$(Lv)(z) = \int_{S_zM} \chi(\beta)v(\beta)\,d\nu(\beta).$$

The new method introduced in this paper is based on *semiclassical analysis* and eliminates the need for using sufficiently small domains and layer stripping for obtaining the injectivity and stability results, assuming natural geometric conditions are satisfied. It applies both to the localized problems (in the sense of the artificial boundary), in which case it still uses Melrose's scattering algebra [7], and to global problems, in which case one uses a variant of the standard semiclassical algebra. As injectivity or stability statements, the results are the same as one would obtain with the original techniques of [14], but with a more transparent and streamlined proof. This is reflected by the stronger technical theorems on the modified normal operator for the X-ray transform. The analytic heart of the approach is again to introduce a localizer χ_h, which now also depends on h; for h small this again localizes very close to geodesics tangential to level sets of $\tilde{x}$:

$$(L_hv)(z) = \int_{S_zM} \chi_h(\beta)v(\beta)\,d\nu(\beta).$$

Concretely, on manifolds of dimension ≥ 3, in the case of no conjugate points but with a convex foliation still, we directly obtain a modified normal operator that is invertible; this involves the use of a *semiclassical foliation pseudodifferential algebra* (the aforementioned variant), but not the scattering algebra, and it also eliminates the need for making small steps (thin layers) in the layer stripping approach. One in fact needs a weaker requirement on the lack of conjugate points for curves from point of tangency to the foliation, which in dimension > 3 can be further weakened similarly to the work of Stefanov and Uhlmann [11]. Since in this case there is no need to renormalize $\tilde{x}$ as there is no artificial boundary, in order to simplify the notation we write $x = \tilde{x}$; this allows a notationally uniform treatment later.

Theorem 1.1. *Suppose M is a compact Riemannian manifold with boundary of dimension ≥ 3 equipped with a function x with strictly convex level sets and dx*

nonzero. Suppose also that geodesics do not have points conjugate to their points of tangency to the level sets of x. Then the semiclassically modified normal operator A, see (3.6) *with $\Phi(x) = -x$, of the geodesic X-ray transform is a* left invertible *elliptic order -1 pseudodifferential operator.*

Here left invertible means that there is an order 1 pseudodifferential operator G on $\tilde{M}$ such that $GA = \mathrm{Id}$ on $\dot{H}^s$ for all s (this is the space of distributions in H^s supported in the domain, using Hörmander's notation [3]). An immediate consequence, due to the fact that the operator L (see (3.5)) used in the definition of our modified normal operator A is a standard Fourier integral operator for fixed non-zero h, with appropriate order and canonical relation, is:

Corollary 1.1. *Let $s \in \mathbb{R}$. Under the hypotheses of the theorem, the X-ray transform I is injective on $\dot{H}^s$ and we have stability estimates: there exists $C > 0$ such that for all $f \in \dot{H}^s$, we have $\|f\|_{\dot{H}^s(M)} \leq C\|If\|_{H^{s+1/2}(SM)}$.*

On the other hand, if conjugate points are present, one can still work in appropriately small layers as determined by the geometry (to eliminate the conjugate points), but without the need to further shrink the size of the steps to obtain invertibility as required in [14]; this approach still uses the scattering algebra at the artificial boundary.

Theorem 1.2. *Suppose M is a compact Riemannian manifold with boundary of dimension ≥ 3 equipped with a function $\tilde{x}$ with strictly concave level sets in $\tilde{x} \geq -c$, from the super-level sets, and $d\tilde{x}$ nonzero. Suppose also that geodesics contained in the region $\tilde{x} \geq -c$ do not have points conjugate to their points of tangency to the level sets of $\tilde{x}$. Then, with $x = \tilde{x} + c$, the semiclassically modified normal operator A, see* (3.6)*, with $\Phi(x) = x^{-1}$ and with cutoff $\tilde{\chi}$ given in* (3.11)*, of the geodesic X-ray transform is a* left invertible *elliptic order* $(-1, -2)$ *scattering pseudodifferential operator.*

The method is also applicable to other X-ray transform problems, such as the X-ray transform on asymptotically conic, e.g. asymptotically Euclidean, spaces, studied in work with Zachos, based in part on Zachos' work [17], where it is harder to implement the original 'thin layer' approach of [14].

Notice that for nonlinear problems there is an additional role in localizing near ∂M, not addressed by the semiclassicalization, namely if one uses a Stefanov-Uhlmann pseudolinearization formula [10], one needs to make sure that the coefficients of the transform in the formula are close to known values, typically at the boundary. This need for localization can be eliminated if the unknown quantity is a priori globally close to a given background, in which case one can obtain injectivity results provided one has injectivity results for the background, without having to introduce the additional small semiclassical parameter, but in order to obtain the prerequisite injectivity results for the background, the semiclassical approach is still very useful.

The plan of this short paper is the following. In Section 2 we introduce the analytic ingredients, namely the semiclassical foliation pseudodifferential algebras, and then in Section 3 we use this for the analysis of the X-ray transform. The whole of Section 3 consists of the proofs of Theorems 1.1 and 1.2.

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2. The semiclassical algebra

In this section we discuss an inhomogeneous pseudodifferential semiclassical algebra associated to a foliation $\mathcal{F}$ on a manifold M. As is usual, it depends on a semiclassical parameter, traditionally denoted by $h \in [0,1]$; it is an h-dependent family of operators on M. The standard semiclassical algebra is built from vector fields hV, $V \in \mathcal{V}(M)$ (see e.g. [18]), so semiclassical differential operators are in the algebra over $C^\infty(M)$ generated by these, i.e. locally finite sums of finite products of these with $C^\infty(M)$. Thus, in local coordinates $z \in O \subset \mathbb{R}^n$, $P \in \mathrm{Diff}^m_\hbar(M)$ means that

$$P = \sum_{|\alpha| \leq m} a_\alpha(z,h)(hD_z)^\alpha,$$

with $a_\alpha \in C^\infty(\mathbb{R}^n \times [0,1])$, supported in O. Thus, as a traditional differential operator, P degenerates at $h = 0$, but as a semiclassical operator it does not; its semiclassical principal symbol is

$$p(z,\zeta) = \sum_{|\alpha| \leq m} a_\alpha(z,0)\zeta^\alpha,$$

obtained by replacing hD_z by ζ, and evaluating the coefficients at $h = 0$, and the operator is semiclassically elliptic if there is $c > 0$ such that

$$|p(z,\zeta)| \geq c\langle\zeta\rangle^m,$$

i.e. p is non-vanishing and is elliptic in the standard sense.

Our new semiclassical foliation algebra is built from vector fields which are either semiclassical in the sense above, i.e. hV, $V \in \mathcal{V}(M)$, or $h^{1/2}$-semiclassical and tangent to the foliation:

$$\mathcal{V}_{\hbar,\mathcal{F}}(M) = h\mathcal{V}(M) + h^{1/2}\mathcal{V}(M;\mathcal{F}),$$

where $\mathcal{V}(M;\mathcal{F})$ denotes the Lie algebra of vector fields tangent to the foliation. In local coordinates, in which the foliation is locally given by $x = (x_1,\ldots,x_k)$ being constant, and remaining coordinates (which are thus coordinates along the leaves) are $y_1,\ldots,y_{n-k}$, this means that the semiclassical foliation vector fields are

$$\sum_{j=1}^k a_j(x,y,h)hD_{x_j} + \sum_{j=1}^{n-k} b_j(x,y,h)h^{1/2}D_{y_j}.$$

Correspondingly, elements of the algebra of semiclassical foliation differential operators of order m, $\mathrm{Diff}_{\hbar,\mathcal{F}}(M)$, are of the form

$$P = \sum_{|\alpha|+|\beta| \leq m} a_{\alpha\beta}(x,y,h)(hD_x)^\alpha(h^{1/2}D_y)^\beta,$$

and the semiclassical foliation principal symbol is

$$p(x,y,\xi,\eta) = \sum_{|\alpha|+|\beta| \leq m} a_{\alpha\beta}(x,y,0)\xi^\alpha\eta^\beta,$$

with semiclassical ellipticity meaning that there is $c > 0$ such that

$$|p(x,y,\xi,\eta)| \geq c\langle(\xi,\eta)\rangle^m.$$

A somewhat different perspective of $\mathcal{V}_{\hbar,\mathcal{F}}(M)$ is that it is a conformal version of the adiabatic Lie algebra. In the latter, with ϵ the adiabatic parameter, one considers a fibration (rather than just a foliation) $\mathcal{F}$, and the sum of $\epsilon\mathcal{V}(M)$ and $\mathcal{V}(M;\mathcal{F})$, so in local coordinates as above the vector fields are

$$\sum_{j=1}^{k} a_j(x,y,\epsilon)\epsilon D_{x_j} + \sum_{j=1}^{n-k} b_j(x,y,\epsilon) D_{y_j}.$$

Thus, with $\epsilon = h^{1/2}$, our Lie algebra is ϵ times the adiabatic Lie algebra, i.e. is a conformal, more precisely, a 1-conformal (in that the conformal factor is the first power of the adiabatic parameter ϵ) version of the adiabatic Lie algebra. The conformal factor makes this algebra more localized, just like in the comparison of the more microlocalized scattering [7] and the more global b-algebras of Melrose [5, 6]; this is what allows for relaxing the requirements on $\mathcal{F}$ to being a foliation.

Returning to our semiclassical perspective, we turn this into a pseudodifferential operator algebra $\Psi_{\hbar,\mathcal{F}}(M;\mathcal{F})$ as follows. First starting locally with $\mathbb{R}^n$ and the foliation $\mathcal{F}_{\mathbb{R}^n}$ given by the joint level sets of the x_j, we consider symbols a with

$$|(D_z^\alpha D_\zeta^\beta a)(z,\zeta,h)| \leq C_{\alpha\beta}\langle\zeta\rangle^{m-|\beta|},$$

i.e. the standard semiclassical class (one can also require differentiability in h; since this is a parameter, i.e. there is no differentiation in it, the choice is mostly irrelevant), but quantizing it according to the foliation as

$$\begin{aligned}(A_h u)(x,y) &= (Au)(x,y,h)\\ &= (2\pi)^{-n} h^{-n/2-k/2} \int e^{i(x-x')\cdot\xi/h + i(y-y')\cdot\eta/h^{1/2}} a(x,y,\xi,\eta,h) u(x',y',h)\, d\xi\, d\eta\, dx'\, dy'.\end{aligned}$$

This gives a class $\Psi_{\hbar,\mathcal{F}}(\mathbb{R}^n;\mathcal{F}_{\mathbb{R}^n})$, where the uniformity statement of the symbolic estimates in z on $\mathbb{R}^n$ is not shown in the notation.

Of course, one can change variables as $\tilde\eta = h^{1/2}\eta$, to obtain the usual semiclassical quantization

$$(2\pi)^{-n} h^{-n} \int e^{i(x-x')\cdot\xi/h + i(y-y')\cdot\tilde\eta/h} a(x,y,\xi,h^{-1/2}\tilde\eta,h) u(x',y',h)\, d\xi\, d\tilde\eta\, dx'\, dy'$$

of the symbol

$$\tilde a(x,y,\xi,\tilde\eta,h) = a(x,y,\xi,h^{-1/2}\tilde\eta,h);$$

regarded as a semiclassical symbol, $\tilde a$ is of a weaker type:

$$|D_x^\alpha D_y^\beta D_\xi^\gamma D_{\tilde\eta}^\delta \tilde a(x,y,\xi,\tilde\eta,h)| \leq C_{\alpha\beta\gamma\delta} h^{-|\delta|/2} \langle(\xi, h^{-1/2}\tilde\eta)\rangle^{m-|\gamma|-|\delta|},$$

and while this is sufficient to push the semiclassical algebra through, it is more precise to consider the foliation setup above.

Remark 2.1. As an aside, one can also consider this as a 'blown-down' 2-microlocal coisotropic semiclassical algebra corresponding to the coisotropic $\tilde\eta = 0$, which corresponds exactly to the joint characteristic set of the semiclassical vector fields tangent to $\mathcal{F}$. The type of this algebra is a 1/2-type, in that from the semiclassical perspective one blows up of the coisotropic parabolically at $h = 0$ (the parabolic direction being tangent to $h = 0$), so the scaling is $h^{-1/2}\tilde\eta$. More singular (with homogeneity 1) and thus delicate coisotropic algebras, corresponding to hypersurfaces, were introduced by Sjöstrand and Zworski [9]. (Though it was used for a different purpose and from a different perspective, the work [2] of Gannot and Wunsch introduced semiclassical paired Lagrangian distributions to extend the work of de Hoop, Uhlmann and Vasy [1] from the non-semiclassical setting, and this relates closely to 1-homogeneous 2-microlocalization at a coisotropic.) The 'blown-down' adjective refers to the fact that from this blow-up perspective, the standard semiclassical behavior (i.e. what happens away from $\tilde{\eta} = 0$) is blown down, since $\tilde{\eta} \neq 0$ corresponds to $|\eta| \to \infty$ as $h \to 0$, and we have placed joint symbolic demands on a in (ξ, η).

We now return to a discussion of the basic properties of the semiclassical foliation algebra. One can more generally allow a in (2) to depend on $z' = (x', y')$ as well. The standard left- and right-reduction arguments, removing the z', resp. z, dependence apply, see e.g. [4, 16], and give asymptotic expansions, so for instance the right-reduced version of $a(z, z', \zeta)$ is

$$b(z', \zeta', h) \sim \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!}(-hD_\xi)^\alpha(-h^{1/2}D_\eta)^\beta \partial_x^\alpha \partial_y^\beta a|_{z=z'=(x',y')},$$

while the left-reduced version is

$$c(z, \zeta, h) \sim \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!}(hD_\xi)^\alpha(h^{1/2}D_\eta)^\beta \partial_{x'}^\alpha \partial_{y'}^\beta a|_{z'=z=(x,y)}.$$

This gives (again, see [4, 16]) that the semiclassical foliation pseudodifferential operators form a filtered $*$-algebra (with respect to the L^2-inner product):

$$\begin{aligned}
&A \in \Psi^m_{\hbar,\mathcal{F}}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}),\ B \in \Psi^{m'}_{\hbar,\mathcal{F}}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}) \Rightarrow AB \in \Psi^{m+m'}_{\hbar,\mathcal{F}}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}),\\
&A \in \Psi^m_{\hbar,\mathcal{F}}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}) \Rightarrow A^* \in \Psi^m_{\hbar,\mathcal{F}}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}),
\end{aligned}$$

and moreover that with $\sigma_m(A) = [a] \in S^m/h^{1/2}S^{m-1}$, so for smooth (in $h^{1/2}$) a,

$$\sigma_m(A)(z, \zeta)|_{h=0} = a(z, \zeta, 0),$$

we have

$$\sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B), \qquad \sigma_m(A^*) = \overline{\sigma_m(A)}.$$

One also has the standard elliptic parametrix construction. One says that A, and its principal symbol a, are elliptic if a has an inverse $b \in S^{-m}/h^{1/2}S^{-m-1}$ in the sense that $ab - 1 \in h^{1/2}S^{-1}$; this is equivalent to the lower bound

$$|a(z, \zeta, h)| \geq c\langle\zeta\rangle^m, \qquad c > 0,$$

for h small (i.e. there exists $h_0 > 0$ such that the estimate holds for $h < h_0$). Then there is a parametrix $B \in \Psi^{-m}_{\hbar,\mathcal{F}}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n})$ such that $AB - I, BA - I \in h^\infty\Psi^{-\infty}_{\hbar,\mathcal{F}}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n})$.

Furthermore, $\Psi_{\hbar,\mathcal{F}}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n})$ is invariant under local diffeomorphisms preserving the foliation as is easily seen by the standard Kuranishi trick; this allows the introduction of the class $\Psi_{\hbar,\mathcal{F}}(M, \mathcal{F})$ on manifolds, which has all the analogous properties to $\Psi_{\hbar,\mathcal{F}}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n})$ discussed above. In addition, with $H^s_{\hbar,\mathcal{F}}(M)$ the foliation semiclassical Sobolev space, i.e. the standard Sobolev space $H^s(M)$ but with the natural h-dependent family of norms, so for $s \geq 0$ integer, locally,

$$\|u\|^2_{H^s_{\hbar,\mathcal{F}}} = \sum_{|\alpha|+|\beta|\leq s} \|(hD_x)^\alpha(h^{1/2}D_y)^\beta u\|^2_{L^2},$$

for negative integer s by duality, in general by interpolation (or via the foliation Fourier transform, or via elliptic ps.d.o's),

$$\Psi^m_{\hbar,\mathcal{F}}(M; \mathcal{F}) \subset \mathcal{L}(H^s_{\hbar,\mathcal{F}}(M), H^{s-m}_{\hbar,\mathcal{F}}(M))$$

uniformly in h.

A great advantage of the semiclassical algebra, which is maintained by the foliation semiclassical algebra, is that the error of the elliptic parametrix construction is $O(h^\infty)$, thus small for h small, as an element of $\mathcal{L}(H^s_{\hbar,\mathcal{F}}(M), H^s_{\hbar,\mathcal{F}}(M))$, so e.g. $BA = I + E$ as the output of the elliptic parametrix construction means that there exists $h_0 > 0$ such that $(\mathrm{Id} + E)^{-1}$ exists for $h < h_0$ (and differs from Id by an element of $h^\infty \Psi^{-\infty}_{\hbar,\mathcal{F}}(M, \mathcal{F})$), and thus A actually has a left inverse, and similarly it also has a right inverse.

As in [14], we actually work on an ambient manifold $\tilde{M}$ with M a domain with smooth boundary in it, and $A \in \Psi^m_{\hbar,\mathcal{F}}(\tilde{M}; \mathcal{F})$ is elliptic on a neighborhood of M. Then there exists $B \in \Psi^{-m}_{\hbar,\mathcal{F}}(\tilde{M}; \mathcal{F})$ such that $AB - I, BA - I \in \Psi^0_{\hbar,\mathcal{F}}(\tilde{M}; \mathcal{F})$ are in fact in $h^\infty \Psi^{-\infty}_{\hbar,\mathcal{F}}(\tilde{M}; \mathcal{F}_{\mathbb{R}^n})$ when localize to a sufficiently small neighborhood of M, i.e. for suitable cutoffs ψ, identically 1 in M,

$$\psi(AB - I)\psi, \psi(BA - I)\psi \in h^\infty \Psi^{-\infty}_{\hbar,\mathcal{F}}(\tilde{M}, \mathcal{F}).$$

So in particular, with $E = BA - I$, for v supported in M, $\psi v = v$, so $\psi BA\psi = \psi^2 + \psi E\psi$ shows that

$$(\mathrm{Id} + \psi E\psi)v = \psi BAv.$$

Now $\mathrm{Id} + \psi E\psi$ is invertible for sufficiently small h, so $(\mathrm{Id} + \psi E\psi)^{-1}\psi B$ is a left inverse for A on distributions supported in M.

There is an immediate extension of this algebra to the scattering setting of Melrose [7]; this algebra actually can be locally reduced to a standard Hörmander algebra, which in turn was studied earlier by Parenti [8] and Shubin [13]. For simplicity, since this is the only relevant case for us, we consider only the case of a codimension one foliation given by a boundary defining function x. Recall that scattering vector fields $V \in \mathcal{V}_{\mathrm{sc}}(M)$ on a manifold with boundary M are of the form xV', V' is a b-vector field, i.e. a vector field on M tangent to ∂M, so in local coordinates, they are of the form

$$a_0(x, y)x^2 D_x + \sum a_j(x, y)xD_{y_j}.$$

As mentioned, we take our foliation to be given by the level sets of x, so the foliation tangent sc-vector fields are locally

$$\sum a_j(x, y)xD_{y_j}.$$

The semiclassical version of $\mathcal{V}_{\mathrm{sc}}(M)$ is simply $\mathcal{V}_{\mathrm{sc},\hbar}(M) = h\mathcal{V}_{\mathrm{sc}}(M)$ (for which pseudodifferential operators were introduced by Vasy and Zworski [15], but in the local, Euclidean, setting this has a much longer history); the semiclassical foliation version is

$$\mathcal{V}_{\mathrm{sc},\hbar,\mathcal{F}}(M; \mathcal{F}) = h\mathcal{V}_{\mathrm{sc}}(M) + h^{1/2}\mathcal{V}_{\mathrm{sc}}(M; \mathcal{F}).$$

Thus, the semiclassical foliation scattering differential operators take the form

$$\sum_{\alpha+|\beta|\le m} a_{\alpha\beta}(x, y, h)(hx^2 D_x)^\alpha (h^{1/2} xD_y)^\beta.$$

The corresponding pseudodifferential operators $A \in \Psi^{m,l}_{\mathrm{sc},\hbar,\mathcal{F}}(M, \mathcal{F})$ again arise by a modified semiclassical quantization of standard semiclassical symbols a, i.e. ones satisfying (conormal in x) symbol estimates

$$|(xD_x)^\alpha D_y^\beta D_\tau^\gamma D_\mu^\delta a(x, y, \tau, \mu, h)| \le C_{\alpha\beta\gamma\delta} \langle(\tau, \mu)\rangle^m x^{-l},$$

namely

$$A_h u(x,y) = Au(x,y,h)$$
$$= (2\pi)^{-n} h^{-n/2-1/2} \int e^{i\left(\frac{x-x'}{x^2}\frac{\tau}{h}+\frac{y-y'}{x}\frac{\mu}{h^{1/2}}\right)} a(x,y,\tau,\mu,h)\, u(x',y')\, \frac{dx'\,dy'}{(x')^{n+1}}\, d\tau\, d\mu.$$

Thus, in $x > 0$, these are just the standard semiclassical foliation operators, in $h > 0$ the standard scattering pseudodifferential operators, with the combined behavior near $x = h = 0$. In particular we have an elliptic theory as in the semiclassical foliation setting: if A is elliptic, meaning

$$|a(x,y,\tau,\mu,h)| \ge c x^{-l} \langle(\tau,\mu)\rangle^m, \qquad c > 0,$$

for h sufficiently small, then there is a parametrix $B \in \Psi^{-m,-l}_{\mathrm{sc},\hbar,\mathcal{F}}(M,\mathcal{F})$ with

$$AB - \mathrm{Id}, BA - \mathrm{Id} \in h^\infty \Psi^{-\infty,-\infty}_{\mathrm{sc},\hbar,\mathcal{F}}(M,\mathcal{F}),$$

and there exists $h_0 > 0$ such that for $h < h_0$, $A \in \mathcal{L}(H^{s,r}_{\mathrm{sc},\hbar,\mathcal{F}}, H^{s-m,r-l}_{\mathrm{sc},\hbar,\mathcal{F}})$ is invertible with uniform bounds. One can again proceed with localizing the elliptic parametrix construction as above in case one has a smooth domain M in an ambient space $\tilde{M}$.

3. Global X-ray transform

We now consider the inverse problem for the X-ray transform

$$If(\beta) = \int_{\gamma_\beta} f(\gamma_\beta(t))\, dt,$$

where for $\beta \in SM$, γ_β is the geodesic through β (or in fact other similar families of curves work equally well), i.e. $\beta = (\gamma_\beta(0), \dot\gamma_\beta(0)) \in S_{\gamma_\beta(0)}M$ (with the dot denoting t-derivatives) utilizing the notation of [14]. We overall follow the approach of [12, Section 4-5] via oscillatory integrals, rather than the blow-up analysis of [14]. Concretely, the approach of the non-semiclassical proof of Proposition 4.2 in [12] underlies most of the local arguments near $t = 0$ and as we follow these quite closely, we will be relatively brief.

With x the function giving the foliation, writing $x(\gamma_\beta(t)) = \gamma^{(1)}_\beta(t)$, the concavity hypothesis is that

$$\dot\gamma^{(1)}_\beta(t) = 0 \Longrightarrow \ddot\gamma^{(1)}_\beta(t) > 0. \tag{3.1}$$

By compactness considerations this implies that there exist $\epsilon > 0$ and $C_0 > 0$ such that

$$|\dot\gamma^{(1)}_\beta(t)| \le \epsilon \Longrightarrow \ddot\gamma^{(1)}_\beta(t) \ge C_0.$$

It is convenient to take advantage of this also holding in a neighborhood M' of M in $\tilde{M}$.

Remark 3.1. We actually do not need to make any convexity assumptions on ∂M. However, if it not strictly convex, we need to consider the geodesic segments as those in M', and some of these may intersect M in a number of segments. This is not an issue below since knowing If in the sense of integrals along geodesic segments in M, one also obtains $I'f$, the integrals along geodesic segments in M' when f is supported in M. *We do not make this distinction explicit below; thus I actually refers to I' from this point on.* (There is no such issue if ∂M is strictly convex and one chooses $\partial M'$ appropriately.)

We write, relative to our convex foliation and some coordinates, denoted by y, along the level sets, $\beta = (x, y, \lambda, \omega)$, so we write tangent vectors as

$$\lambda\,\partial_x + \omega\,\partial_y,$$

and use $\gamma^{(1)}$ to denote the x component of γ, and similarly $\gamma^{(2)}$ to denote the y component of γ to avoid confusion. Then we have

$$\begin{aligned}\gamma_{x,y,\lambda,\omega}(t) &= (\gamma^{(1)}_{x,y,\lambda,\omega}(t), \gamma^{(2)}_{x,y,\lambda,\omega}(t))\\ &= \big(x + \lambda t + \alpha(x, y, \lambda, \omega)t^2 + t^3\Gamma^{(1)}(x, y, \lambda, \omega, t),\\ &\qquad y + \omega t + t^2\Gamma^{(2)}(x, y, \lambda, \omega, t)\big)\end{aligned} \tag{3.2}$$

with $\Gamma^{(1)}, \Gamma^{(2)}$ smooth functions of x, y, λ, ω, t, α a smooth function of x, y, λ, ω, and $\alpha(x, y, 0, \omega) \geq C > 0$ by the concavity from the super-level sets hypothesis; see [14, Section 3] and [12, Proof of Proposition 4.2]. For us the relevant regime will be λ small; we shall restrict to an arbitrarily small neighborhood of $\lambda = 0$ via the semiclassical localization.

This implies the following bound:

Lemma 3.1. *There exists $T > 0$ such that every geodesic reaches $\partial M'$ (thus ∂M) in affine parameter $\leq T$.*

Moreover, there exist $\lambda_0 > 0$ and $C > 0$ such that for all $\beta = (x, y, \lambda, \omega)$ with $|\lambda| < \lambda_0$ and for t in the closed interval on which γ_β is defined,

$$\gamma^{(1)}_{x,y,\lambda,\omega}(t) \geq x + \lambda t + Ct^2/2. \tag{3.3}$$

Proof. The concavity hypothesis implies that any critical point of $\gamma^{(1)}$ in M' is a strict local mimimum, and $\dot\gamma^{(1)}$ can only change sign once and do so non-degenerately since immediately to the left of any zero of $\dot\gamma^{(1)}$ it is negative, and immediately to the right it is positive by the concavity. Thus, for any geodesic either the sign of $\dot\gamma^{(1)}$ is constant (non-zero) or there is a unique point on it with minimal $\gamma^{(1)}$ in M' (so either in $(M')^\circ$ or on $\partial M'$).

Moreover, in case the minimum of $\gamma^{(1)}$ is reached at some t_0, $\dot\gamma^{(1)}(t)$ has the same sign as $t - t_0$. Indeed, this is so for sufficiently small $|t - t_0|$ by either the concavity or by the minimum being on $\partial M'$, and if it vanished for some $t > t_0$ (with $t < t_0$ similar), taking the infimum $t_1 > t_0$ of the values of t at which this happens one concludes that $\dot\gamma^{(1)}(t_1) = 0$, and $\dot\gamma^{(1)}(t) > 0$ for $0 < t < t_0$, which is a contradiction in view of the concavity hypothesis.

In addition, by the uniform concavity estimate, if $\dot\gamma^{(1)}(t_1) \geq \epsilon$, then $\dot\gamma^{(1)}(t) \geq \epsilon$ for $t \geq t_1$, and similarly if $\dot\gamma^{(1)}(t_1) \leq -\epsilon$ then $\dot\gamma^{(1)}(t) \leq -\epsilon$ for $t \leq t_1$. Note that by the uniform concavity estimate, $|\dot\gamma^{(1)}(t)| \leq \epsilon$ can only hold for an affine parameter interval $2\epsilon/C_0$; and if the minimum of $\gamma^{(1)}$ is reached at t_0, then for $|t - t_0| \geq \epsilon/C_0$ one necessarily has $|\dot\gamma^{(1)}(t)| \geq \epsilon$.

Taking into account that M' is compact so x is bounded, we conclude that there exists $T > 0$ such that every geodesic γ_β reaches $\partial M'$ in both directions in affine parameter $\leq T$: if $|x| \leq C_1$ on M', say, then this holds with $T = 2\epsilon/C_0 + 4C_1/\epsilon$.

Turning to (3.3), it suffices to prove this for $\lambda = 0$, and then it follows for sufficiently small λ by compactness taking into account that it holds near $t = 0$ by (3.2). For $\lambda = 0$, we have seen that $\dot\gamma^{(1)}(t)$ has the same sign as t. Then by compactness one obtains a positive lower bound for $\dot\gamma^{(1)}$ on any compact subset of $(0, \infty)$. Since we have the estimate (3.3) for sufficiently small t, say $0 < t < \delta$, using the positive lower bound $\dot\gamma^{(1)}(t)$ for $t \geq \delta/2$ proves (3.3) for $t \geq 0$ at the cost of reducing C; $t \leq 0$ is analogous. □

Remark 3.2. We recall from [14] that we needed to work in a sufficiently small region so that there are no geometric complications, thus there the interval $[-T, T]$ of integration in t, i.e. T, is such that $\ddot\gamma^{(1)}(t)$ is uniformly bounded below by a positive constant in the region over which we integrate, see the discussion in [14] above Equation (3.1), and then further reduced in Equations (3.3)-(3.4) so that the map sending $(x, y, \lambda, \omega, t)$ to the lift of $(x, y, \gamma_{x,y,\lambda,\omega}(t))$ in the resolved space $\tilde M^2$ with the diagonal being blown up, is a diffeomorphism in $t \geq 0$, as well as $t \leq 0$. In the present paper the appearance of no conjugate points assumptions occurs in a closely related manner, when dealing with the stationary phase expansion, though we use the weaker concavity condition (3.1), so even geometrically we reduce the conditions impose. In addition, the extra restrictions in [14] that arise from making the smoothing (thus 'trivial') error removable disappear here.

We now introduce the weight Φ for the exponential conjugation of our normal operator. Below we consider weights $\Phi = \Phi(x)$ which are decreasing functions of x; in the global context, $\Phi(x) = -x$ will be used, in the scattering context (in which $x > 0$) $\Phi(x) = x^{-1}$. Now, for $\Phi(x) = -x$,

$$\Phi(\gamma^{(1)}_{x,y,\lambda,\omega}(t)) - \Phi(x) \leq -\lambda t - Ct^2/2 \leq -\frac{C}{2}\Big(t + \frac{\lambda}{C}\Big)^2 + \frac{\lambda^2}{2C} \leq \frac{\lambda^2}{2C}.$$

Hence, with $\hat\lambda = \lambda/\sqrt{h}$, $\hat t = t/\sqrt{h}$, the rescaling which plays a key role below,

$$h^{-1}(\Phi(\gamma^{(1)}_{x,y,\lambda,\omega}(t)) - \Phi(x)) \leq -\frac{C}{2}\Big(\hat t + \frac{\hat\lambda}{C}\Big)^2 + \frac{\hat\lambda^2}{2C},$$

so for $\hat\lambda$ in a fixed compact set (and h sufficiently small, as is always assumed),

$$\exp(h^{-1}(\Phi(\gamma^{(1)}_{x,y,\lambda,\omega}(t)) - \Phi(x))) \tag{3.4}$$

is uniformly bounded above by a Gaussian in $\hat t$.

We consider now the operator L defined by

$$Lv(z) = \int \tilde\chi(x, y, \lambda/h^{1/2}, \omega) v(\gamma_{x,y,\lambda,\omega})\, d\lambda\, d\omega, \tag{3.5}$$

with $\tilde\chi$ having compact support in the third variable, thus localizing to $\lambda \sim h^{1/2}$, and hence to geodesics almost tangent to the level sets of x at the base point (x, y), for h small. Further we consider the operator

$$A_h = e^{-\Phi(x)/h} L_h I e^{\Phi(x)/h}, \tag{3.6}$$

which is thus given by

$$A_h f(z) = \int e^{-\Phi(x(z))/h} e^{\Phi(x(\gamma_{z,\lambda,\omega}(t)))/h} \tilde\chi(z, \lambda/h^{1/2}, \omega) f(\gamma_{z,\lambda,\omega}(t))\, dt\, |d\nu|,$$

where A_h is understood to apply only to f with support in M, thus for which the t-integral is in a fixed finite interval, say $[-T, T]$, where $|d\nu|$ is a smooth positive density in (λ, ω), such as $|d\lambda\, d\omega|$. The first step is to prove:

Proposition 3.1. *$A_h \in h\Psi^{-1}_{\hbar,\mathcal F}(\tilde M; \mathcal F)$.*

Proof. The operator A_h is the left quantization of the (a priori tempered distributional) symbol a_h where a_h is the inverse Fourier transform in the second variable z' of the integral kernel: if K_{A_h} is the Schwartz kernel of A_h, then in the sense of oscillatory integrals (or directly if the order of a is sufficiently low)

$$K_{A_h}(z,z') = (2\pi)^{-n} h^{-n/2-1/2} \int e^{i(x-x')\xi/h+(y-y')\eta/h^{1/2}} a_h(x,y,\xi,\eta)\, d\xi\, \mathrm{d}\eta,$$

i.e. $(2\pi)^{-n}$ times the semiclassical foliation Fourier transform in (ξ,η) of

$$(x,y,\xi,\eta) \mapsto e^{ix\cdot\xi/h+iy\cdot\eta/h^{1/2}} a_h(z,\zeta),$$

so taking the semiclassical foliation inverse Fourier transform in (x',y') yields $(2\pi)^{-n} a_h(x,y,\xi,\eta) e^{ix\cdot\xi/h+iy\cdot\eta/h^{1/2}}$, i.e.

$$a_h(z,\zeta) = (2\pi)^n e^{-ix\cdot\xi/h-iy\cdot\eta/h^{1/2}} (\mathcal{F}^{-1}_{h,\mathcal{F}})_{(x',y')\to(\xi,\eta)} K_{A_h}(x,y,x',y'). \tag{3.7}$$

Here we are using local coordinates; we comment below on the considerations when z and z' are far apart and cannot be analyzed in the same coordinate chart.

Now,

$$\begin{aligned} K_{A_h}(x,y,x',y') &= \int e^{-\Phi(x)/h} e^{\Phi(x(\gamma_{z,\lambda,\omega}(t)))/h} \tilde{\chi}(z,\lambda/h^{1/2},\omega) \\ &\qquad \delta(z'-\gamma_{z,\lambda,\omega}(t))\, dt\, |d\nu| \\ &= (2\pi)^{-n} h^{-n/2-1/2} \int e^{-\Phi(x)/h} e^{\Phi(x(\gamma_{z,\lambda,\omega}(t)))/h} \tilde{\chi}(z,\lambda/h^{1/2},\omega) \\ &\qquad e^{-i\xi'\cdot(x'-\gamma^{(1)}_{z,\lambda,\omega}(t))/h} e^{-i\eta'\cdot(y'-\gamma^{(2)}_{z,\lambda,\omega}(t))/h^{1/2}}\, dt\, |d\nu|\, |d\xi'|\, |d\eta'|; \end{aligned}$$

as remarked above, the t integral is actually over a fixed finite interval, say $|t| < T$, or one may explicitly insert a compactly supported cutoff in t instead. (So the only non-compact domain of integration is in (ξ',η'), corresponding to the Fourier transform.) Thus, taking the semiclassical foliation inverse Fourier transform in x', y' and evaluating at ξ,η gives

$$\begin{aligned} a_h(x,y,\xi,\eta) &= \int e^{-\Phi(x)/h} e^{\Phi(x(\gamma_{z,\lambda,\omega}(t)))/h} \tilde{\chi}(z,\lambda/h^{1/2},\omega) \\ &\qquad e^{i\xi\cdot(\gamma^{(1)}_{z,\lambda,\omega}(t)-x)/h} e^{i\eta\cdot(\gamma^{(2)}_{z,\lambda,\omega}(t)-y)/h^{1/2}}\, dt\, |d\nu|. \end{aligned} \tag{3.8}$$

The proof of the proposition is completed by showing that the right hand side is actually h times a symbol of order -1.

Notice that technically we are using local coordinates in (3.8). For the semiclassical foliation pseudodifferential operators for symbolic statements we should be considering z, z' in the same chart as well as when z and z' are apart and cannot be analyzed in the same chart. In the latter case $|t|$ bounded below by a positive constant, and we show that K_{A_h} is smooth and $O(h^\infty)$. This is implied by the semiclassically Fourier transformed, in z', expression being Schwartz, i.e. a_h (and its derivatives) being $O(h^\infty \langle(\xi,\eta)\rangle^{-\infty})$; for this we do not need to explicitly consider a coordinate chart in $z = (x,y)$. We prove this decay below by stationary phase considerations in (λ,ω,t), meaning that the phase is not actually stationary; in this case the oscillatory factor $e^{-i(\xi x/h+\eta\cdot y/h^{1/2})}$ is actually irrelevant.

We first consider the $|t|$ small behavior, say $|t| < T_0$, so a single chart can be used in the analysis (z, z' are in the same chart). We change the variables of integration to $\hat{t} = t/\sqrt{h}$, and $\hat{\lambda} = \lambda/\sqrt{h}$, so the $\hat{\lambda}$ integral is in fact over a fixed compact interval, but the $\hat{t}$ one is over $|\hat{t}| < T_0/\sqrt{h}$ which grows as $h \to 0$; in this process we obtain an additional factor of h from the change of density. We deduce that the phase is

$$\begin{aligned}\xi\cdot(\gamma^{(1)}_{z,\lambda,\omega}(t) - x)/h + \eta \cdot (\gamma^{(2)}_{z,\lambda,\omega}(t) - y)/h^{1/2}\\ = \xi(\hat{\lambda}\hat{t} + \alpha\hat{t}^2 + h^{1/2}\hat{t}^3\Gamma^{(1)}(x, y, h^{1/2}\hat{\lambda}, \omega, h^{1/2}\hat{t}))\\ + \eta \cdot (\omega\hat{t} + h^{1/2}\hat{t}^2\Gamma^{(2)}(x, y, h^{1/2}\hat{\lambda}, \omega, h^{1/2}\hat{t})),\end{aligned}$$

while the exponent of the exponential damping factor (which we regard as a Schwartz function, part of the amplitude, when one regards $\hat{t}$ as a variable on $\mathbb{R}$) is

$$\begin{aligned}-\Phi(x)/h + \Phi(x(\gamma_{z,\lambda,\omega}(t)))/h\\ = x/h - \gamma^{(1)}_{x,y,\lambda,\omega}(t)/h\\ = -h^{-1}(\lambda t + \alpha t^2 + t^3\Gamma^{(1)}(x, y, \lambda, \omega, t))\\ = -(\hat{\lambda}\hat{t} + \alpha\hat{t}^2 + \hat{t}^3h^{1/2}\hat{\Gamma}^{(1)}(x, y, h^{1/2}\hat{\lambda}, \omega, h^{1/2}\hat{t})),\end{aligned}$$

with $\hat{\Gamma}^{(1)}$ a smooth function. Thus, after the rescaling of t and λ to $\hat{t}$ and $\hat{\lambda}$, the integrand of (3.8) is a smooth function of all variables. In view of the Gaussian decay of the exponential damping factor around (3.4), the lack of compactness in the $\hat{t}$ integration domain is not an issue, and we conclude that for ξ, η in a bounded region we conclude that a_h is h times a C^∞ function.

We now consider the behavior of $|(\xi, \eta)| \to \infty$ to complete showing that a_h is a symbol. The only subtlety in applying the stationary phase lemma is that the domain of integration in $\hat{t}$ is not compact, so we need to explicitly deal with the region $|\hat{t}| \geq 1$, say, assuming that the amplitude is Schwartz in $\hat{t}$, uniformly in the other variables (as it is in our case thanks to the exponential weight factor). Notice that as long as the first derivatives of the phase in the integration variables have a lower bound $c|(\xi, \eta)|\,|\hat{t}|^{-k}$ for some k, and for some $c > 0$, the standard integration by parts argument gives the rapid decay of the integral in the large parameter $|(\xi, \eta)|$. At $h = 0$ the phase is $\xi(\hat{\lambda}\hat{t} + \alpha\hat{t}^2) + \hat{t}\eta\cdot\omega$; if $|\hat{t}| \geq 1$, say, the $\hat{\lambda}$ derivative is $\xi\hat{t}$, which is thus bounded below by $|\xi|$ in magnitude, so the only place where one may not have rapid decay is at $\xi = 0$ (meaning, in the spherical variables, $\frac{\xi}{|(\xi,\eta)|} = 0$). In this region one may use $|\eta|$ as the large variable to simplify the notation slightly. The phase is then with $\hat{\xi} = \frac{\xi}{|\eta|}$, $\hat{\eta} = \frac{\eta}{|\eta|}$,

$$|\eta|(\hat{\xi}(\hat{\lambda}\hat{t} + \alpha\hat{t}^2) + \hat{t}\hat{\eta}\cdot\omega),$$

with parameter differentials (ignoring the overall $|\eta|$ factor)

$$\hat{\xi}\hat{t}\,d\hat{\lambda}, (\hat{t}\hat{\eta} + \hat{t}^2\hat{\xi}\partial_\omega\alpha)\cdot\,d\omega, (\hat{\xi}(\hat{\lambda} + 2\alpha\hat{t}) + \hat{\eta}\cdot\omega)\,d\hat{t}.$$

With $\hat{\Xi} = \hat{\xi}\hat{t}$ and $\rho = \hat{t}^{-1}$ these are

$$\hat{\Xi}\,d\hat{\lambda}, \hat{t}(\hat{\eta} + \hat{\Xi}\partial_\omega\alpha)\cdot\,d\omega, (\hat{\Xi}(\rho\hat{\lambda} + 2\alpha) + \hat{\eta}\cdot\omega)\,d\hat{t},$$

and now for critical points $\hat{\Xi}$ must vanish (as we already knew from above), then the last of these gives that $\hat{\eta}\cdot\omega$ vanishes, but then the second gives that there cannot be a critical point (in $|\hat{t}| \geq 1$). While this argument was at $h = 0$, the full phase derivatives are

$$(\hat{\xi}\hat{t}(1+h^{1/2}\hat{t}\partial_\lambda\alpha+h\hat{t}^2\partial_\lambda\Gamma^{(1)})+\hat{\eta}\cdot h\hat{t}^2\partial_\lambda\Gamma^{(2)})\,d\hat{\lambda},$$
$$(\hat{t}\hat{\eta}+h^{1/2}\hat{t}^2\hat{\eta}\cdot\partial_\omega\Gamma^{(2)}+\hat{t}^2\hat{\xi}\partial_\omega\alpha+h^{1/2}\hat{t}^3\hat{\xi}\partial_\omega\Gamma^{(1)})\cdot\,d\omega,$$
$$(\hat{\xi}(\hat{\lambda}+2\alpha\hat{t}+3h^{1/2}\hat{t}^2\Gamma^{(1)}+h\hat{t}^3\partial_t\Gamma^{(1)})+\hat{\eta}\cdot\omega+2h^{1/2}\hat{t}\Gamma^{(2)}+h\hat{t}^2\partial_t\Gamma^{(2)})\,d\hat{t},$$

i.e.

$$(\hat{\Xi}(1+t\partial_\lambda\alpha+t^2\partial_\lambda\Gamma^{(1)})+\hat{\eta}\cdot t^2\partial_\lambda\Gamma^{(2)})\,d\hat{\lambda},$$
$$\hat{t}(\hat{\eta}+\hat{\eta}\cdot t\partial_\omega\Gamma^{(2)}+\hat{\Xi}\partial_\omega\alpha+t\hat{\Xi}\partial_\omega\Gamma^{(1)})\cdot\,d\omega,$$
$$(\hat{\Xi}(\hat{\lambda}\rho+2\alpha+3t\Gamma^{(1)}+t^2\partial_t\Gamma^{(1)})+\hat{\eta}\cdot\omega+2t\Gamma^{(2)}+t^2\partial_t\Gamma^{(2)})\,d\hat{t},$$

and now all the additional terms are small if T_0 is small, where we assume $|t|<T_0$, so the lack of critical points in the $h=0$ computation implies the analogous statement (in $|\hat{t}|>1$) for the general computation *assuming T_0 is sufficiently small.*

Now, if $t\in[T_0,T]$ is not so small, the same result can be achieved under a no-conjugate points assumption, i.e. that the Jacobian $\frac{\partial\gamma}{\partial(t,\lambda,\omega)}$ is full rank for t away from 0. Notice that we might use different coordinate charts for z,z' as discussed at the end of the paragraph of (3.8), with the factor $e^{-i(\xi x/h+\eta\cdot y/h^{1/2})}$ of the integrand irrelevant if one is to prove rapid decay. In this case one can run the non-stationary phase argument directly for t (as opposed to $\hat{t}$) away from 0 and (λ,ω), showing that there are no stationary points, and thus, as $|(\xi,\eta)|\to\infty$ or $h\to 0$, one can reduce to the case $\hat{t}=0$ discussed below (as there are no other non-trivial contributions). Concretely, we need to keep in mind that the exponential weight is bounded by $e^{-\epsilon/h}$ for some $\epsilon>0$ when t is bounded away from 0 (and h is sufficiently small), thus is rapidly decaying as $h\to 0$; in particular this assures the smoothness of a_h, and its rapid decay in h, for bounded (ξ,η). Now, there is $C_0>0$ such that if $|\xi|/h^{1/2}>C_0|\eta|$ then ∂_t of the phase is bounded from below by a positive constant multiple of $|\xi|/h$, since $\partial_t\gamma^{(1)}\neq 0$ by the convexity properties of the foliation. Thus, in this case we deduce rapid decay of the integral in $|\xi|/h$, hence also in $|\eta|/h^{1/2}$. So let $\tilde{\xi}=\xi/h^{1/2}$, and assume that $|\tilde{\xi}|<2C_0|\eta|$. Then the phase becomes

$$h^{-1/2}(\tilde{\xi}(\gamma^{(1)}_{x,y,\lambda,\omega}(t)-x)+\eta\cdot(\gamma^{(2)}_{x,y,\lambda,\omega}(t)-y)),$$

which is a standard $h^{1/2}$-semiclassical phase, whose non-stationarity is assured by the no-conjugate points hypothesis. Due to the exponential weight, the amplitude is rapidly decreasing in h regardless of the singular λ-dependence of $\tilde{\chi}$, which makes stationary phase applicable, giving the desired result of rapid decay in $|\eta|/h^{1/2}$, hence also in $|\tilde{\xi}|/h^{1/2}=|\xi|/h^{1/2}$ in view of the constraint on $\tilde{\xi}$.

This discussion implies that we may work in $|\hat{t}|<2$, say, and one can use the standard *parameter-dependent* stationary phase lemma, see e.g. [3, Theorem 7.7.6] for our stationary phase computation in $(\hat{t},\hat{\lambda},\omega)$. At $h=0$, the stationary points of the phase are $\hat{t}=0$, $\xi\hat{\lambda}+\eta\cdot\omega=0$, which remain critical points for h non-zero due to the $h^{1/2}\hat{t}^2$ vanishing of the other terms, and when $|\hat{t}|<2$ and h is sufficiently small, so $h^{1/2}\hat{t}$ is small, there are no other critical points. (One can see this in a different way: above we worked with $|\hat{t}|\geq 1$, but for any $\epsilon>0$, $|\hat{t}|\geq\epsilon$ would have worked equally.) These critical points lie on a smooth codimension 2 submanifold of the parameter space. For the following argument it is useful to consider (ξ,η) jointly, and write $\hat{\xi}=\frac{\xi}{|(\xi,\eta)|}$, $\hat{\eta}=\frac{\eta}{|(\xi,\eta)|}$. Moreover, we write $\theta=(\hat{\lambda},\omega)$, and decompose it into a parallel and orthogonal component relative to $(\hat{\xi}, \hat{\eta})$: $\theta^{\|} = (\hat{\xi}, \hat{\eta}) \cdot (\hat{\lambda}, \omega)$, resp. $\theta^{\perp}$. At $h = 0$, the $(\hat{t}, \theta^{\|})$-Hessian matrix of the phase

$$|(\xi, \eta)|(\hat{\xi}(\hat{\lambda}\hat{t} + \alpha\hat{t}^2) + \hat{t}\hat{\eta} \cdot \omega) = |(\xi, \eta)|(\hat{t}\hat{\theta}^{\|} + \hat{\xi}\alpha\hat{t}^2)$$

within $\hat{t} = 0$ (where the critical set lies) is

$$|(\xi, \eta)| \begin{pmatrix} 2\hat{\xi}\alpha & 1 \\ 1 & 0 \end{pmatrix},$$

which is always invertible. This also implies, by continuity (and homogeneity in (ξ, η)) of the Hessian the invertibility for small h. We thus use the stationary phase lemma in the $(\hat{t}, \theta^{\|})$ variables, which shows that a_h is (h times, due to the integration variable change, already mentioned for the finite ξ, η discussion!) a symbol of order -1, since the stationary phase is with respect to a 2-dimensional space. □

We now proceed to compute the semiclassical principal symbol:

Proposition 3.2. *There exists $\tilde{\chi}$ of compact support such that the operator $A_h \in h\Psi^{-1}_{\hbar,\mathcal{F}}(\tilde{M}; \mathcal{F})$ is elliptic on M.*

Proof. For the semiclassical principal symbol computation we may simply set $h = 0$ in the above rescaled expression used for the stationary phase argument. Thus, with $\tilde{\chi} = \chi(\lambda/h^{1/2}) = \chi(\hat{\lambda})$, we have that

$$\begin{aligned} a_h(x, y, \xi, \eta) = h \int & e^{i(\xi h^{-1}(\gamma^{(1)}_{x,y,x\hat{\lambda},\omega}(h^{1/2}\hat{t}) - x) + \eta h^{-1/2} x^{-1}(\gamma^{(2)}_{x,y,x\hat{\lambda},\omega}(h^{1/2}\hat{t}) - y))} \\ & e^{-(\hat{\lambda}\hat{t} + \alpha\hat{t}^2)} \chi(\hat{\lambda})\, d\hat{t}\, d\hat{\lambda}\, d\omega \\ = h \int & e^{i(\xi(\hat{\lambda}\hat{t} + \alpha\hat{t}^2 + h^{1/2}\hat{t}^3\Gamma^{(1)}(x,y,h^{1/2}\hat{\lambda},\omega,h^{1/2}\hat{t})) + \eta\cdot(\omega\hat{t} + h^{1/2}\hat{t}^2\Gamma^{(2)}(x,y,h^{1/2}\hat{\lambda},\omega,h^{1/2}\hat{t})))} \\ & e^{-(\hat{\lambda}\hat{t} + \alpha\hat{t}^2)} \chi(\hat{\lambda})\, d\hat{t}\, d\hat{\lambda}\, d\omega, \end{aligned} \tag{3.9}$$

up to errors that are $O(h^{1/2}\langle\xi, \eta\rangle^{-1})$ relative to the a priori order, -1, arising from the 0th order symbol in the oscillatory integral and the 2-dimensional space in which the stationary phase lemma is applied, and semiclassical order -1, corresponding to the factor of h in front. Factoring out the overall h, this in turn becomes, modulo $O(h^{1/2})$ errors, i.e. at the semiclassical foliation principal symbol level,

$$a_0(x, y, \xi, \eta) = \int e^{i(\xi(\hat{\lambda}\hat{t} + \alpha\hat{t}^2) + \eta\cdot\omega\hat{t})} e^{-(\hat{\lambda}\hat{t} + \alpha\hat{t}^2)} \chi(\hat{\lambda})\, d\hat{t}\, d\hat{\lambda}\, d\omega. \tag{3.10}$$

Now computing the principal symbol of this, i.e. the behavior as $|(\xi, \eta)| \to \infty$, we consider the critical points of the phase, $\hat{t} = 0$, $\theta^{\|} \equiv \hat{\xi}\hat{\lambda} + \hat{\eta} \cdot \omega = 0$, where $\theta^{\perp}$ is the variable along the critical set (where $\hat{t}$ and $\theta^{\|}$ vanish), which gives, up to an overall elliptic factor, keeping in mind that $\hat{\lambda}$ depends on $\theta^{\perp}$ along this equator (namely along $\theta^{\|} = 0$),

$$\int_{\mathbb{S}^{n-2}} \chi(\hat{\lambda}(\theta^{\perp}))\, d\theta^{\perp},$$

which is elliptic for $\chi \geq 0$ with $\chi(0) > 0$ since the codimension one planes (intersected with a sphere) $\theta^{\|} = 0$ and $\hat{\lambda} = 0$ necessarily intersect in at least a line (intersected with the sphere) as the dimension is $n \geq 2 + 1 = 3$.

This of course implies that for finite, but sufficiently large, (ξ, η), the semiclassical symbol a_0 is elliptic. For general finite (ξ, η) it is harder to compute a_0 explicitly for general χ. However, when χ is a Gaussian, the computation is straightforward. We write

$$\begin{aligned} a_0(x, y, \xi, \eta) &= \int e^{-(\alpha(1-i\xi)\hat{t}^2+\hat{t}(\hat{\lambda}(1-i\xi)-i\eta\cdot\omega)}\chi(\hat{\lambda})\,d\hat{t}\,d\hat{\lambda}\,d\omega \\ &= \int e^{-\alpha(1-i\xi)(\hat{t}+\frac{\hat{\lambda}(1-i\xi)-i\eta\cdot\omega}{2\alpha(1-i\xi)})^2} e^{\frac{(\hat{\lambda}(1-i\xi)-i\eta\cdot\omega)^2}{4\alpha(1-i\xi)}}\chi(\hat{\lambda})\,d\hat{t}\,d\hat{\lambda}\,d\omega \\ &= c\int \alpha^{-1/2}(1-i\xi)^{-1/2} e^{\frac{(\hat{\lambda}(1-i\xi)-i\eta\cdot\omega)^2}{4\alpha(1-i\xi)}}\chi(\hat{\lambda})\,d\hat{\lambda}\,d\omega \\ &= c\int \alpha^{-1/2}(1-i\xi)^{-1/2} e^{\frac{\hat{\lambda}^2(1-i\xi)}{4\alpha}-i\frac{\hat{\lambda}}{2\alpha}\eta\cdot\omega-\frac{(\eta\cdot\omega)^2}{4\alpha(1-i\xi)}}\chi(\hat{\lambda})\,d\hat{\lambda}\,d\omega, \end{aligned}$$

with c a non-zero constant. Now a particularly helpful choice is $\chi(\hat{\lambda}) = e^{-\hat{\lambda}^2/(2\alpha)}$, for then we have

$$\begin{aligned} a_0(x, y, \xi, \eta) &= c\int \alpha^{-1/2}(1-i\xi)^{-1/2} e^{-\frac{\hat{\lambda}^2(1+i\xi)}{4\alpha}-i\frac{\hat{\lambda}}{2\alpha}\eta\cdot\omega-\frac{(\eta\cdot\omega)^2}{4\alpha(1-i\xi)}}\,d\hat{\lambda}\,d\omega \\ &= c\int \alpha^{-1/2}(1-i\xi)^{-1/2} e^{-\frac{1+i\xi}{4\alpha}(\hat{\lambda}+i\frac{\eta\cdot\omega}{1+i\xi})^2-\frac{(\eta\cdot\omega)^2}{4\alpha(1+i\xi)}-\frac{(\eta\cdot\omega)^2}{4\alpha(1-i\xi)}}\,d\hat{\lambda}\,d\omega \\ &= c'\int (1+\xi^2)^{-1} e^{-\frac{(\eta\cdot\omega)^2}{2\alpha(1+\xi^2)}}\,d\omega \end{aligned}$$

and the integral is now positive since the integrand is such, while c' is a new non-zero constant. It follows immediately that the same positivity property is maintained if χ is close to the Gaussian in the space of Schwartz functions, which can be achieved by taking a compactly supported χ. □

In view of the errors of the elliptic parametrix construction being small in the semiclassical Sobolev spaces, for sufficiently small h (but h can be fixed to such a small value, so A_h is a standard pseudodifferential operator then, and the Sobolev spaces are standard Sobolev spaces with an equivalent norm!), as discussed in Section 2, *this proves Theorem 1.1.*

The scattering version is quite similar; recall that $x = \tilde{x} + c$ in terms of the original foliation function $\tilde{x}$. The cutoff scaling we use in this case is

$$\tilde{\chi}(z, \lambda/(xh^{1/2}), \omega). \tag{3.11}$$

Thus, writing scattering covectors as $\xi_{\mathrm{sc}}\,\frac{dx}{x^2} + \eta_{\mathrm{sc}}\,\frac{dy}{x}$, i.e. substituting

$$\xi_{\mathrm{sc}} = x^2\xi,\ \ \eta_{\mathrm{sc}} = x\eta,$$

into (3.8)

$$\begin{aligned} a_h(x, y, \xi_{\mathrm{sc}}, \eta_{\mathrm{sc}}) = \int & e^{-\Phi(x)/h} e^{\Phi(x(\gamma_{z,\lambda,\omega}(t)))/h}\tilde{\chi}(z, \lambda/(xh^{1/2}), \omega) \\ & e^{ix^{-2}\xi_{\mathrm{sc}}\cdot(\gamma^{(1)}_{z,\lambda,\omega}(t)-x)/h} e^{ix^{-1}\eta_{\mathrm{sc}}\cdot(\gamma^{(2)}_{z,\lambda,\omega}(t)-y)/h^{1/2}}\,dt\,|d\nu|, \end{aligned} \tag{3.12}$$

with $\Phi(x) = x^{-1}$ in this case.

Proposition 3.3. *Let* $\tilde{M}_c = \tilde{M} \cap \{\tilde{x} \geq -c\} = \tilde{M} \cap \{x \geq 0\}$. *Then* $A_h \in h\Psi^{-1,-2}_{\mathrm{sc},\hbar,\mathcal{F}}(\tilde{M}_c; \mathcal{F})$.

Proof. We change the variables of integration to $\hat{t} = t/(h^{1/2}x)$, and $\hat{\lambda} = \lambda/(h^{1/2}x)$, so again the $\hat{\lambda}$ integral is in fact over a fixed compact interval, but the $\hat{t}$ one is over $|\hat{t}| < T/(x\sqrt{h})$ which grows as $h \to 0$ or $x \to 0$. We get that the phase is

$$\xi_{\mathrm{sc}}(\hat{\lambda}\hat{t} + \alpha\hat{t}^2 + xh^{1/2}\hat{t}^3\Gamma^{(1)}(x, y, xh^{1/2}\hat{\lambda}, \omega, xh^{1/2}\hat{t})) \\ + \eta_{\mathrm{sc}} \cdot (\omega\hat{t} + xh^{1/2}\hat{t}^2\Gamma^{(2)}(x, y, h^{1/2}\hat{\lambda}, \omega, xh^{1/2}\hat{t})),$$

while the exponential damping factor (which we regard as a Schwartz function, part of the amplitude, when one regards $\hat{t}$ as a variable on $\mathbb{R}$) is

$$\begin{aligned} &-1/(hx) + 1/(h\gamma^{(1)}_{x,y,\lambda,\omega}(t)) \\ &= -h^{-1}(\lambda t + \alpha t^2 + t^3\Gamma^{(1)}(x, y, \lambda, \omega, t))x^{-1}(x + \lambda t + \alpha t^2 + t^3\Gamma^{(1)}(x, y, \lambda, \omega, t))^{-1} \\ &= -(\hat{\lambda}\hat{t} + \alpha\hat{t}^2 + \hat{t}^3 xh^{1/2}\hat{\Gamma}^{(1)}(x, y, xh^{1/2}\hat{\lambda}, \omega, xh^{1/2}\hat{t})), \end{aligned}$$

with $\hat{\Gamma}^{(1)}$ a smooth function. Thus, for ξ, η in a bounded region we conclude that a_h is a C^∞ function. Furthermore, we observe that with $(\xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})$ in place of (ξ, η), and in the new integration variables $\hat{t}$ and $\hat{\lambda}$, (3.12) has the same form as (3.8), so identical stationary phase arguments are applicable. In particular, the t bounded away from 0 case proceeds analogously with $xh^{1/2}$ playing the role of $h^{1/2}$, keeping in mind that the exponential weight is bounded by $e^{-\epsilon/(xh)}$ for t bounded away from 0, so is rapidly decaying in xh; in this case $|\xi_{\mathrm{sc}}|/(xh^{1/2}) > C_0|\eta_{\mathrm{sc}}|$ assures the possibility of t-integration by parts to obtain rapid decay in xh, while if $|\widetilde{\xi_{\mathrm{sc}}}| = |\xi_{\mathrm{sc}}|/(xh^{1/2}) < 2C_0|\eta_{\mathrm{sc}}|$, then one can apply a standard no-stationary point argument as above under the no-conjugate point assumption since the phase is $x^{-1}h^{-1/2}$ times a usual homogeneous degree 1 phase in $(\widetilde{\xi_{\mathrm{sc}}}, \eta_{\mathrm{sc}})$, giving rapid decay in $x^{-1}h^{-1/2}|\eta_{\mathrm{sc}}|$, thus in $x^{-2}h^{-1}|\xi_{\mathrm{sc}}|$ as well. □

Finally, it remains to compute the semiclassical foliation scattering principal symbol, which is, taking into account the density factor hx^2 from the change of variables,

$$a_0(x, y, \xi_{\mathrm{sc}}, \eta_{\mathrm{sc}}) = hx^2 \int e^{i(\xi_{\mathrm{sc}}(\hat{\lambda}\hat{t}+\alpha\hat{t}^2)+\eta_{\mathrm{sc}}\cdot\omega\hat{t})} e^{-(\hat{\lambda}\hat{t}+\alpha\hat{t}^2)} \chi(\hat{\lambda})\, d\hat{t}\, d\hat{\lambda}\, d\omega. \tag{3.13}$$

Then completely analogously to the above computation yields that as $|(\xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})| \to \infty$, up to an overall elliptic factor, we have

$$\int_{\mathbb{S}^{n-2}} \chi(\hat{\lambda}(\theta^\perp))\, d\theta^\perp,$$

which is elliptic for $\chi \geq 0$ with $\chi(0) > 0$. Further, the ellipticity at finite points follows the same computation as above, for the same choice of χ, $\chi(\hat{\lambda}) = e^{-\hat{\lambda}^2/(2\alpha)}$, with (ξ, η) replaced by $(\xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})$. Again, in view of the errors of the elliptic parametrix construction being small in the semiclassical Sobolev spaces as discussed in Section 2, *this proves Theorem 1.2.*

References

[1] Maarten de Hoop, Gunther Uhlmann, and András Vasy. Diffraction from conormal singularities. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(2):351–408, 2015.

[2] Oran Gannot and Jared Wunsch. Semiclassical diffraction by conormal singularities. *Preprint, arXiv:1806.01813*, 2018.

[3] L. Hörmander. *The analysis of linear partial differential operators,* vol. 1-4. Springer-Verlag, 1983.

[4] R. B. Melrose. Lecture notes for '18.157: Introduction to microlocal analysis'. Available at `http://math.mit.edu/ rbm/18.157-F09/18.157-F09.html`, 2009.

[5] Richard B. Melrose. Transformation of boundary problems. *Acta Math.*, 147(3-4):149–236, 1981.

[6] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1993.

[7] Richard B. Melrose. Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces. In *Spectral and scattering theory (Sanda, 1992)*, volume 161 of *Lecture Notes in Pure and Appl. Math.*, pages 85–130. Dekker, New York, 1994.

[8] Cesare Parenti. Operatori pseudo-differenziali in R^n e applicazioni. *Ann. Mat. Pura Appl. (4)*, 93:359–389, 1972.

[9] Johannes Sjöstrand and Maciej Zworski. Fractal upper bounds on the density of semiclassical resonances. *Duke Math. J.*, 137(3):381–459, 2007.

[10] Plamen Stefanov and Gunther Uhlmann. Rigidity for metrics with the same lengths of geodesics. *Math. Res. Lett.*, 5(1-2):83–96, 1998.

[11] Plamen Stefanov and Gunther Uhlmann. Integral geometry of tensor fields for a class of non-simple riemannian manifolds. *Amer. J. Math.*, 130:239–268, 2008.

[12] Plamen Stefanov, Gunther Uhlmann, and Andras Vasy. Local and global boundary rigidity and the geodesic X-ray transform in the normal gauge. *Preprint, arXiv:1702.03638*, 2017.

[13] M. A. Šubin. Pseudodifferential operators in R^n. *Dokl. Akad. Nauk SSSR*, 196:316–319, 1971.

[14] Gunther Uhlmann and András Vasy. The inverse problem for the local geodesic ray transform. *Invent. Math.*, 205(1):83–120, 2016.

[15] A. Vasy and M. Zworski. Semiclassical estimates in asymptotically Euclidean scattering. *Commun. Math. Phys.*, 212:205–217, 2000.

[16] András Vasy. A minicourse on microlocal analysis for wave propagation. In *Asymptotic analysis in general relativity*, volume 443 of *London Math. Soc. Lecture Note Ser.*, pages 219–374. Cambridge Univ. Press, Cambridge, 2018.

[17] Evangelie Zachos. *The X-ray transform on asymptotically Euclidean spaces*. PhD thesis, Stanford University, 2020.

[18] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

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